First Course on Partial Differential Equations - II
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Lecture - 13
Conservation Law

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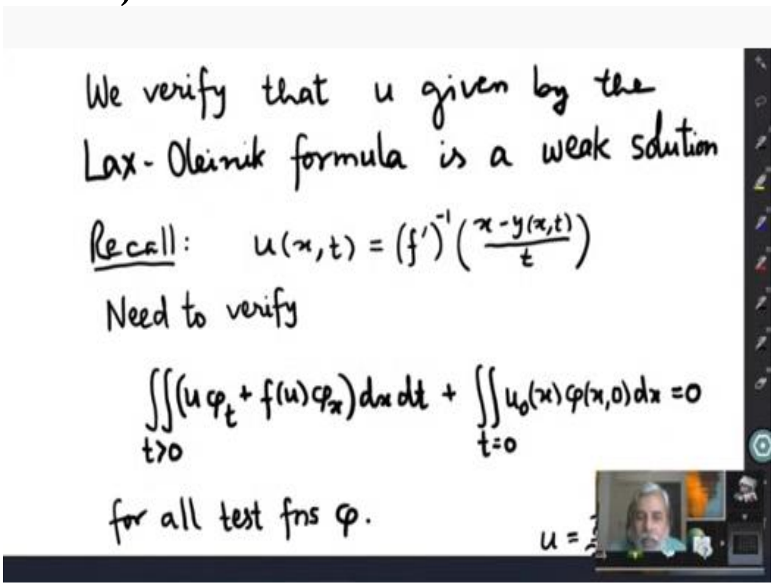

Hello everyone welcome back in the previous class we derived the Lax-Oleinik formula and we also verified that the solution given by the Lax-Oleinik formula is a weak solution of our conservation law. So, we continue read more discussion on that function given by the Lax-Oleinik formula as I stated there are the third part of the theorem is very much technical I will not handle here and same is with uniqueness theorem. So, that is also technical so, any weak solution satisfying the entropy inequality is unique.

So, we also saw examples where non uniqueness was present so this entropy inequality, so, that is an additional condition on the weak solution. So, if we impose that condition, then there is uniqueness. So, another important inequality namely entropy inequality, so now we will discuss. So, again recall that Lax-Oleinik formula so it is given by u x t = f prime inverse x - y x t / t, so where y x t is the minimizer of the functional in the Hopf-Lax formula and that is the connection with Hamilton Jacobi Equation.

So, verified this that it is a weak solution again using Hamilton Jacobi Equation. So, you see that constant relation between the Hamilton Jacobi equation and conservation law.

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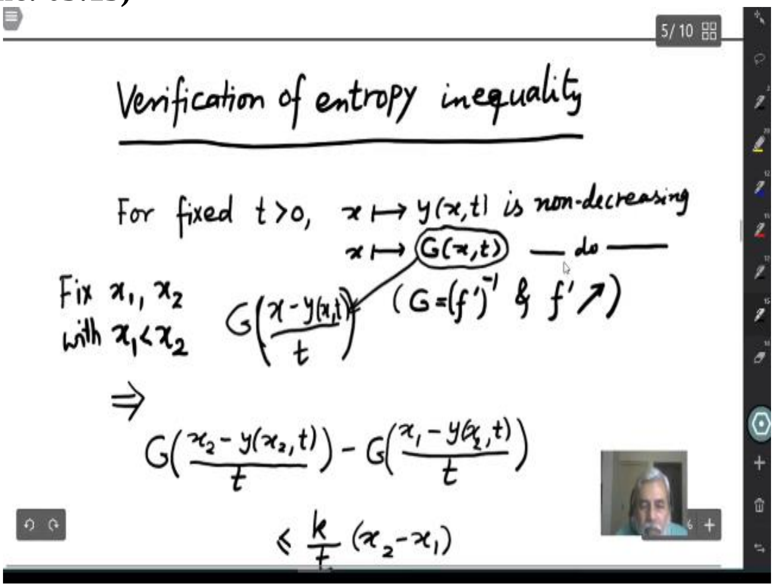

Now we proceed with the proof of entropy inequality. So, what do we have to show again just recall so, u given by the Lax entropy Lax-Oleinik formula satisfies this inequality or equality.

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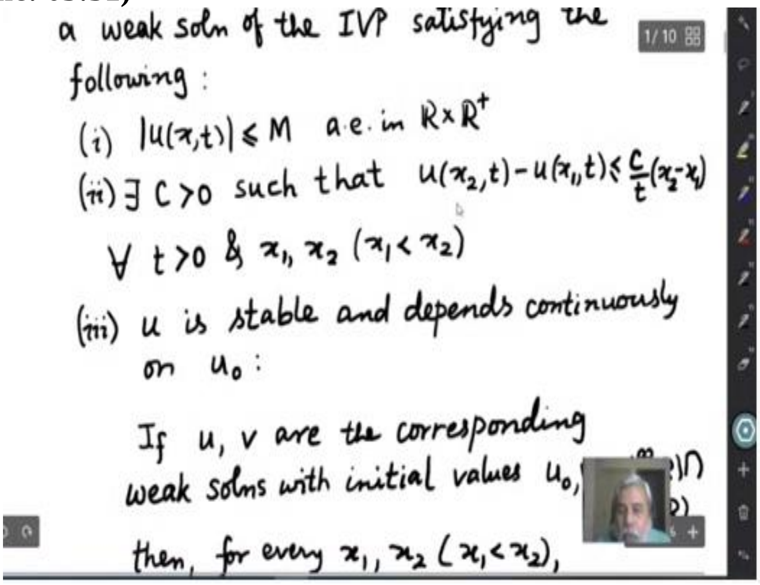

For any x 1, x 2 x 1 less than x 2 there is a constant such that u of x 2 t - u of x 1 t is less than or equal to C / t x 2 - x 1. So, let us begin with that. So, the important properties of this minimizer we already seen that so, this for each fixed t positive, this function x going to y x t is non-decreasing and since this G is equal to f prime inverse and f prime is also increasing, so, this functional inverse. So, this there is something here so, this should not there direction there, so this is replaced by G of x – y x t / t. Now using that a non-decreasing nature of these functions, so we again you fix x 1, x 2 with x 1 less than x 2.

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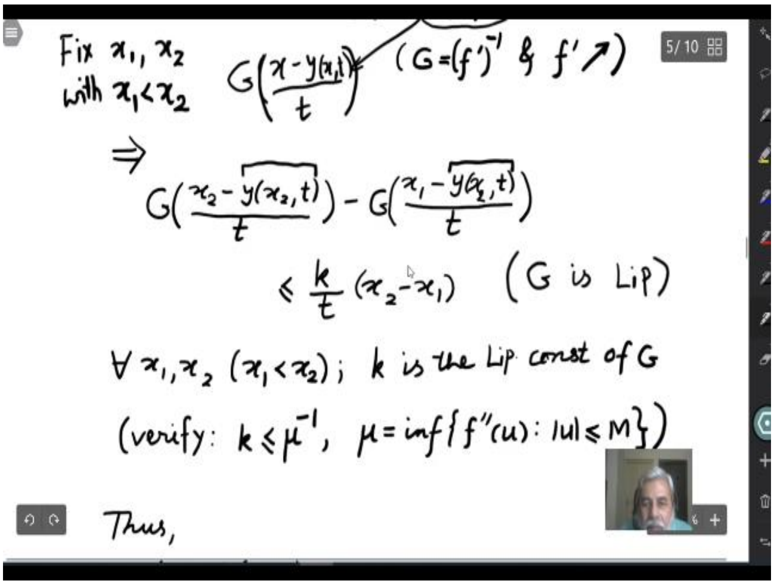

So, consider this expression this G of x 2 - y of x 2 t / t - G of x 1 - y of x 2 t / t so, just notice here this second factor with this same. So, G in particular since f double prime is strictly positive, so you can easily check that this G is Lipschitz. So, since and then you use so, Lipschitz property so G is Lipschitz continuous. So, this forms so since this is same so, the difference is just x 2 - x 1 / t and k is the Lipschitz and that is related to f double prime and other things. So, you can just verify that and this also appears in the proof of Hamilton Jacobi Equation so, you can also get this result from there.

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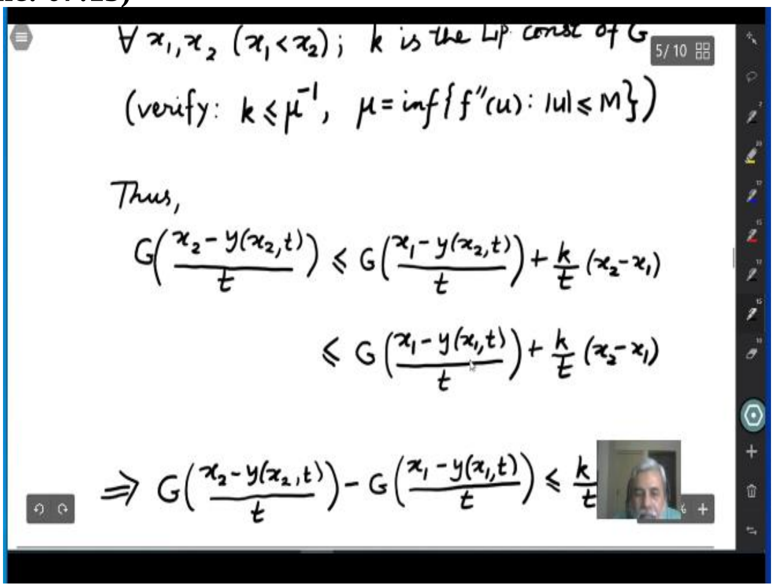

Now, we are using that this function x is going to y x t is non-decreasing so, rewrite again this one, so, you take this next to term the other side. So, G of x 2 - y x 2 t / t less than or equal to G of x 1 - y x 2 t / t + k / t x 2 – x 1 that is the first line and now you that this is not increasing and same with G. So, this first term is less than or equal to G of x 1 - y of x 1, so here we were using term. And now we will take that y of x 1 t the other side and you can see this formula.

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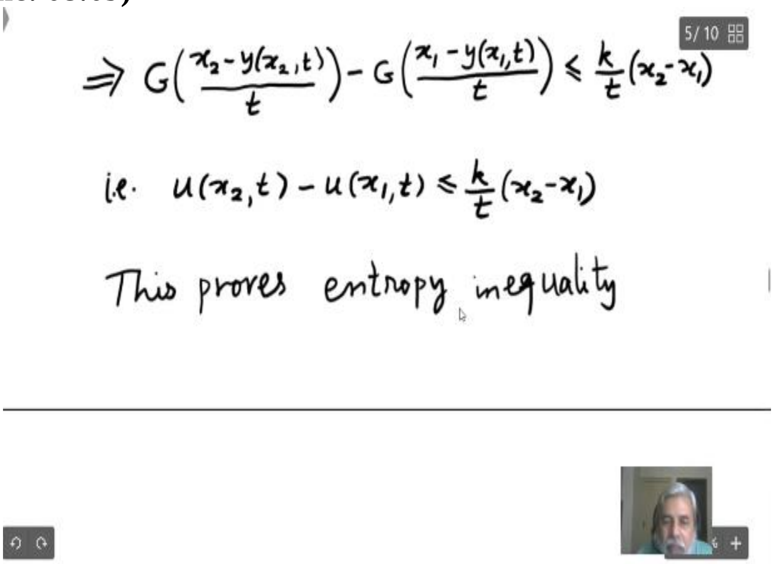

And this is precisely the solution by Lax-Oleinik formula this is u of x 2 t and this is u of x 1 t that is less than or equal to k / t x 2 – x 1 so, this proves entropy inequality. Now let us do some examples simple examples and see the use of Lax-Oleinik formula.

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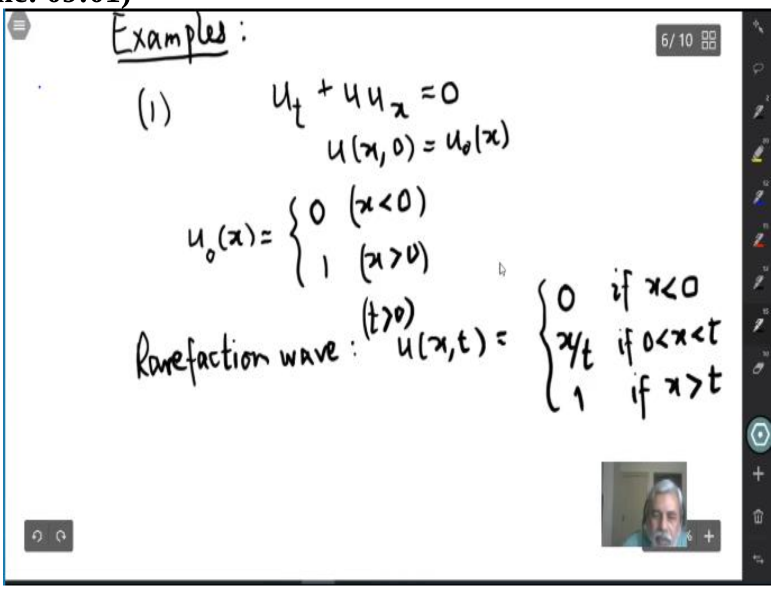

So, these examples we already done from a different point of view, so, again let me read to them using Lax-Oleinik formula. The first one, again Burger's equation u of x 0 is u 0 x. So, the first example consists of rarefaction wave so where u 0 x is 0 x is less than 0 and 1 x greater than 0. So, earlier already we solve this problem using method of characteristics so and you got the solution at this rarefaction wave, so u of x t for t positive. So, 0 if x less than 0 x / t; if 0 less than x less than t and 1 if x is bigger than t, so, this is the rarefaction wave.

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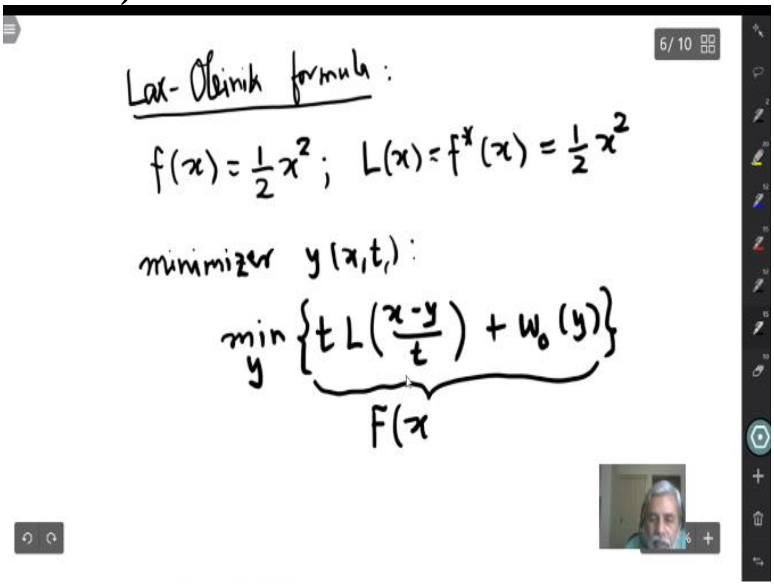

And let us see know, this Lax-Oleinik formula. So, since the function defined by Lax-Oleinik formula automatically satisfies the entropic condition, we should get back this rarefaction wave and let us see that. So, here f of x = half x square and Legendre transform also L of x is F star x, so easy to verify this is also half x square and we are interested in the minimizer y x t. So, minimize this, this function, so minimize y t L of x – y / t + w 0 y. So, for fix x and t you consider this function so, let me call by some name F of x y t.

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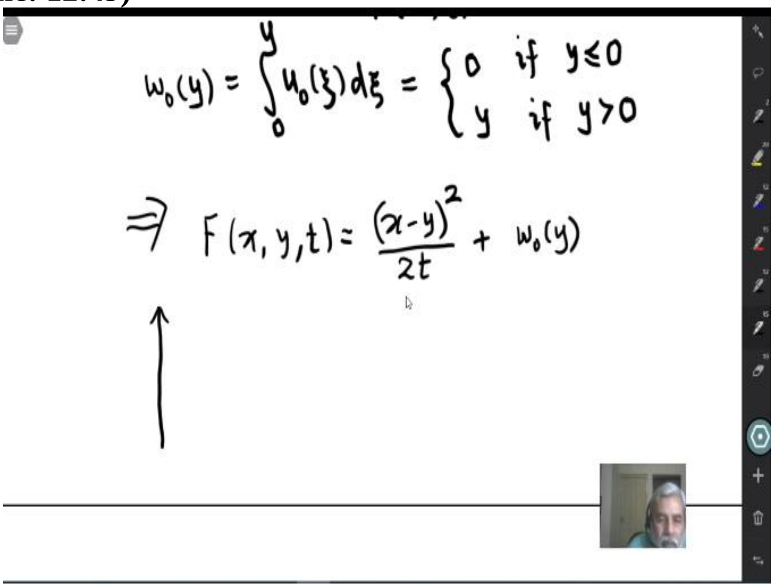

So, in this case so w 0 y is integral 0 to y so u 0 Xi dXi so this is 0 if y is less than 0 y, if y is positive because u 0 just see is 0 for x less than 0 and 1 for x positive. So, very simple integral but this w 0 y is not differentiable at y = 0, but that will not create any problem in finding the minimizer. So, therefore, that implies F of x y t so, this is t into L of x - y / t but L of F of x is again half x square so, that will be the x - y square by half t square and 1 t cancels there. So, you just have the first part x - y square / 2t + w 0 just take a look at the w 0 y.

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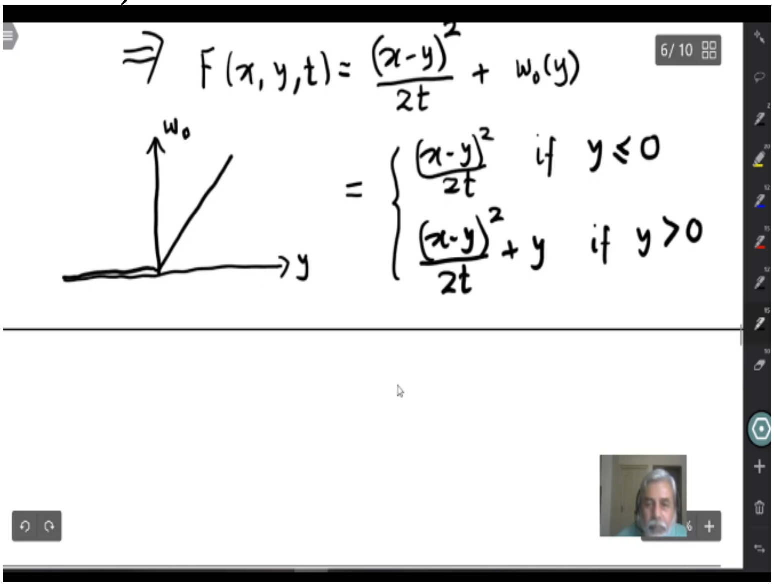

So, just for illustration so, this is y this w 0, so it is 0 here and then set y = 0 it is not differentiable. In fact, it is a very good exercise so you do need some calculus here, but you also need to analyze this so explicitly written. So, this has 2 parts now so x - y square / 2 t if y less than or equal to 0 and x - y square / 2t + y if y is positive. So, this minimizer y certainly depends on x and t, t is positive but x can vary over the real line and it is a very good exercise.

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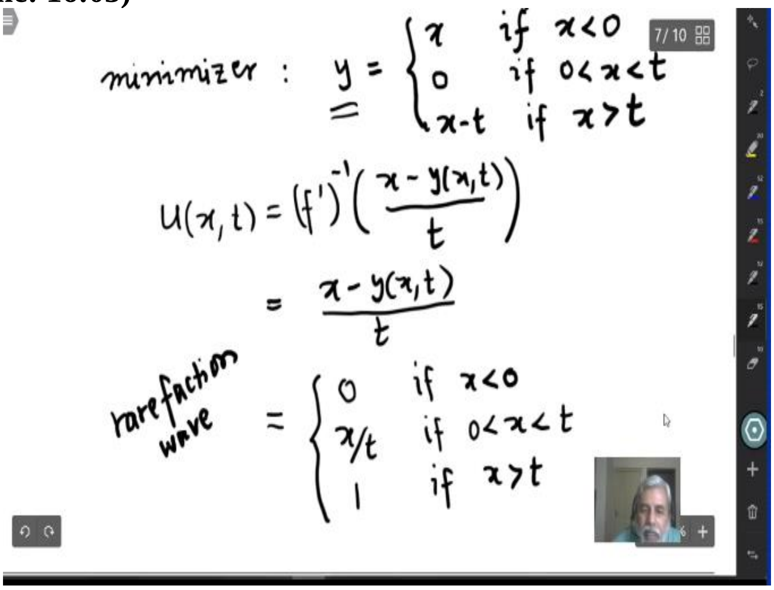

So, just this minimizer y equal to so it is very much depends on the position of the x, so this equal to x if x is less than 0, 0 if 0 x less than t and x - t if x is bigger than t. So, it is not immediately clear, there are 2 expressions for this functional F of x y t. So, you have to be careful in finding the minimum of this function. So, we are interested in the minimizer not exactly the minimum value.

But minimum value comes into picture in determining this minimize. I know you are just plug-in in the Lax-Oleinik formula. So, this Lax-Oleinik formula so this, by the by so just one more thing here, so f prime x is x and so f inverse is also x because this is what we need in the Lax-Oleinik formula. So, u of x t is f prime inverse of x - y x t / t and this is simply x – y of x t with minimizer that and looking at the expression of the minimizer. So, just plugged so this so x is less than 0 it is x so this is you will be 0 if x less than 0.

And if x is between 0 and t, then it is 0 so you get x / t and finally, if x is bigger than t then it is x - t. So, you just get 1, this is precisely the rarefaction. So, let me consider the shock wave also that the second example.

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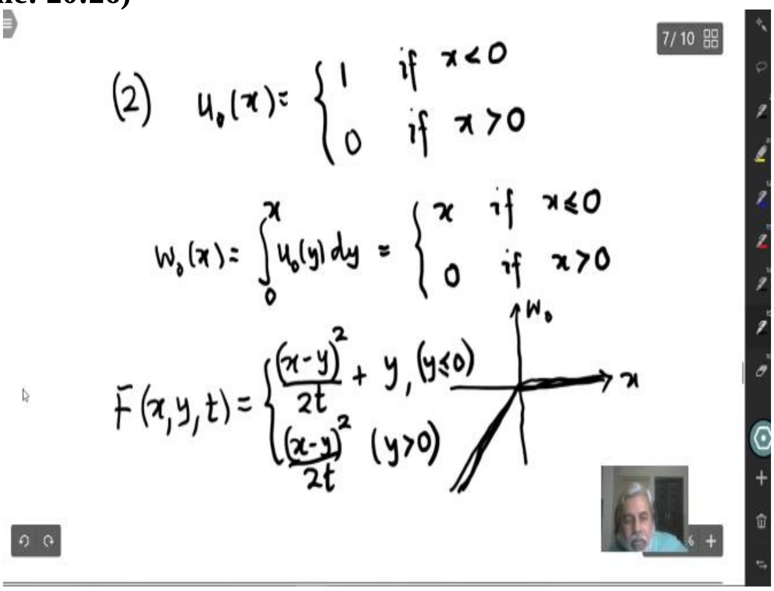

So, here again Burger's equation only we change the initial condition. So, this is 1 if x less than 0, 0 if x bigger than 0 so, everything remains same. So, only now, this functional F of x y t changes because this w 0 changes. So, now w 0 of x is 0 to x u 0 y dy and that is equal to now this x if x less than or equal to 0, again this is not differentiable at x = 0, so again it is now in this case the w 0.

So, we have the other solution that it is x obviously this here and then x axis, this is function w 0. And in this case you will see that again F of x y t this it is x - y square / 2t + y if y is less than or equal to 0 and x - y square / 2t if y is positive. So, again for F of x y t, t positive, so, you find the minimizer and here that minimum value also comes into picture

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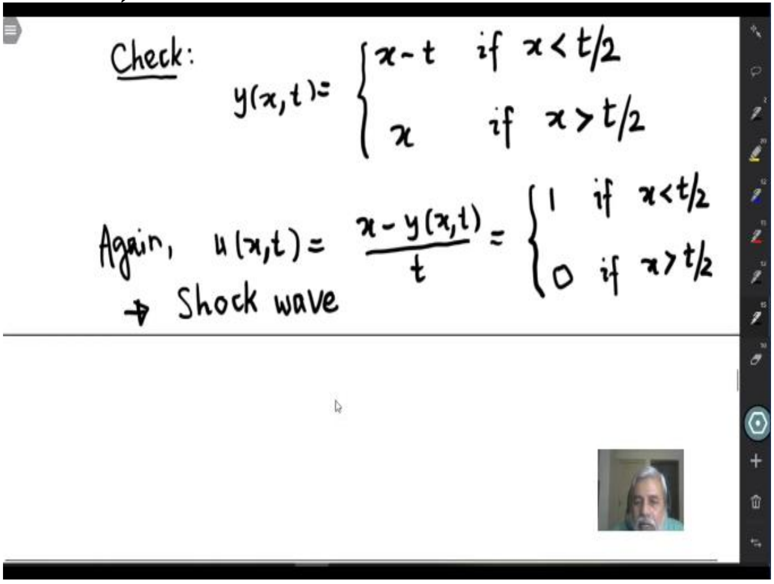

So, this I will write here some check so, y x t is x - t if x is less than t / 2 and x if so and then you plugged in that. So, again we have so, this u of x t = x - y minus there is no change here, because we are not change the equation. So, this point that is 1 if x less than t / 2 and 0 if x greater than t / 2 this is same wave shock wave. So, you can take some simple different initial functions u 0 and try to work out this minimization problem, in general, it is not easy somewhat we have to compute.

So, even in the simplest case, you have to really work to get these values have to minimizer. They are not straightforward so, you have to use some calculus also some analysis of this function. How it is profile for y less than or equal to 0 and y bigger than 0 then you have to combine the analysis of these 2 regions separately and arrive at this minimizer. So, if this w 0 were differentiable also could just differentiate directly this F of x y t, this w 0 were differentiable.

So, the usual calculus, so you differentiate with respect to y and equated to 0 find the critical points and then check whether that use a minimum or maximum and do just choose the minimum here. So, with that illustration I come to an end of this discussion on conservation law. Thank you.