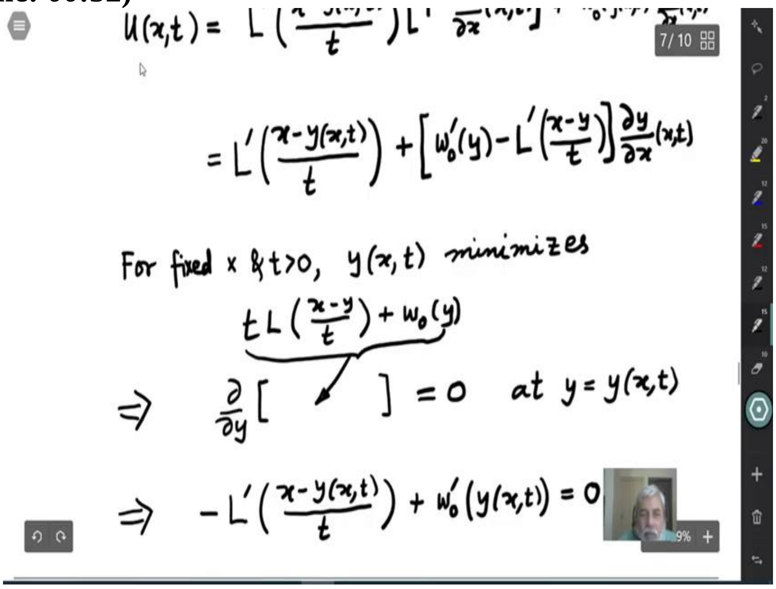
First Course on Partial Differential Equations - II
Prof. A.K. Nandakumaran
Department of Mathematics
Indian Institute of Science - Bengaluru
Prof. P.S. Datti
Former Faculty, TIFR-CAM - Bengaluru

Lecture - 12
Conservation Law

So, in the previous class we proceeded with the derivation of the Lax-Oleinik formula. So, one of the important ingredients namely the minimizer of the Lax-Oleinik formula, so, we studied these properties and now it is easy to derive the actually the formula.

(Refer Slide Time: 00:52)



So, this important ingredient y x, t which is the minimizer in the Hopf-Lax formula. So, we showed that this function x going to y of t is a non-decreasing function and hence by Lebesgue theorem, it is a differentiable function almost everywhere. So, at those points of differentiability of this function x, so, we can define this function. So, u x t is equal to d / dx of this t L x - y x, t.

This is the function that is minimized in the Hopf-Lax formula and y is the minimizer and this for each t positive u is defined almost everywhere and that is sufficient for us to verify it is a weak solution of the conservation law. Let us perform the this derivation and let us simplify this formula a little bit and by divide this w 0 we are taking as the integral of u 0. So, u 0 is the initial function for the conservation law and we can check that this so, this is locally integrable.

So, this w 0 is absolutely continuous function. So, w 0 is also differentiable almost everywhere and again with the Lebesgue calculus, so, it is a tricky issue. So, here we are taking for example, composition of 2 functions, so, one is non decreasing and another one is absolutely continuous function. So, this composition is also differentiable almost everywhere. So, there are some certain issues we have to pay attention because we are using some finer aspects of Lebesgue theory.

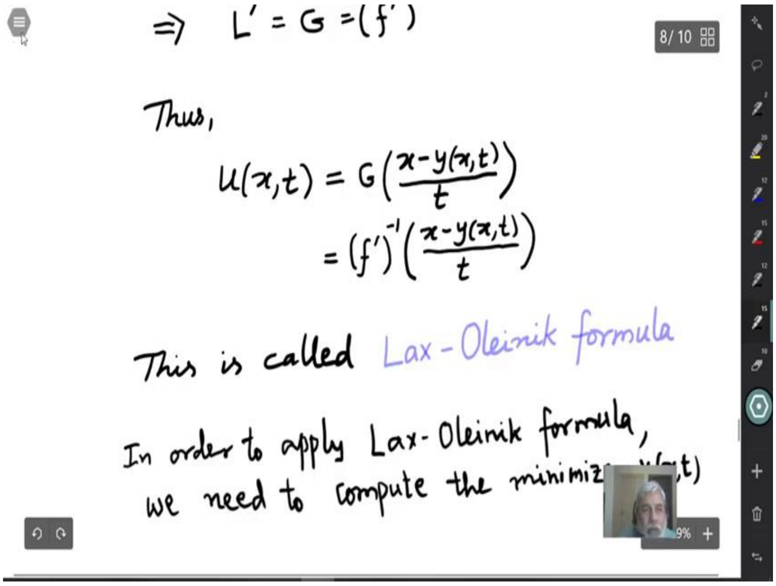
And same thing is true here L is a smooth function, but this is only non-decreasing function. So, again the composition is well defined. The composition function is almost everywhere differentiable. So let us perform this integration, so chain tool again applies, so, this L prime, so this is d / dx or dy is nothing to do that t comes out. So, once I differentiate with respect to x. So, this L prime again this x y x - y t / t and that produces this one.

1 / t and that t t cancels and then I have the derivative of this with respect to x so that I get 1 - dy / dx. So, remember this is only our most every L and similarly with the discomforted function w 0 with y. So, w 0 prime into dy / dx. Again you simplify a little bit so you take out this L prime x – y x, t and then you combine these 2 terms, w 0 prime y I am just writing y is y of x, t and - L prime x - y / t into this dy / dx that partial derivatives.

So now we will see what this term in the bracket is so fixed x and t positive this any real x and the positive this y x t minimises this functional. So, by elementary calculus again this derivative with respect to y must be 0. So, here again now, we are losing that part of the calculus this must be 0 at y = y x, t and now again you perform this derivative and plug in this y = y x, t. So, what we get is -L prime at this x - y x, t / t + w 0 prime y x, t = 0.

And this is precisely the term in the bracket within a -L prime w 0. So, that term in the bracket vanishes. So, this lacks this formula for you simplifies. So, hence u of x, t simply the derivative of L evaluated at x - y x, t / t. Again this is referred to as Lax-Oleinik formula and now, we expect this L prime in terms of the given s.

(Refer Slide Time: 06:01)



So, again recall so, this last time also I stated that so, if f double prime is bounded away from 0, then your frame is strictly increasing function. So, this f prime inverse exists in the functional inverse exists and again recall the Legendre transform. So, L of y supremum xy - f of x, again supremum is achieved at the start, so, you just take the derivative of that and equate it to 0 and that gives you y - f prime x star = 0.

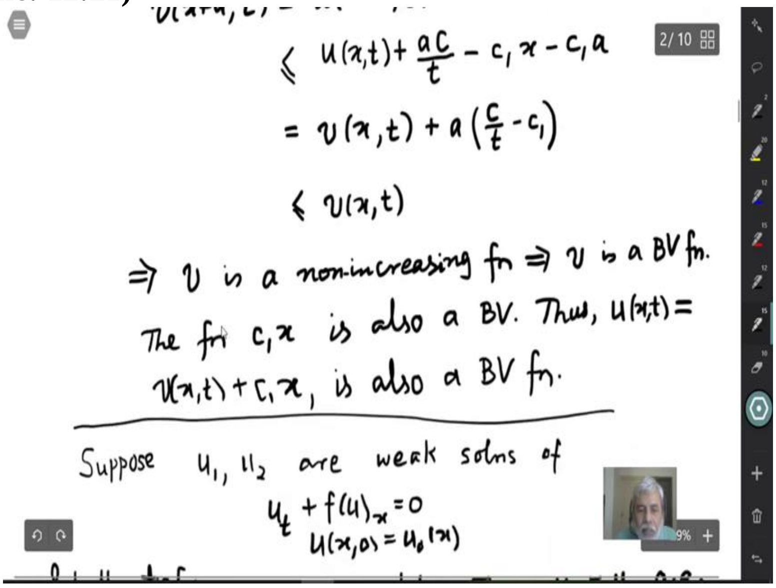
o, our x star is equal to f prime inverse y, so, this is functionally inverse just remember that and we will use this notation so, G = f prime inverse. So, you should operating all the time that the inverse simplify the notation I know you plug in this value of x star here, so, that is why the supremum is achieved. So, L of y is y G y. So, this x is x star and x star is G of y - f of G y. So, taking the; supremum effect that this x star is simply G of y.

So, this L of y is y G y - f of G y and now, you take the derivative with respect to y. So, L prime y is G y + y G prime y minus so, this is composition of f and G. So, get f prime G of y into G prime but by the definition of the functional inverse this f prime at G of y = y. So, the last term is also y into G prime y with a negative sign and here we have y G prime y for the true sides so, that is equal to G of y.

So, this L prime y is nothing but G of y and so, we can now substitute that expression in this formula. So, our u x, t finally becomes u of x, t equal to G of x - y x, t / t and just again recall that G is f prime inverse. So, everything the right hand side just involves the given f the of the scalar conservation law that convention is given and the right hand side only depends on that. This is called Lax-Oleinik formula.

So, again as you notice here, the important thing is played by this minimizer, so, in order to apply that Lax-Oleinik formula, we need to compute this minimizer and sometimes that may not be easy. We see an example. So by any given x and t positive, so we have to only compute this y x, t and just substitute it and again remember this y x, t minimizer of this function, so this earlier we will call this w x, t which is a solution of the Hamilton Jacobi equation. So now we proceed with the verify the solution given by this function given by the Lax-Oleinik formula is indeed the weak solution now a confirmation, so let me just state that.

(Refer Slide Time: 12:21)



So here is the verification. So now, I state the main theorem as follows. So we stated it many times. But let me state it once again. So we have this conservation law. So initial value problem of the conservation law, u t + f of u x = 0, u x, 0 = u 0 x and these are the assumptions or the initial data, so it is bounded measurable function with M as its essential supremum R.

Now and with regard to this non linearity f, we assume that c 2 function, which is whose second derivatives, is bounded below by positive numbers, since we are only interested in this interval and the real lag and f double prime is continuous, so this minimum will be positive. So that is not explicitly stated, but it sounds good, then the statement is the conclusion of the theorem, then there is a weak solution of the IVP satisfying the following conditions.

So u is also bounded by the same M and the second property that is important, that is what we will call it, entropy inequality. So there exists a constant C, such that u of x 2 t – u of x 1 t less than or equal to C / t into x 2 - x 1 for all x t positive and x 1 x 2 with x 1 less than x 2. So, this is one sided inequality there is no x 2 value here, so, it is only one sided inequality and I also state the third property which is continuous dependence on the initial condition.

So u is stable and depends continuously on u 0. So, the statement is arithmetically stated as follows. So, if u and v are the corresponding weak solution, so, within initial values u 0 and v 0. So, u in the weak solution within the initial condition u 0 and v the initial v solution with initial condition v 0 and u 0, v 0 are assumed to be not only bounded, but are also integrable L 1 R. Then for x 1 x 2 satisfying x 1 less than x 2 and t positive the following estimate holds.

So, the right hand, left hand side is the integral of u - v at time t integrated over the interval x 1 x 2 and right hand side is the integral of the initial values u 0 and v 0 integrated over x 1 - At + x 2 + At. So, there is a spread of the initial interval. So, this is so, you can see again finite speed of propagation and speed is given by this number A which is maximum f prime mod f prime u again in that same interval mod u less than or equal to M and we also put this mu as the infimum of f prime u mod u less than or equal to M.

And since f prime f double prime is continuous and we are assuming that it is greater than 0 in this interval. So, infimum is positive and the constant C in the second statement this entropy inequality depends only on this M, mu and A. This inequality 2 is termed as entropy inequality as you see is really an elegant inequality which does not involve directly at least any condition on the characteristics or the values of the function as it approaches a discontinuity curve. So, they are all hidden and end up in this nice inequality and this inequality can also be rewritten as instead of x 2 x 1.

So, we use only one variable and then use only A positive. So, u of x + at – u x t / a is less than or equal to C / t. So, maybe need a form of entropy conditions. Earlier we will discuss that in terms of characteristics. That shocker, but here you do not see any of those things it directly involves the solution itself that weak solution. So, before going to prove the statements in the theorem, so, I will not be handling this part 3 that is very deep and technical and takes many hours of calculations and presentation.

So, you can refer to some literature for that and that is an important. So, put together say that this initial value problem is well posed in the sense of hard work. So, you have like a uniqueness or not discuss, discuss that thing in little bit, but there is a continuous dependence on that. So uniqueness also is just, let me test discuss that uniqueness. So, any weak solution satisfying the entropy inequality is unique.

So, let me just elaborate a little bit on that. So, suppose u 1 u 2 are weak solutions of this conservation law and same initial condition on force and both satisfying entropy inequality then u 1 = u 2 almost everywhere. So that is the uniqueness and again this uniqueness proof of uniqueness is quite technical and uses some more tools from the analysis so, that smoothening and other things so, I will not do that unit this part also.

So, it is very cumbersome. So, you can just look into our book where a detailed uniqueness proof is given. So you have to use modifiers to smooth the. So, these remember these are only almost everywhere defined functions. Again, not at all differentiable in the usual sense, so, we have to first smoothen them out and then derive lots of inequalities. So, it is very very technical, this proof of this uniqueness theorem is very technical, so, I will not venture into that. So, let me just mention this smoothness region that is coming out of this theorem.

So, namely this entropy inequality, so, it contains some regularity result for the solution. So, remember to begin with u is defined only almost everywhere and u satisfies this integral equation in order to the weak solution, so you do not see anything further in the Lax - Oleinik formula, but this Lax this entropy inequality allows us to say something more on the solution.

So that is what now I want to highlight this, for each t positive this function x going to u x t. So, this is given by the last word in the formula, which satisfy the entropy inequality is a BV function. So, let me stress there on any interval, so it is a local result so it is locally bounded a reason. So you cannot take it to the whole of R that it could be bounded, unless you put more assumptions on the initial condition.

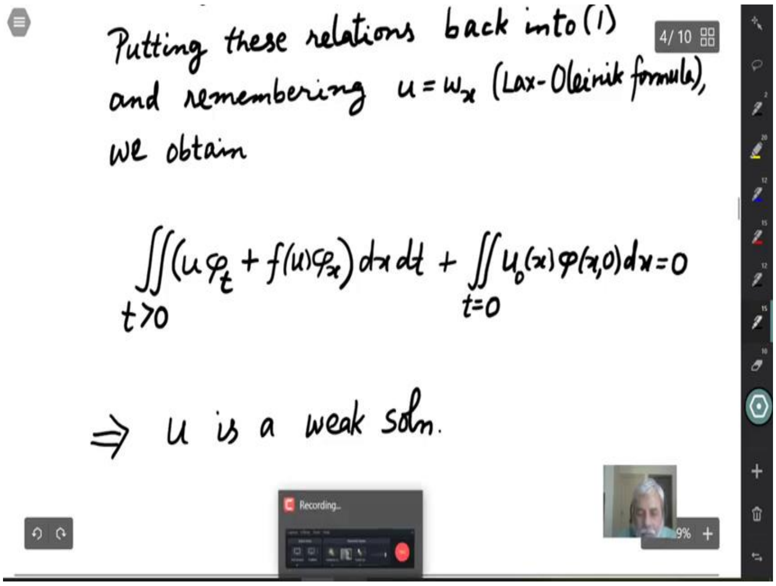
So this new will not be a bounded deviation function on the entire arc but this entropy inequality gives us a local result and that is easy to prove that is easy to fix t positive and take any c 1 constant such that c 1 is bigger than c / t and define this new function v of x, t is equal to u of x, t - c 1 x. So, c 1 is the constant you have chosen and now, we show that this v is non-increasing function.

So, for that you just compute this take any a positive and you compute this v of x + a at t and that is by definition u of x + a, t - c 1 x + a and now use entropy inequality. So, u of x +a, t is less than or equal to u of x, t + ac / t. So, that c is appearing here. So, you multiply by here and take that u x, t other side and now this u x, t - c 1 x that is nothing but v of x, t and this one I take a common and c / t - c 1 and by our choice of the constant c 1.

So, this is always less than or equal to 0 and that gives us v of x + a t less than or equal to v of x, t. So, v is a non-decreasing function. So, it is a function of bounded variation that we already seen and same thing is true with c 1 x. So, this is again a monotonically increasing or decreasing function depending on the sign of c 1 hence, that is also a bounded variation function and u will be the sum of these 2 functions are bounded variation is also a bounded variation function.

So, bounded was the class of bounded variation functions they form a vector space. So, this sum results where bounded variation function. So, this entropy inequality implies certain regularity on the solution u of course for t positive. So, similar to d 1 we observed in heat equation. So, initially u 0 may be very very rough it is only assume that is a bounded measurable function and even in the case of heat equation, so, the initial condition can be just a bounded function but there for t positive the solution suddenly becomes solution given by the Fourier, Poisson's formula becomes c infinity of course, here c infinity is not there but some smoothness is there.

(Refer Slide Time: 28:11)



Now, we continue to verify this function given by the Lax-Oleinik formula is a weak solution. So, again recall that u of x, t is given by this f prime universe. So, this is again functional inverse let me repeat that. So, at the x - y x, t / t. So, we need to verify this integral relation for all test functions phi and again can I solve between Hamilton Jacobi equation and conservation law comes into play.

So, you remember so this u is d / dx of w = w x and w is the solution of the Hamilton Jacobi equation and w is a Lipschitz function. So, it is differentiable almost everywhere. So, it satisfies this Hamilton Jacobi equation point wise. Now you pick up any test function phi and multiply this Hamilton Jacobi equation by phi x. So this is a partial derivative with respect to x. So if I remember phi is function of x and t and it has supported in a rectangle in R 2 plane.

So, this notation I have already introduced in a previous class just miss this double integral t positive means this. So, u multiply by phi x and integrate over t positive that means this. So, again now we want so, you consider for example the first term here I want to integrate by parts. So, you have to justify that. So, integrate these are some certain issues, but you have to pay attention otherwise we start thinking that these are all smooth functions.

So, there is no need to worry, but they are not it take me some parts justified that is why I listed those facts from the analysis. So, why t is justified? So, w is a Lipchitz function. So, hence it is absolutely continuous function. So, first I integrate by parts with respect to t variable here. So, just consider only t variable here. So, that will not test the x integration. So, that w t so, I take that t to and phi is just a smooth function that is a test function. So, there is a c 1 function so, integration by parts is justified.

So, I take that t derivative to phi and since it is a t derivative there is a boundary 0 here it is not minus infinity. So, a boundary term appears here and that is the initial condition. So, w t so, this is integration by parts with respect to t variable and now you will do integration by parts with respect to x. So, just so, essentially what we are doing is we are exchanging this t derivative to phi and x derivative to w.

Only thing is we have to exercise caution and justify the integration by parts and similar thing I do in the second integral. So, if you know do integration by parts with respect to x variable and now, the x integral varies from minus infinity to infinity and phi is a test function, it vanishes at both infinity and minus infinity so, there are no boundary terms here. So, we will only get minus zero to infinity, minus infinity to infinity w x phi t to dx dt and similarly with the single integral here.

So, we get minor w 0 x and remember this w 0 is absolutely continuous just pay attention to all these things that in place w 0 is absolutely continuous and that is the minimum sufficient condition for the integration by parts total. Now, you put together all these things and again in this one, so, we are not touching this second term at all so, only the first term. So, now we have computed that.

Now you put back again here I will return again w 0 x = u 0 almost everywhere and now we put back all these conditions all these computations into one and remember u is w x that is Lax-Oleinik formula to get u phi t + f of u phi x dx dt and integral t = 0 u 0 x phi x, 0 dx and that is precisely the definition of weak solution. So u given by the Lax-Oleinik formula is indeed a weak solution of the conservation law. I will take up the entropy inequality next time. Thank you.