First Course on Partial Differential Equations - II
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Lecture – 01
Laplace Perron’s Method

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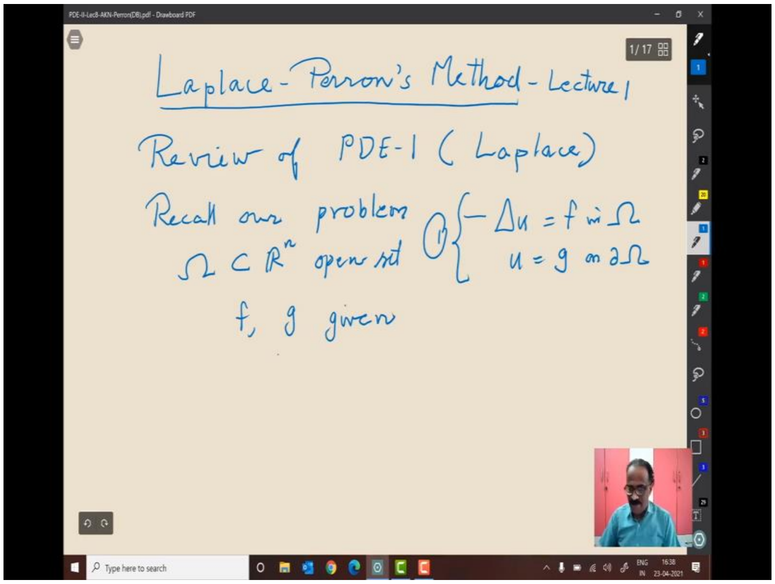

Good morning, so now we are going to discuss about the some material from the Laplace equation. So, in this set of lectures we have 2 things one is about the Perron's method of existence of boundary variable problem and then we will also discuss about the Newtonian potential namely potential theory related to existence of Laplace equation. So, in this first lecture we will recall about the Laplace equation as it is some of its properties which we have discussed in the PDE 1 course.

So, let me recall, so this lecture is going to be the review of PDE 1 about the Laplace equation we just recall the results. So, if you are not familiar with you should go through before continuing this course. So, let us recall our problem of study recall our problem. So, you have an Omega subset of R n open set most of the time you will see bounded open set, but start with this Omega open set in R n and you are interested in studying minus Laplace u equal to f in Omega and u = g on d Omega.

So, this is the problem these problems we call it as one where f and g are given f is g given with normally assume some conditions which we will discuss with you as we proceed.

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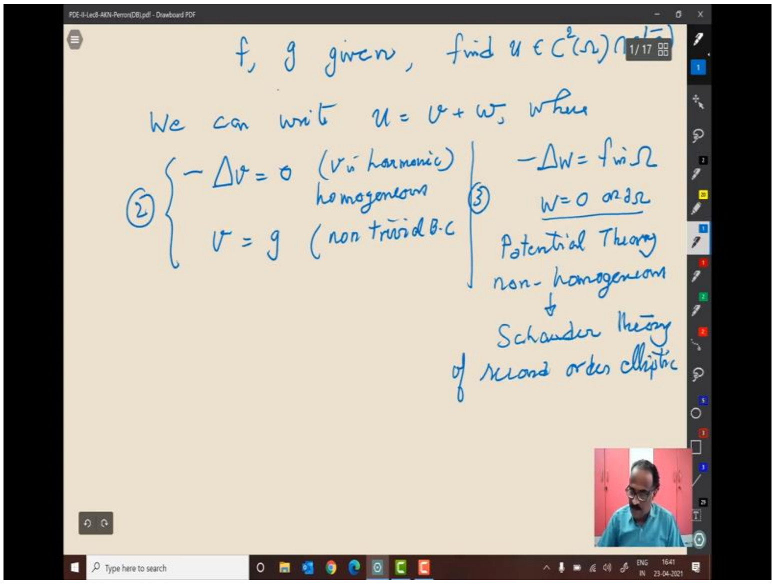

So, f and g are given so the problem is that find the u in appropriate spaces, find u basically you want to find u a classical solution it should be C 2 of Omega to define what is Laplacian of u and it should be intersection of C of Omega bar. So, you need continuity up to the boundary because you want to define what is u = g. So, this problem can be solved we can write these are all we have discussed last time.

We can write u is equal to we split this problem into 2 problems that is what how we are going to study, where we will satisfy a homogeneous equation Laplacian of v that is means that should be v, v is harmonic that is homogeneous equation with non trivial g, so you with the non trivial boundary condition. On the other hand w satisfies minus Laplacian of W = f and W = 0 on d Omega this is in Omega and this is so this is your problem 2 I will be recalling, this we are problem 3 and this is called the potential theory.

So, we will study about that, so we will split our lectures into 2 parts. One is about the potential theory, the other one but trivial boundary conditions, so this is a trivial boundary condition but non homogeneous data non homogeneous this equation. And this is what I want eventually after studying we will not do it in this course. Then there is more general theory about Schauder theory of second order elliptic operator's, so if you want to understand this one.

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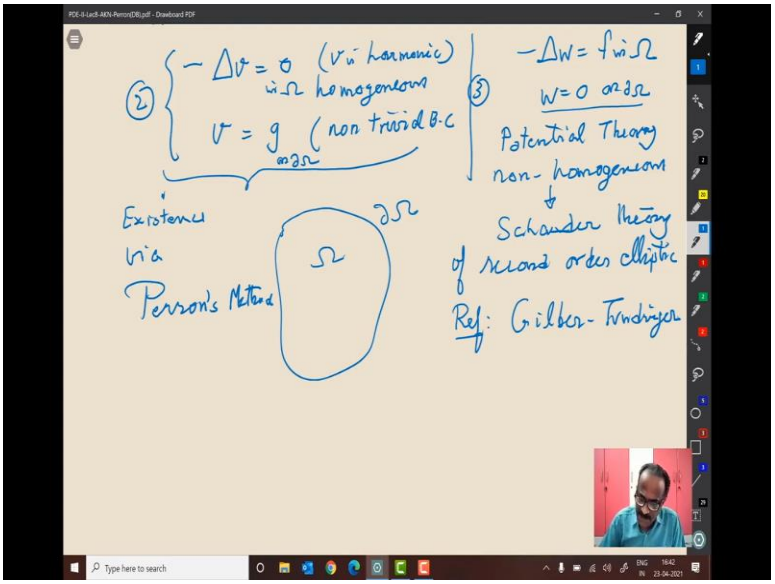

You can see the reference Gilberk and Trudinger if you want to see more about Gilberk-Trudinger, here this is me what we are going to do first in the beginning this is what we are going to study. You have seen some special problems in the upper half plane and above all we will do it more general Omega this in Omega this is on d Omega, d Omega is the boundary. So, you have typically domain Omega this will be going Omega.

And you have your d Omega, this we will prove the existence via Perron's method things are a bit delicate. So, we will try to motivate you some of the things here. So, this is what we are going to do. So, maybe this we will finish of Perron's method in 3 or 4 lectures and another 5 lectures or something we will do potential theory. So, basically we will do the extension of what we have studied in the last PDE 1 course around 9 to 10 lectures.

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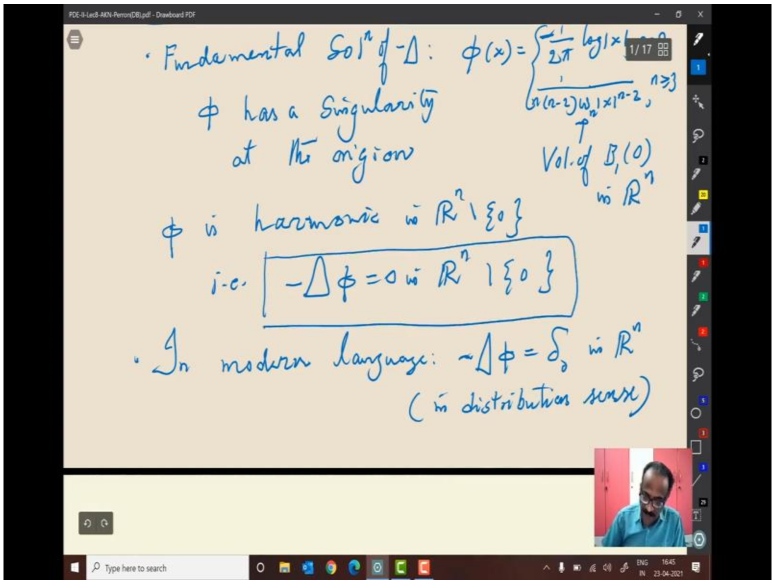

So, let me recall some of the things now this is what essentially we are going to study. So, recall in that one, the one of the first thing we have introduced is fundamental solution. So, go through it, what is the fundamental solution of minus Laplacian? So, what do we have introduced a phi x which is a singularity at the origin this is minus 1 over 2 pi log mod x in the case of dimension 2 depends on dimension.

And then otherwise 1 / n into n - 2 Omega n modulus of x power n -2 this is when n greater than or equal to 3. So, this is the volume of the unit bar value of unit bar radius B 1 0 in R n. So that is how and this is a singularity phi as a singularity at the origin and we know that phi is harmonic in R n - 0. So, you see except at the origin it is harmonic that is Laplacian of phi we write it minus always equal to 0 in R n - 0.

But in modern language which we may not study here but we may hint at the end of this course a little bit about it in modern language it is called minus Laplacian of phi equal to Dirac delta at the origin in R n this is in distribution theory we will tell you later we already mentioned the what is the meaning of this probably we will do it in distribution sense. So that is what we have done. So, minus Laplacian.

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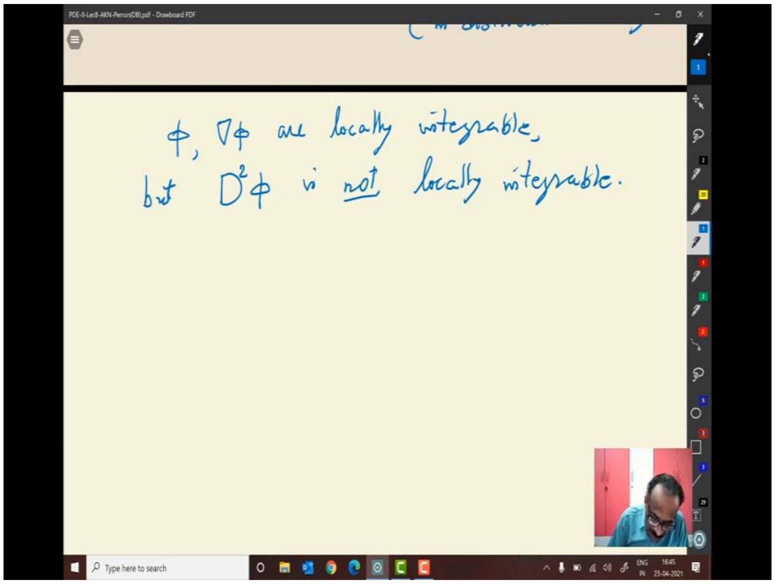

And you have done a lot of problems in this previous case what are the troubles and advantages? If you look at it the function phi even though there is a singularity phi and phi and grade phi singularity are not serious even though there is singularity these are locally integrable, but when you go to second derivative is not locally integrable that is way you cannot take it locally integrable you see. So, the first result we have proved is the first result we have proved in this direction.

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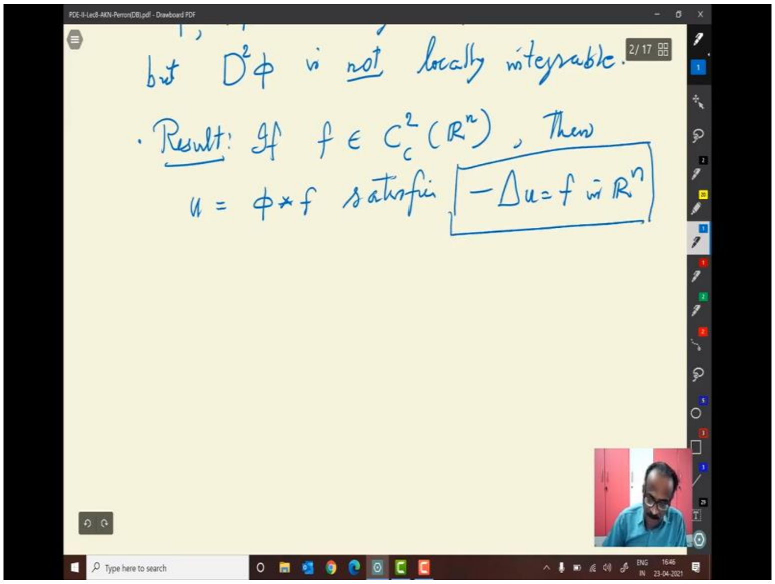

One result we have proved immediately after introducing if f is in twice continuously differentiable function with the compact support it is not a boundary value problem it is in then if you define you as the convolution with the phi satisfies of solves, satisfies minus Laplacian of U equal to f in R n. So, you recovered the source term by convolve in with it, so this is one result we have proved.

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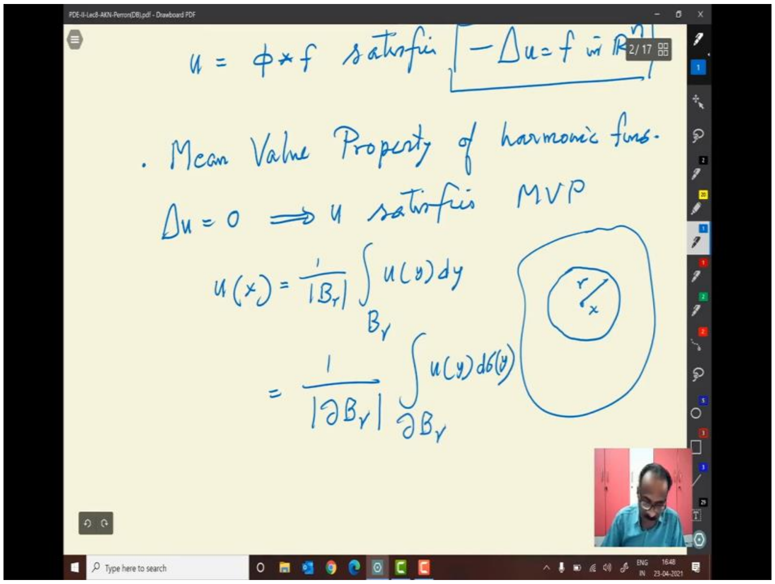

But the major study in the previous course about various properties about it one of the important thing we have studied what is called mean value property of harmonic functions what is that? Laplacian of u = 0 implies mean value property implies u satisfies mean value property that is if you have a domain here you have any point x here you take any ball of radius r you can compute your value in fact these are if and only if property basically eventually we have seen that this is if and only if property.

You can compute the u of x is equal this is let not mentioned that one u of x is equal to the average of B over B this is the any ball of radius. Ball of radius r you integrate over B of r u of y dy this is also equal to you can average it out of the boundary of ball, ball integral over B r u of y d sigma y this you integrate with respect to the surface measure.

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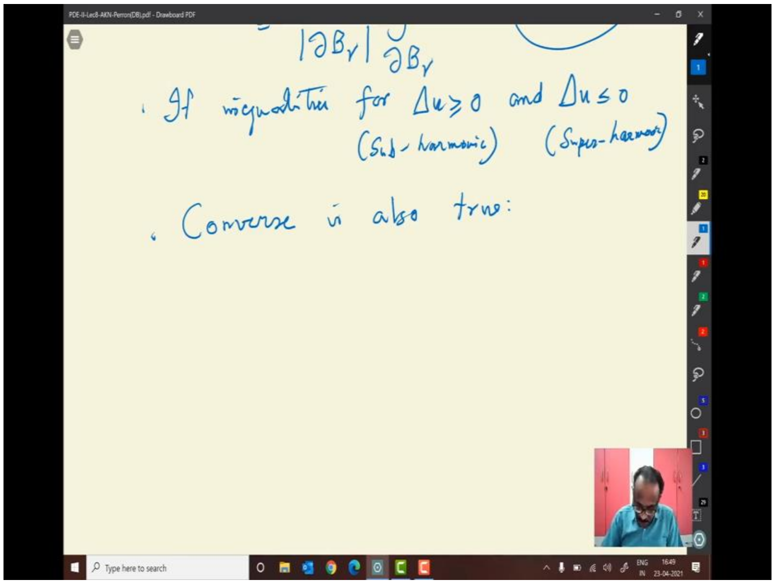

In fact you have inequalities for a Laplacian u greater than or equal to 0 and Laplacian of u less than or equal to 0 these are called sub harmonic, you will see more general definition of sub harmonic function later in Perron's method in the next class. This is called super harmonic we have motivated enough these things in the PDE 1 course. So, if u is a harmonic function and then u satisfy this one. And it is if and only if property that means the converse is also true, that is if u is harmonic you have this equality but you can if this satisfies for all continuous function.

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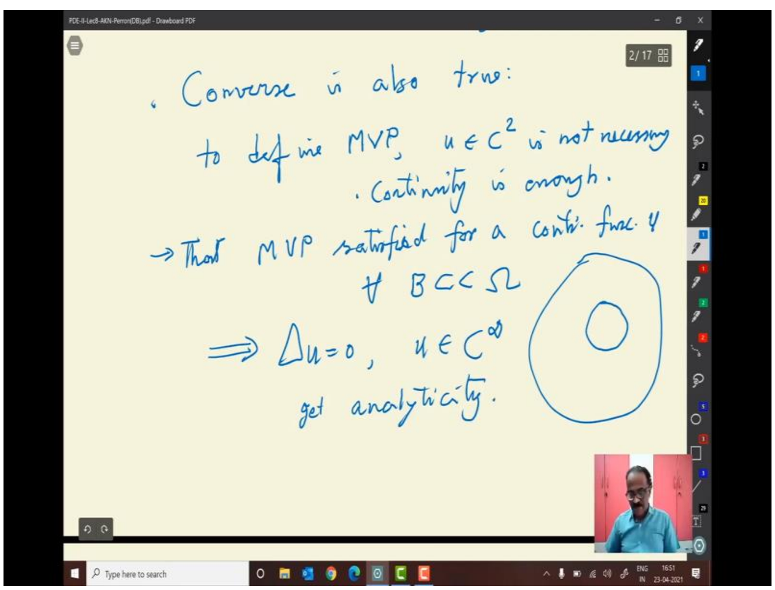

That advantage is that to define these functions you do not need the differentiability of u, to define mean value property u is in C 2 is not necessary you see continuity is enough that is what they have to see that. So, the mean value property actually gives you the smoothness of the product that means that is mean value property for you will not state everything which you there are other wise you can refer the book and recall everything.

Means value property satisfied for continuous function u for all ball inside contain completely in Omega you should not be boundary. So, you have for all ball mean value properties satisfied then you can actually show that Laplacian u = 0 that is what and in fact you can actually show that u is infinity and you can also get analyticity. So, you see you can prove all that in fact we have proved all these facts in the last class.

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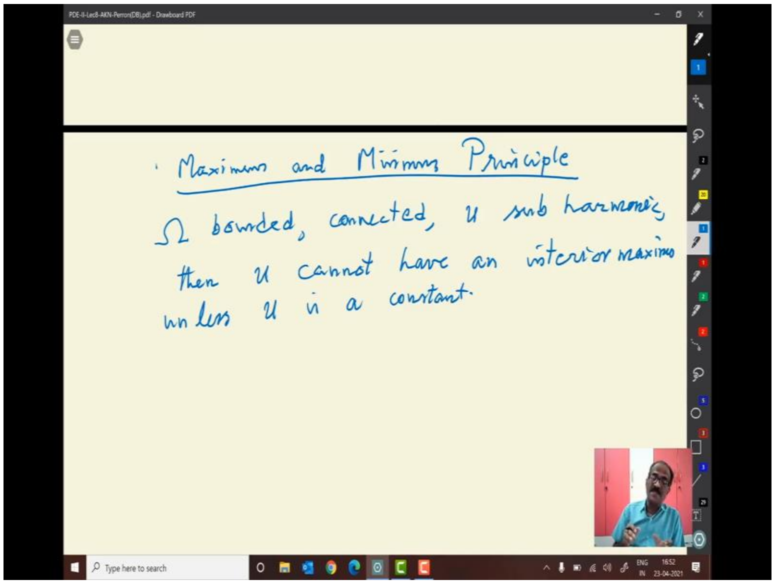

That is what about the mean value property and another important thing we proved in the last class is maximum minimum principles because we are going to use all this maximum and minimum principles. So, let me only write the maximum principle but you can also write down the minimum principle. So, in this case if Omega bounded, connected then u sub harmonic then u cannot have an interior maximum unless u is a constant.

That is the maximum point inside unless u is a constant, of course constant means every point is a maximum and every point is that if u is not a constant function. And if it is a sub harmonic function in a connected bounded set then it cannot have an interior, this has formatting implications.

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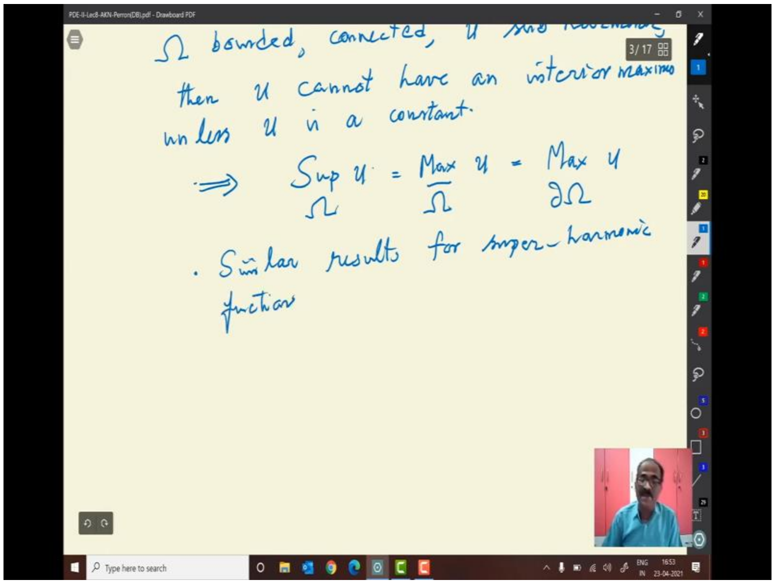

And in fact this also implies immediately if you look at your supremum of u, u is bounded up to the boundary. So, supremum you are taking an Omega will be the same as maximum in Omega u in Omega bar and the maximum it since Omega bar is at close to bounded set it will have a maximum in Omega bar not in Omega. And hence that maximum has to be on the boundary. So, you will get this is same as maximum over u on the boundary you see.

Similar results for super harmonic functions; so in the case of super harmonic function it cannot have an interior minimum harmonic function. And if you happen to be harmonic it will not have an interior maximum and it cannot have an interior minimum and the minimum and maximum attained on the boundary.

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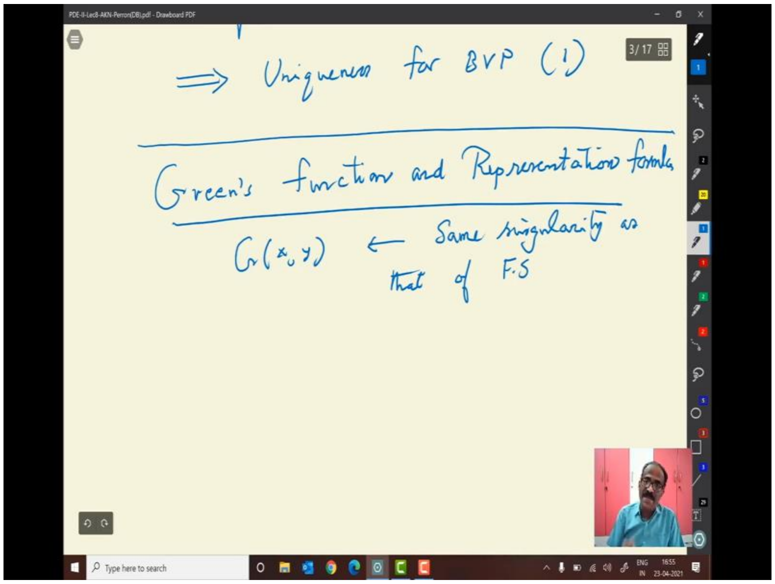

This using this we have immediately proved uniqueness of our initial boundary value problem, uniqueness for initial boundary value problem 1 which we have proved in the last class. After that this is what about the maximum principle we have proved and many other results we have proved in our PDE 1 course, after that we have introduced what is called a Green's function and representation formula. So, Green's function we have introduced as a 2 variable function G x y same type of singularity as fundamental solution.

You know that even though fundamental solution has a singularity and it is not a harmonic function in the entire plane it provides solutions in one of the results we have proved that using fundamental solution you got the solution actually. So that singularity provides you solutions is capture the source term they covered the source term all the things, so the same singularity as that of fundamental solution. So, this is the x y have the same singularity at x = y for that one.

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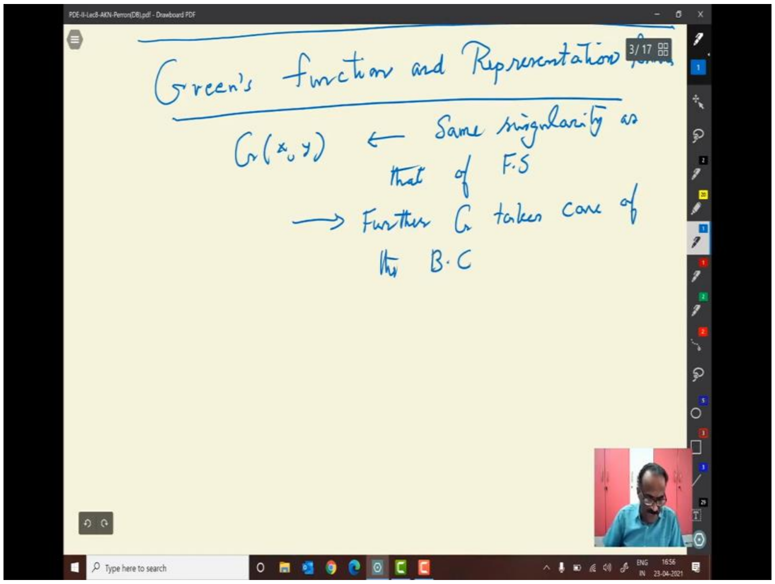

And what is additional thing and further G takes care of the boundary condition because phi you have not imposed in the boundary condition but here G also takes care of the boundary condition. So, we have explained on that thing but let me not get into all the details here but you should understand this Green's function etcetera.

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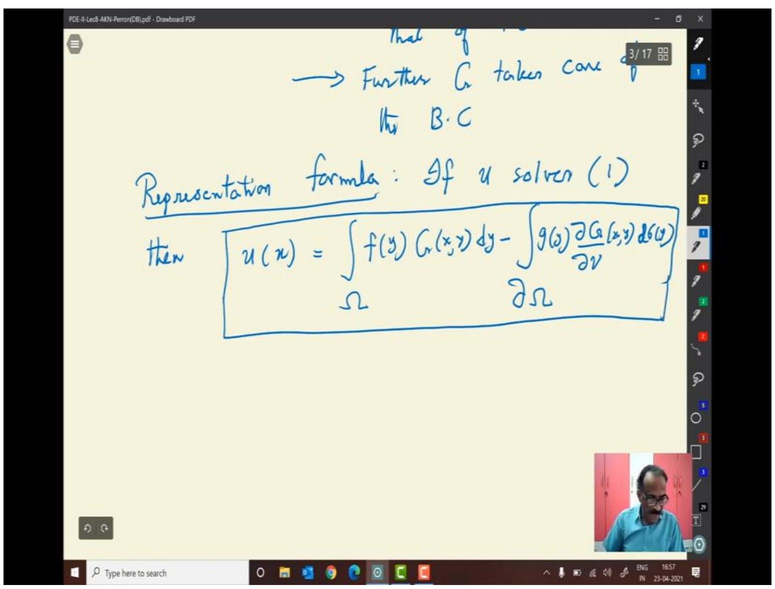

So, with this provider representation formula that is the important thing. So, this is a representation in the sense that if u solves 1 in a classical way solves the problem the boundary minus Laplacian u = f and u = g in Omega then u has a representation that does not mean that it is only if you solve use it this one, that does not mean that this will be a solution we will make this thing immediately u is a representation this is in Omega f of y G of x y dy minus integral over d Omega g of y dG / d nu this is the normal derivative of x y and integrated with respect to the surface measure. So, you see so you have your representation formula.

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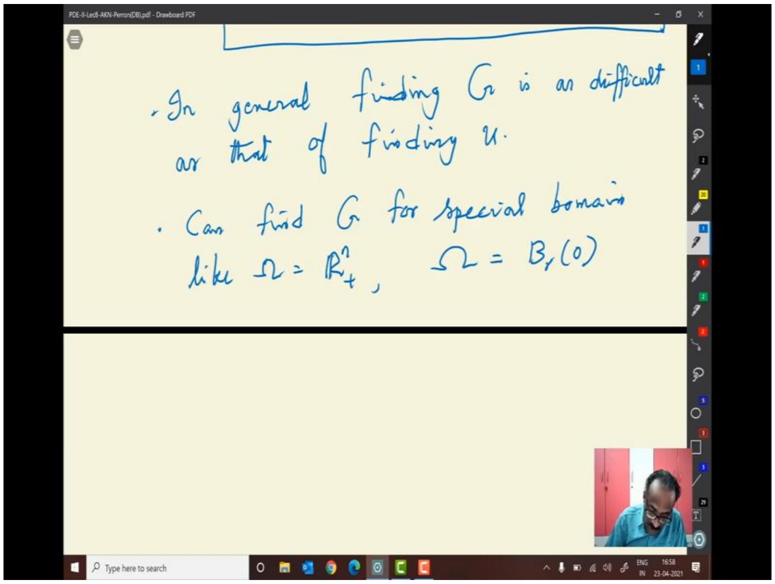

And then in general finding G is hard finding G is as difficult as that of finding u then what is the advantage why do we introduced G but can find G for special domains that is the thing it is you find G for special domains, like a ball like the upper half plane like Omega is equal to the upper half plane R n plus or Omega is equal to a ball of any ball in fact a ball of radius use this to construct solutions.

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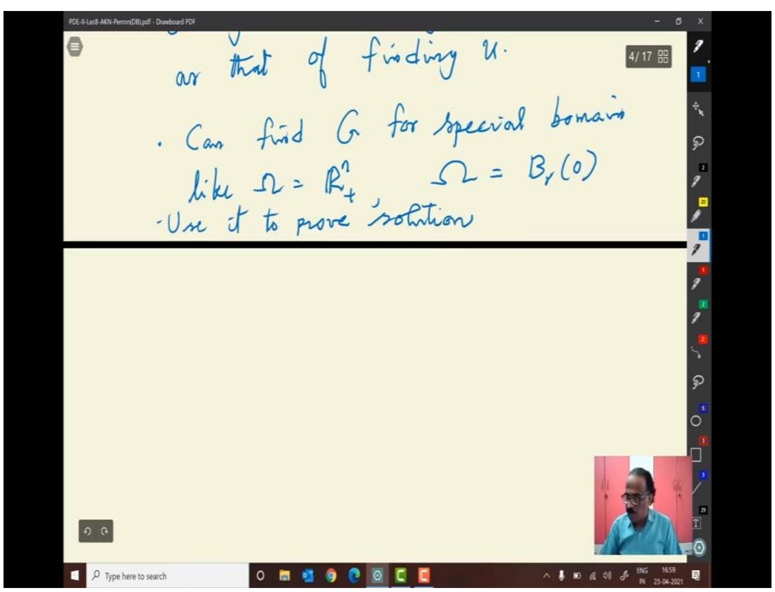

So, we use it to construct solutions use it to prove solutions and that whole idea which we will be doing here, so before completing today's talk.

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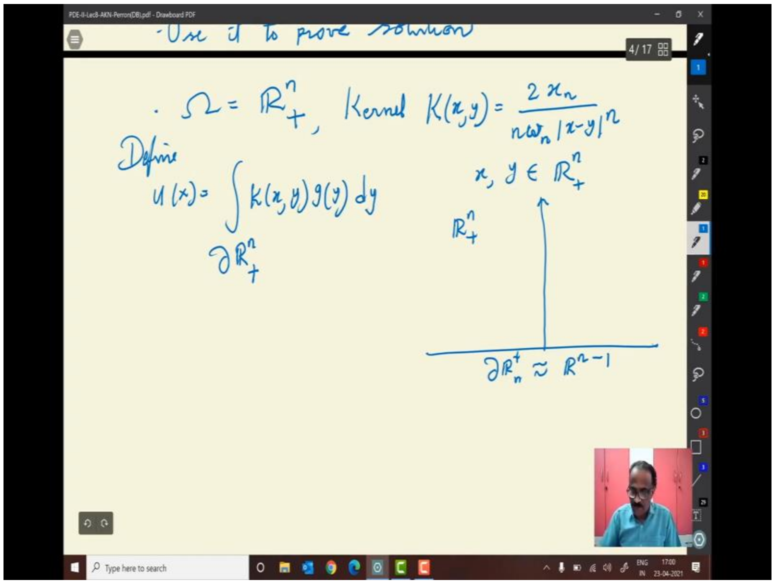

So, let me 2 cases let me do when Omega = R n plus and it will define the kernel this is called the Poisson kernel K x y these are all again introduced there so you defined kernel is equal to 2x n / n Omega n Omega n mod x - y power n this is the kernel for x y in R n plus. So, this is your R n plus this is your domain R n plus and this is your domain boundary R n plus, which of course can be identified with R n -1. And with this you can write, you define by the representation formula defined u of x is equal to integral over the boundary R n + = k of x y g of y dy and taking f = 0 because that is a Newtonian potential which we will study later.

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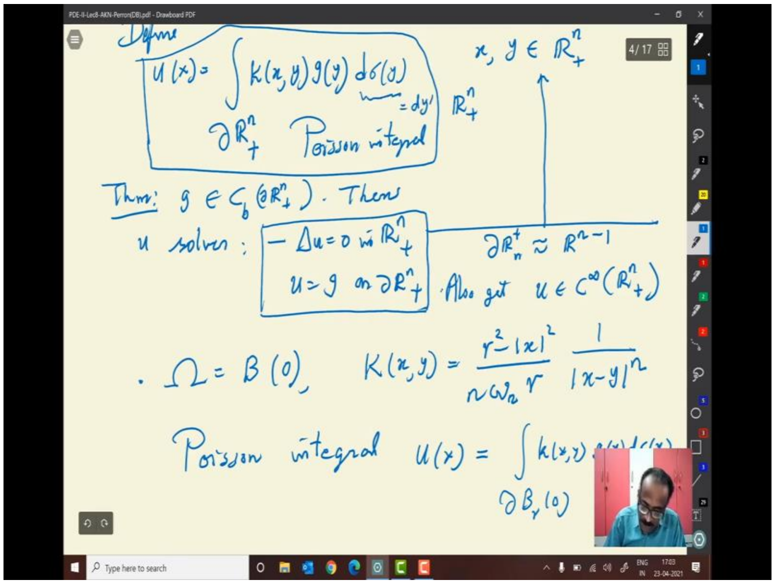

And then this is called the Poisson integral for the upper half plane and then you have proved a theorem what we you have proved? If g is a boundary continuous function in R n plus boundary of R n plus then u solves minus Laplacian of u = 0 in R n plus and u = g on the boundary. So, you have proved an existence for using that fundamental solution because we expressly constructed the Green's function using that you have solved this one.

And in fact we also get actually u is a C infinity functioning in R n. So, you actually have more smoothness that is a property of your Laplacian. So, you have proved one existence theorem but we are going to do it for a general domain Omega.

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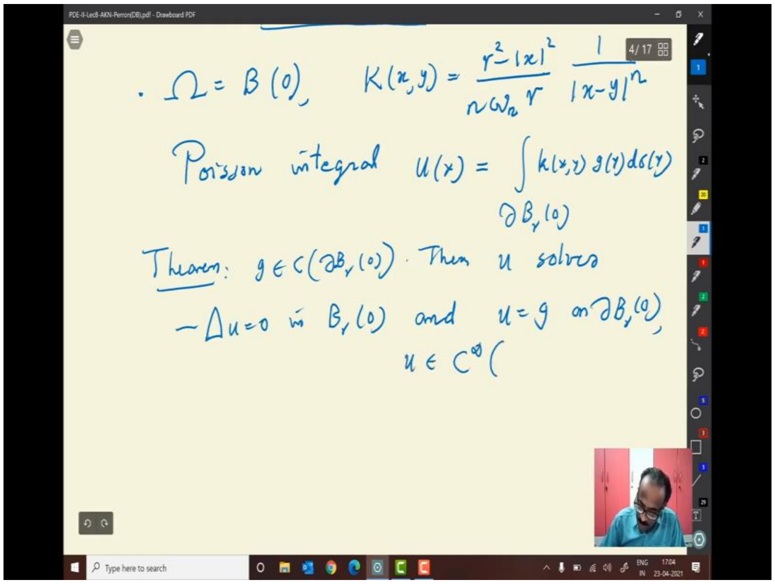

And the second case we have proved is when Omega is a ball Omega is equal to a ball of radius r in this case you are K x y again the kernel is defined to be r square - mod x square / n Omega n r and then 1 / mod x - y over n you see kernel and you are in this case the Poisson integral we use this Poisson integral in the Perron's methods. In the next class you will use and then you define a similar thing of ux you can define that ux exactly like this.

ux you can define to be on the boundary of the ball. So, you have defined boundary of the ball of radius 0 k of x y g of y d sigma y. And here also d sigma y on the boundary, but here in this case you get it as dy prime layer because it is subtract R n - 1, so you can combine what is R n plus. And again you have the theorem exactly the theorem g belongs to continuous function now the boundary of the ball.

Of course you do not have to see the boundedness now because the boundary of the ball itself is a compact set, so it will be automatically solved then u solves minus Laplacian of u = 0 in B r of 0 and u = g on boundary of 0. In fact you have again you end or so u is in C infinity of the B r 0.

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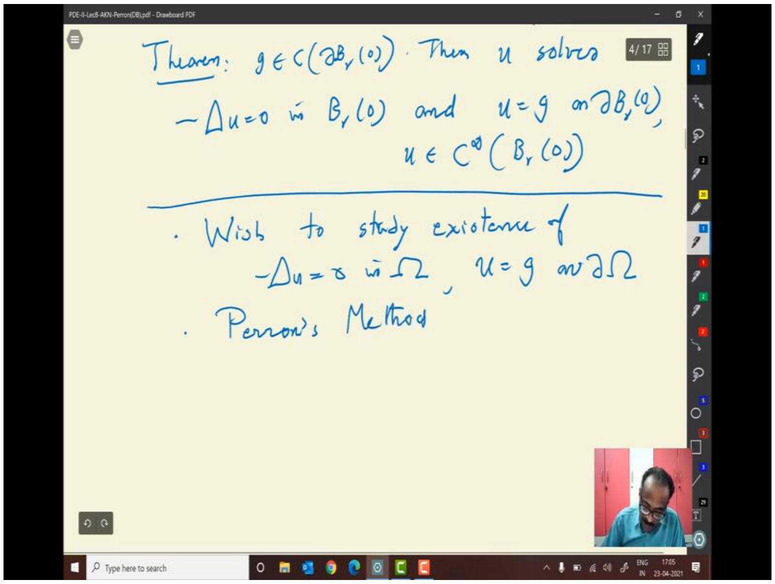

So, as I said so this is a brief about the PDE 1 course which we have PDE 1 about the Laplacian what we have studied. So, what we want to wish to study now is not with easy wish to study existence of my Laplacian u = 0 minus or not does not matter in Omega and u = g on d Omega. So, what we introduce these Perron's method and it is not true for all domains you need.

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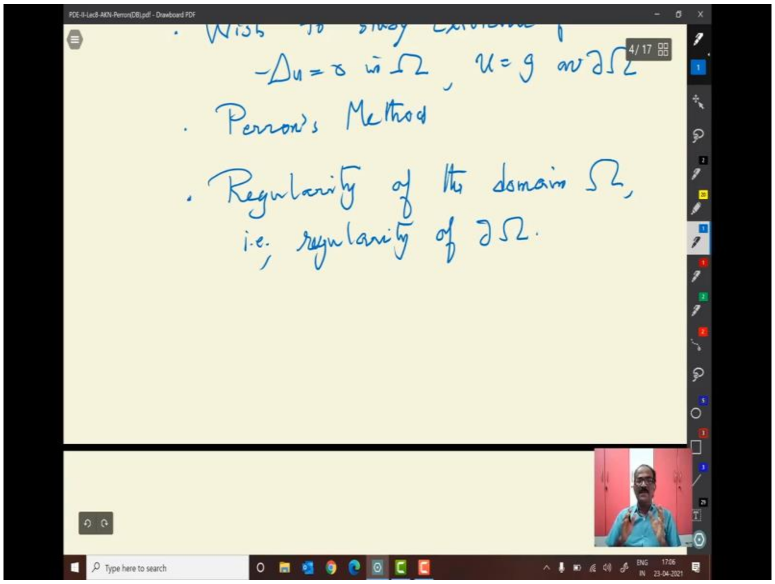

And what we need is the regularity of the results. So, you need the regularity of the domain, the regularity of the domain. The regularity of the domain means the regularity of the boundary that is regularity of d Omega. So, in the next class our aim is to see start introducing the Perron's method. And for that the ingredients which we need is one is about the kind of Poisson's integral we will be using the Poisson's integral. And we will also define the definition of sub harmonic and super harmonic functions with the continuous functions using maximum principles. Fine I will stop here, thank you very much.