First Course on Partial Differential Equations - II
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Lecture - 11
Conservation Law

Hello, everyone welcome back, we will continue the discussion on the conservation law and today we will proceed to derive the Lax-Oleinik formula for a possible solution of the conservation.

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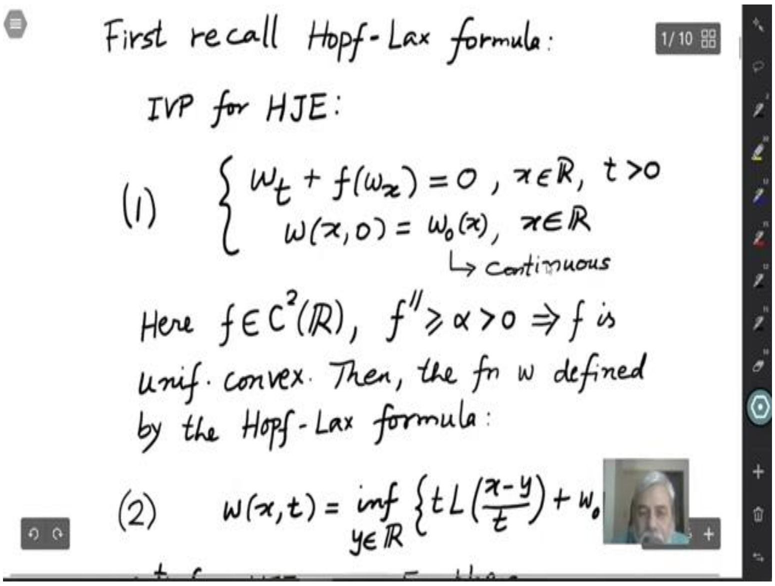

Recall that in the previous class we discussed some special solutions of a conservation law namely the rarefaction waves and shock waves. And these are solutions to an initial value problem called Riemann problem. And Riemann problem consists of the scalar conservation law in which the initial function consists of only 2 constant states. And now we will consider a general bounded measurable initial function u 0 and try to show that this conservation law has a weak solution uniqueness comes later.

So, where we assumed that the f is c to and uniformly convex. In the literature there are many different proofs of this important theorem that this uniformly convex conservation law has a big solution. And our approach here is to use Lax-Oleinik formula and this Lax-Oleinik formula in turn is derived from the Hopf-Lax formula which gives solution to the Hamilton Jacobi Equation.

So, Hamilton Jacobi Equation we have already studied in detail and derive this Hopf - Lax formula and showed that indeed is a solution of the Hamilton Jacobi Equation. So, again, so, this is the Hamilton Jacobi equation w t + f of omega x and again f is uniformly convex and this is the initial condition given w x 0 = w 0 x. To see the connection between Hamilton Jacobi Equation and conservation law we proceed as follows.

First assume that this w the solution of the Hamilton Jacobi Equation is a C 1 function, though that is in general not true we have only proved that w is Lipschitz function and by a theorem of that makers. Then it follows that w is differentiable almost everywhere and the Hamilton Jacobi Equation is satisfied only in the sense of almost everywhere. So, here almost everywhere refers to a certain statement holds true you accept on a set of Lebesgue measure 0 either on the real line or on the in the plane present case.

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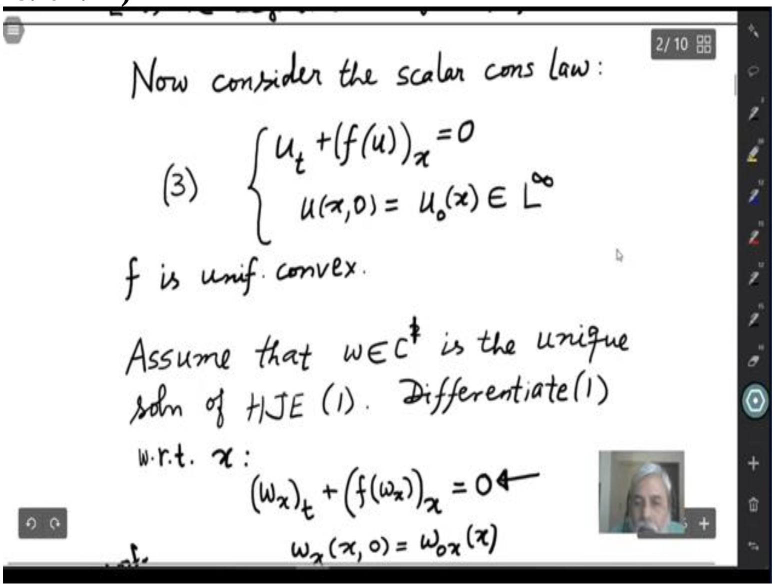

Let us see the connection so this is just to get an idea what the Lax-Olenink formula looks like. So, I assume that the w solution of the Hamilton Jacobi Equation is C 1. So, then we can differentiate the Hamilton Jacobi Equation that equation 1 with respect to x and obtain this equation for the first derivative of w with respect to x, that is w x t + f of w x x = 0 and w x x 0 is w 0 x x. So, if you see, if you compare this equation satisfied by w x and the conservation law, we immediately see that if you take u = w x. Then we have a solution for the conservation law.

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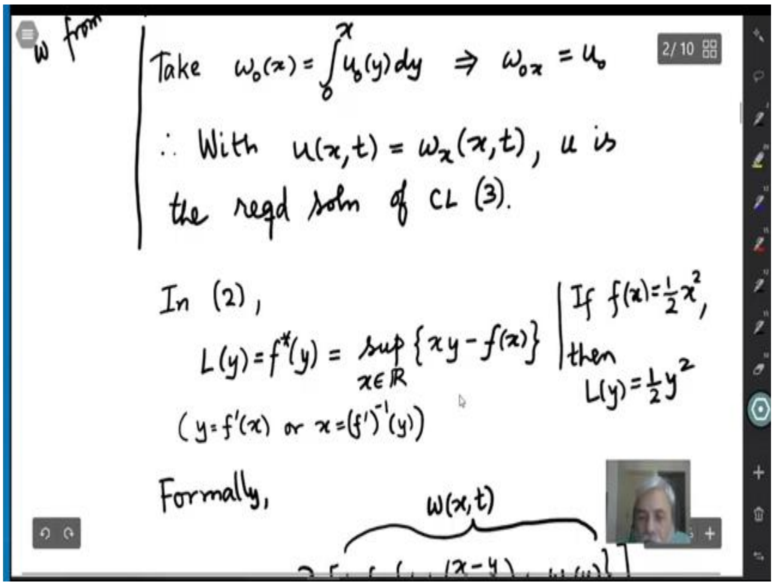

And to satisfy the initial condition we can take w 0 as integral 0 to x u o y dy and that in turn in place, the first derivative of w 0 x is u 0. So, this we have to elaborate a little bit because u 0 is not a continuous function is only a bounded measurable function. So, in what sense, this is true well to see that, we see in due course. So, in order to justify all these statements, we have to borrow many facts from the real analysis and that is Lebesgue theory of measure and integration.

So, I will explain one by one as we proceed, so this heuristic argument showed that if we can somehow differentiate this w, w you remember is given by the Hopf-Lax formula and early the Legendre transform of f this is also denoted by f star. So, this given by this definition supremum over x belongs to R this product xy - f of x. So, this of course can be defined in Rn and that is what we did in the discussion on Hamilton Jacobi Equation. So, there you might replay this multiplication by a scalar output.

So, for example, if f of x = half x square and that is the case of Burger's equation, so any computation show that Ly is equal to also half y square .So, in this case f and its Legendre transform are the same.

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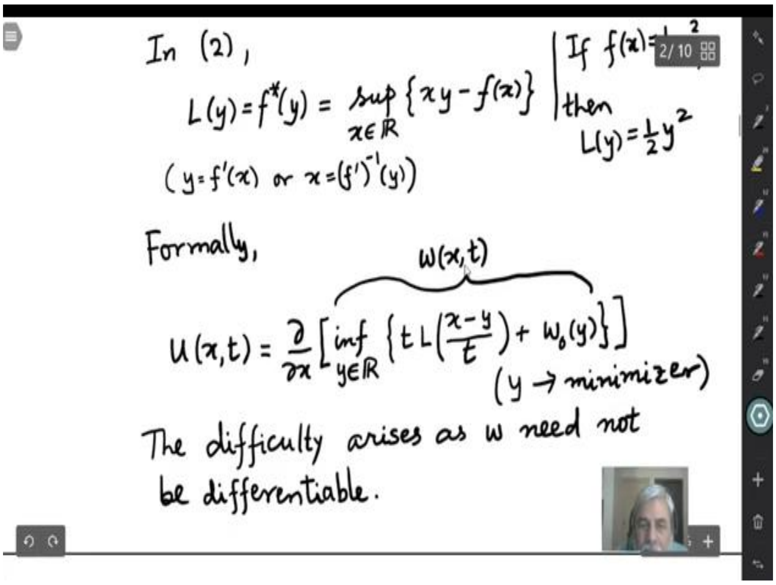

In general, so, since f is assumed to be smooth so, just supremum is attained at y = f prime x, and since we are assuming f is uniformly convex, this f prime has a functional inverse and so, x is given by this universe of f prime y and just you plug in so you get a formula for the Legendre transform. So, formally, you have this u of x t is derivative with respect to x of this w x t. And the only difficulty here is we cannot justify this differentiation directly.

So, we have to study this w x t given by the Hopf-Lax formula and see in what sense this derivative can be taken. And for that, we need to study this minimizer function carefully and this minimizer function also arises when we simplify this formula in the end. So, this place, an important role in the Lax-Oleinik formula. So, we need to study this minimizer of course, that depends on the x t so, its minimizer is a function of the x and t. So, we will first study how that minimizer depends on x and t, so that is our first topic.

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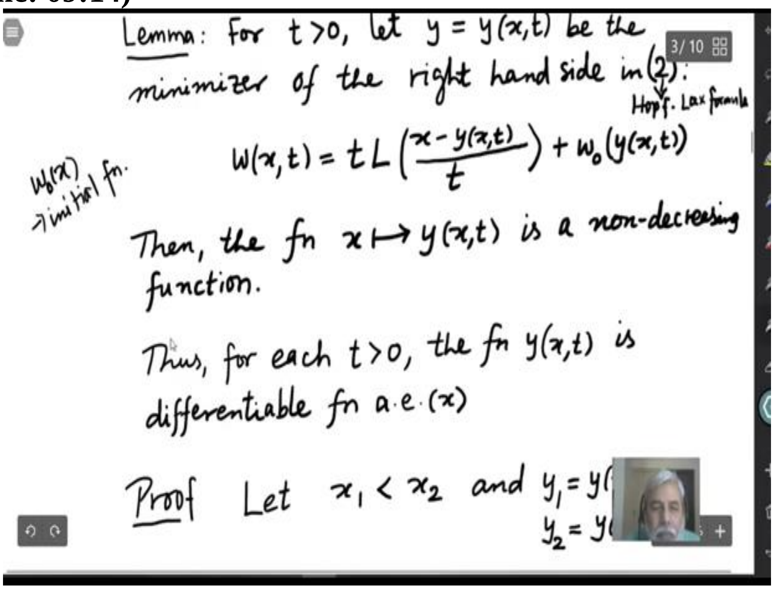

So, this lemma for t positive let y = y x t be the minimizer of the right hand side of 2. So that is 2 is again let me recall that is so you see that connection throughout, so this is called Hopf-Lax formula. So, you have to see this in together, so here I have returned w x t = t L x - y x t + w 0 y of x t. So, this w 0 x will be of course, later on we are taking the x t this is the initial function. So, this initial function we are going to connect it to the initial function for the conservation.

So, then the function x going to y of x t is a non-decreasing function and this allows has to use a theorem of Lebesgue I am again state them in detail below. So, this is Lebesgue theorem so, it is one of the important theorems in the theory of measure and integration. So, any monotonic function is differentiable almost everywhere. So, once that is known, so you can justify this, so this is w x t and this is given by this t of L x – y x t. So, L is a smooth function and w 0 is also an absolutely continuous function, we will see that.

So, this w can be differentiated almost everywhere with respect to x and that makes sense for this u x t and since we are expecting this u to be only a week solution of the conservation law and that weak solution definition in once only an integral relation. So, even almost everywhere defined function will be eligible for that weak solution thanks to again the Lebesgue integration who is allows the integration of functions which are defined only almost everywhere. So, proof of this lemma is quite simple and it just uses the convexity of this L.

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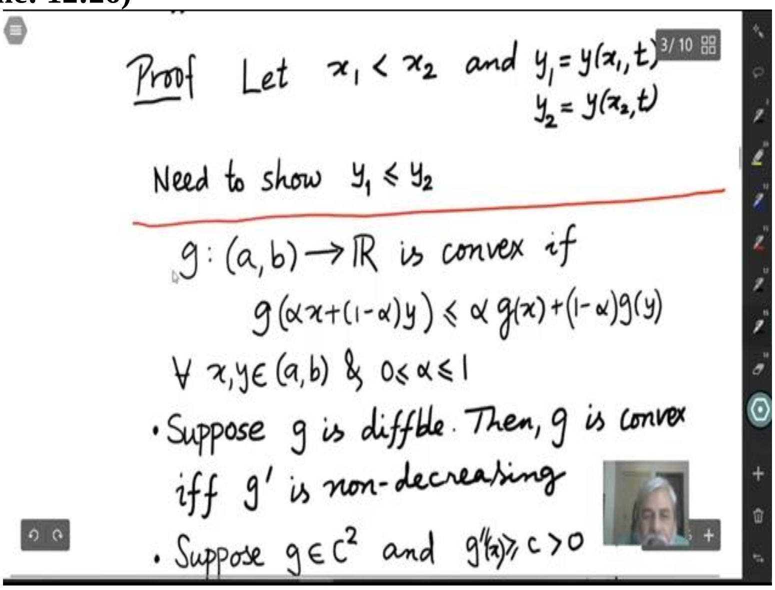

So, for your information I will just recall some definitions of the convex function. So, you take any real value function defined on an interval a b, so it is called convex if this inequality goes to g of alpha x + 1 - alpha y is less than or equal to alpha g x + 1 - alpha g y for all x y in this interval a b and 0 less than alpha less than 1.

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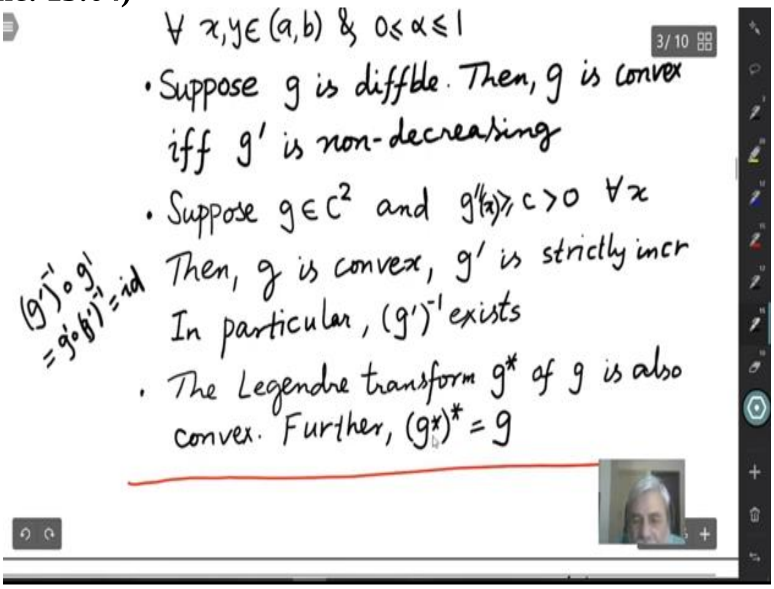

Suppose this g is differentiable function, so we can simplify the convexity condition little bit so, these are easily provable things and those are done in a course on analysis. So, if g is differentiable, then g is convex if and only if g prime is non-decreasing so, this g prime is the derivative of g and that should be a non-decreasing if and only. So, if g prime is non-decreasing then g is convex and it is called with g prime is not decreasing.

So, we can still simplify for that so, if you are assuming more smoothness on g. So, if you assume that g is C 2 and g is bounded away from 0 by pass to constant so, that is g prime g double prime x greater than or equal to c which is positive for all x. Then g is convex and g prime is strictly increased. So, since by this assumption, it will immediately followed that g prime is strictly increasing with a g double prime is strictly positive.

So, g prime is strictly increasing so, g is convex g prime is strictly increasing so, in particular this functional inverse exists, so this is functional inverse. So, g prime inverse composite g this is on their respective domains or if I am not writing that domain equality in that sense. So, g prime would be from this interval a b on to some other interval and g prime inverse will be defined on that range of g prime.

So, this is a simple fact, regarding the convex functions and that we are going to use, this for our f, f is strictly convex, so f prime is strictly increasing. So, this functional inverse exists. And as per the Legendre transform is concerned, so that generally defines only for convex functions. So, Legendre transform of g is also convex. And further, if you take the Legendre transform of g star, then we get back g.

So, again back to the proof of this lemma so, what we have to show in order to show that is non-decreasing, so, take any 2 real numbers x 1and x 2, x 1 less than x 2 and let y 1 be the minimizer at x 1 and y 2 the minimizer y of x 2. So, you just keep this Hopf-Lax formula in mind all the time. So, that is the major thing we are going to use and again and again. So, in order to prove the lemma, we have to show that this y 1 is less than or equal to y 2.

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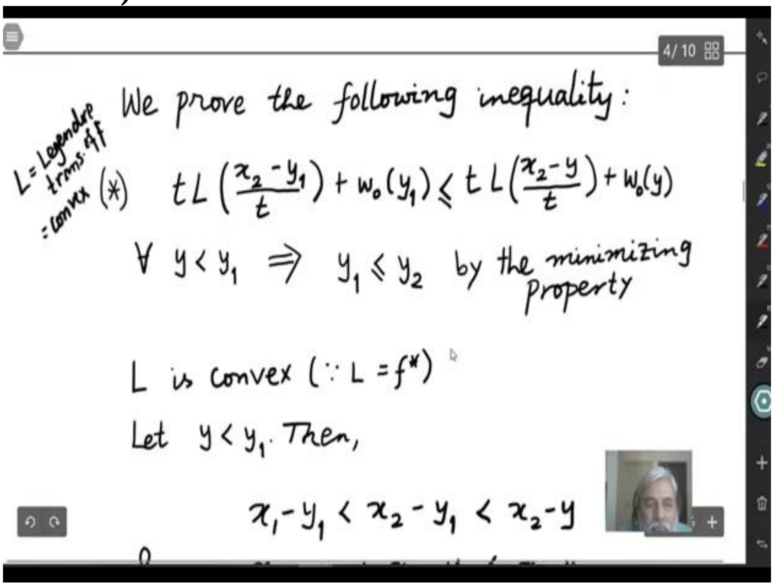

So, in fact, we are going to show a little more so, what we are going to prove is this inequality for the Legendre transform, so this is remember this L is Legendre transform of f so that is convex. So, we are going to show that this inequality course so t of L x 2 - y 1 / t + w 0 y 1 is less than or equal to t L do not need strict inequality, so less than or equal to this side t of L x 2 - y / t + w 0 y and this wholes for all y less than y1.

And once we; prove this inequality star and that implies y 1 less than or equal to y 2 because y 2 is the minimizer corresponding to x 2. And just again, look at this Hopf-Lax formula so, we prove this inequality star.

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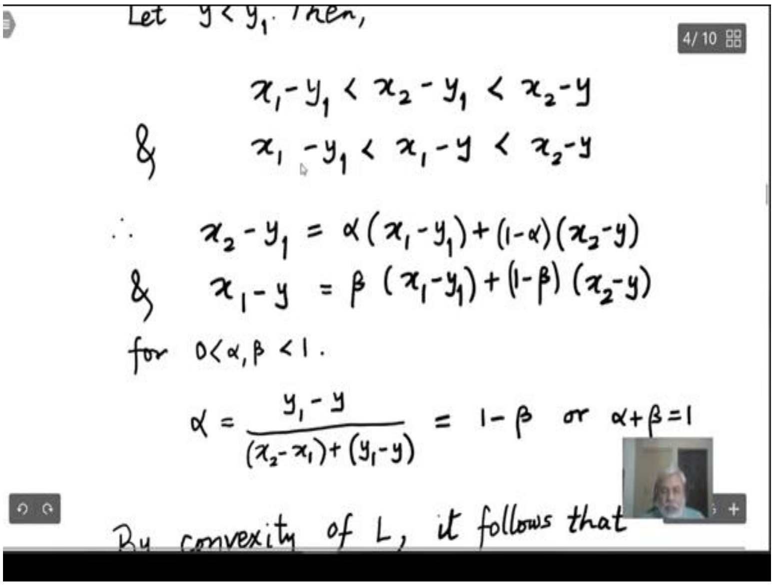

So, you take any y less than y 1 then we have the simple inequalities so, x 2 - y 1 sits in between x 2 - y and x 1 – y 1. So, this x 2 - y 1 is the thing we want here. And x 2 - y you would think we want on that right hand side. So, this x 1 – y also sits in between x 1 - y 1 and x 2 - y. So, therefore, we can write the x 2 - y 1 as a convex combination of x 1 - y 1 and x 2 - y and that what else written here.

And similarly, x 1 - y as a convex combination of x 1 - y 1 and x 2 – y as simple computation showed that this alpha is equal to y 1 - y x 2 - x 1 + y 1 - y by the choice of x 1 x 2 y and y 1 we immediately see that this alpha and beta both line open interval 0 1. That is what I meant by convex combination. So, in this constants alpha and beta are related by so alpha = 1 - beta or alpha beta = 1. So, now you apply convexity on this statement that is x 2 - y 1 is written as convex combination.

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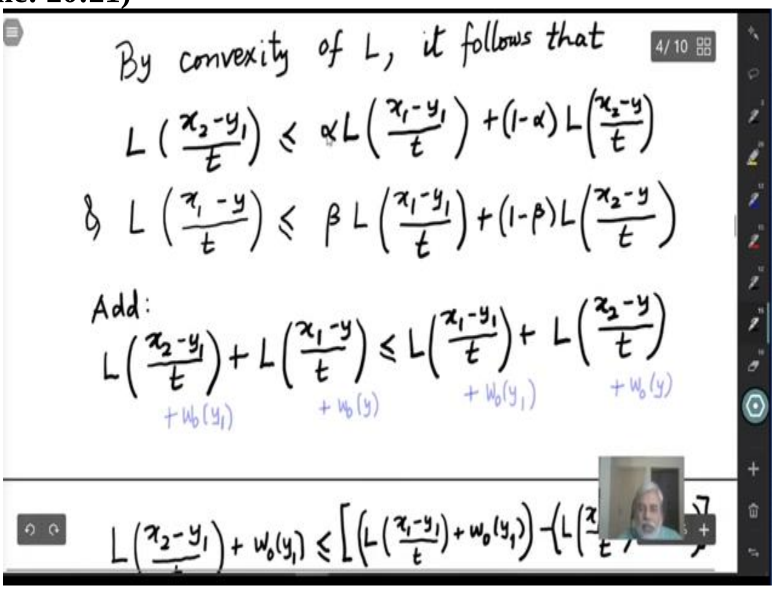

So, by convexity of L, L of somehow you divide throughout by t everywhere it comes, so L of x 2 - y 1 / t less than or equal to alpha L of x 1 - y 1 + 1 - alpha L of x 2 - y / t and similarly, the second one. So, these inequalities follow from the convexity of L. And now, you add these 2 inequalities again remember alpha + beta is 1. So, we get L of x 2 - y 1 / t + L of x 1 - y / t is less than or equal to L of x 1 - y 1 / t because these 2 are same and alpha + beta is 1 and L of x 2 - y / t.

And again look at the Hopf-Lax formula so, there is a term containing the initial condition, so you just have that. So, both sides are at this w 0 y 1 w 0 y w 0 y 1 w 0 y. So, we are not changing the inequality at all. So, even after adding that, we get the same equality.

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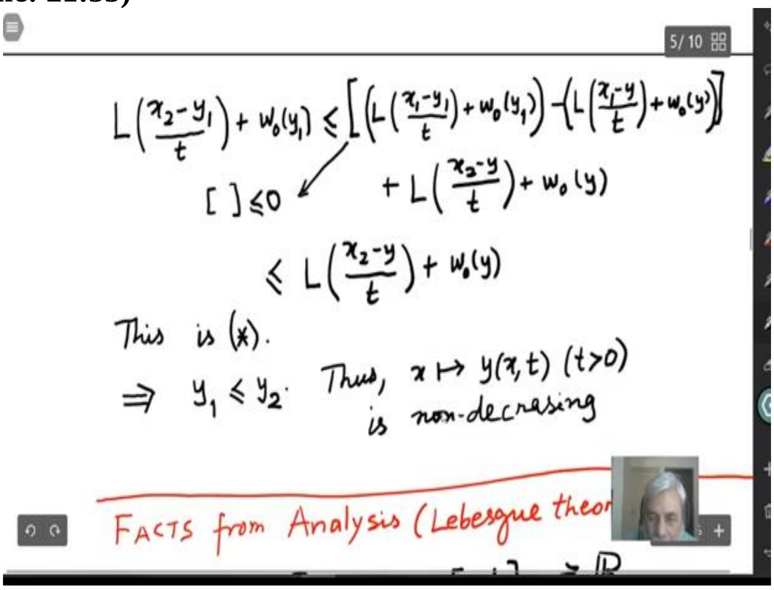

So, now I add this one to this first term so, l of x 2 – y 1 / t + w 0 y 1 is less than or equal to 1. Now, I take this term on the right and club with the first term. So, this L of x 1 – y 1 / t + w 0 y 1 minus I take this one other side. So, L of x 1 - y / t + w 0 y and the last term as it is so, L of x 2 – y / t + w 0 y. And now, y 1 is the minimizer for that function and y is less than y 1. So, the term in the bracket is less than or equal to 0 because y 1 is the minimizer.

So, this is always less than or equal to that for any y. So, this is less than or equal to 0 and so, we get L of x 2 – y 1 / t + w 0 y 1 less than or equal l of x 2 - y / t + w 0 y. And this is the inequality star and that implies that y 1 is less than or equal to y 2and lemma is proved. So, this function x going to y of x t is non-decreasing. Now, before again proceeding further in the derivation of Lax-Olenink formula so, here are some facts from analysis we borrow.

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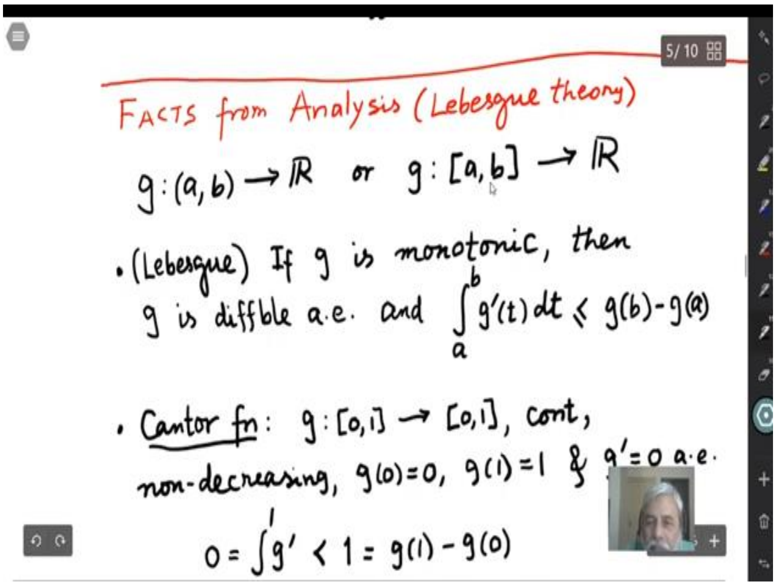

So, here either g is a real valued function defined on an open interval in that case it can be whole real line or it can be defined on a closed interval. So, this is one of the important theorem of the Lebesgue. So, I was telling you, so if g is monotonic then g is differentiable almost everywhere. So, here monotonic means either it is increasing or decreasing. And this inequality inverse integral a to b g prime t dt is less than or equal to.

So, when we have everywhere differentiable function; the fundamental theorem of calculus say that this is the inequality, but in this case, so for almost everywhere differentiable functions there can be strict inequality. So, this is the first difference we observe between everywhere in differential functions and almost everywhere differentiable functions. And this is a standard example the cantor function which is constructed using the cantor set.

So, I hope you all know cantor, what is cantor set so using that cantor set so one construct this Cantor function defined on this closed interval 0 1 to 0 1, it is non-decreasing. In fact g is constant on those intervals removed in the construction of the Cantor set. So, g0 0, g1 = 1. So, in particular g is a non-constant continuous function in fact, by continuity g is all the values between 0 to 1.

So, g is non-constant, but one can show that this g prime so, since it is not decreasing Lebesgue theorem asserts that it is differentiable almost everywhere and g prime is 0 almost everywhere. So, again one more features of almost everywhere differentiable functions. So, the derivative can be 0 yet the function can be very well non constant. So, that will not happen if g is everywhere differentiable that we have in the one elementary calculus we observed that. And we also see that in this case this there is a strict inequality.

So, since g prime is 0 almost everywhere is integrally 0, but the g1 – g0 is just one. So, there is a strict inequality here. The fundamental theorem of calculus as we know from the elementary calculus does not hold for almost everywhere differentiable functions

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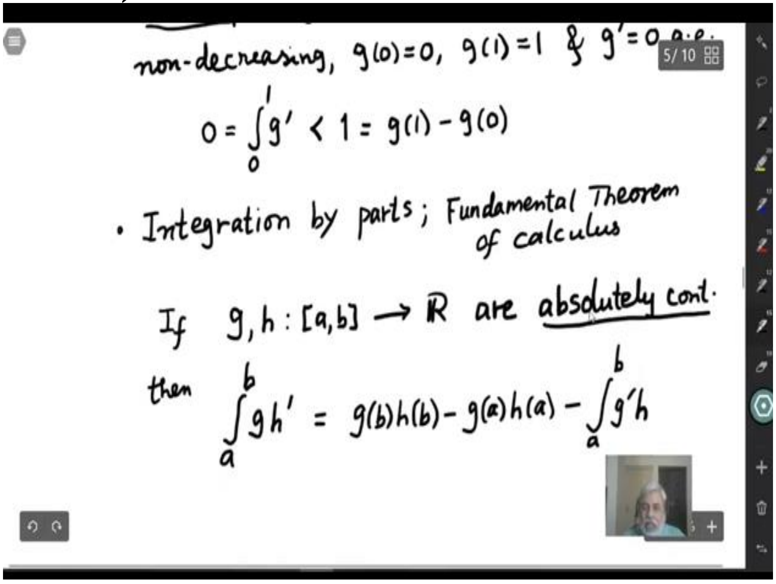

And that is another important distinction. So, for what sort of functions we can expect integration by parts. So, we are in fact going to perform many, integration by parts. So, you should be careful for which functions this integration by parts holds and that in turn do the fundamental theorem of calculus for Lebesgue integral calculus. So, here is one instance where this integration by parts holds true.

So, if g and h are from this closed interval to R are absolutely continuous, so this is another subtle class of continuous functions so, it is more than continuity so, it is called absolutely continuous function. So, I am not going into details of that definition then the integration by parts for so, namely a to b gh prime = gbhb and gaha - integral a to b g prime h. So, if you take for example h is identically equal to 1.

Then we see that the fundamental theorem of calculus for the Lebesgue integral calculus so that was true, so just so this absolute continuity also implies differentiability almost everywhere but then there is a equality here. So, obviously that will tell us that this cantor function is not absolutely continuous.

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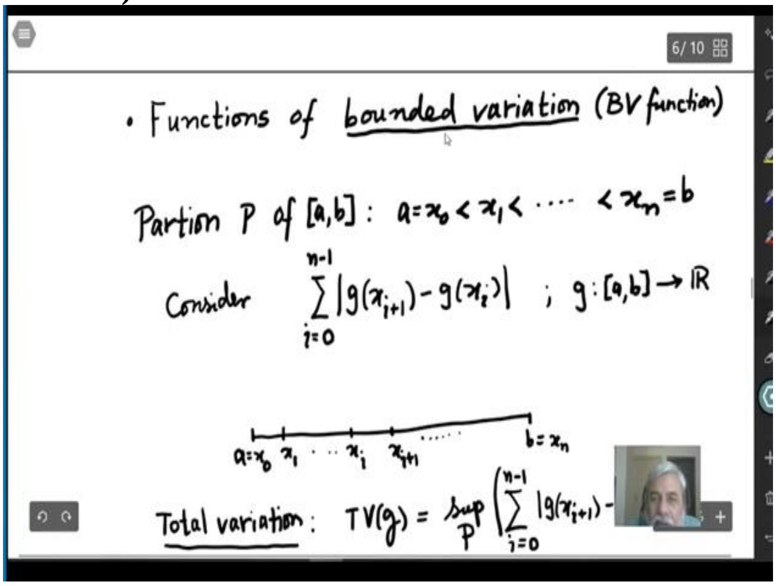

And that brings us to another class of functions which we also need and little later for example you show that the weak solution of the conservation law is a function of bounded variation. So, let me again briefly recall what is meant by a BV function. So, again you take a closed interval and take any partition means so you divide that interval a b into finite number of points. And then you perform this variation.

So, you take the absolute value of g of x i + 1 - g of x i and then some more all the intervals sub intervals. So, this is called variation of g corresponding to the partition. So, how this g oscillates in between the parts, that is what it meant by x. So, sometimes in fact is called bounded oscillation also. So, if for any function certainly this is a finite quantity, but what we want is, so we want to take the sum over all the partitions.

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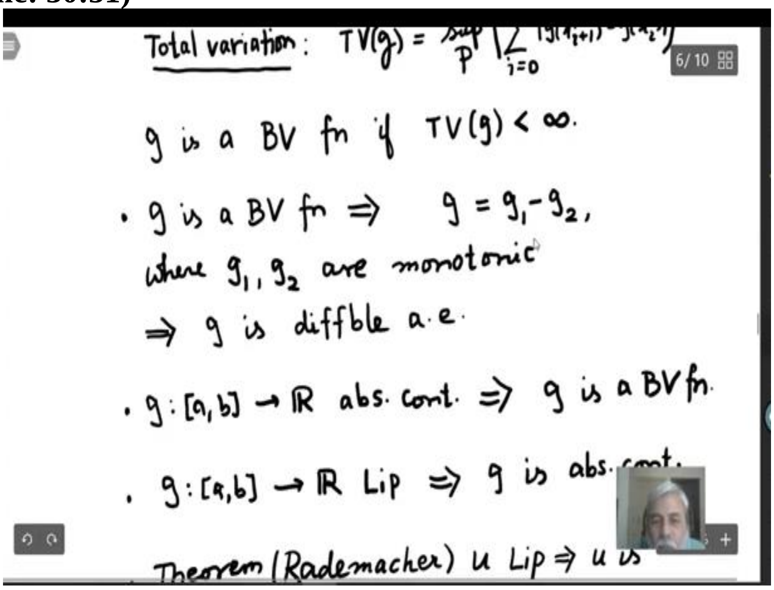

And take supremum of the sum over all the partitions and that is finite, then we will call this g is the BV function and this is called the total variation of g this. So, when you go to higher dimension it will not suffice, so one has to obtain a alternate definition of this total variation. So, this is the definition of the BV function so, if this total variation is finite then you call it a BV function and obviously, if g is either decreasing or increasing so, this we can remove this absolute value and this sum will reduce to only at the end points.

So, obviously a monotonically decreasing or increasing function is a function of bounded variation is a BV function. And conversely that is again in the study of BV functions you might have learned. So, any BV function can be written as difference of 2 monotonic functions and by again applying Lebesgue theorem. So, this g 1 and g 2 are monotonic so, they are differentiable almost everywhere. So, there difference is also almost everywhere differentiable, so this g is differentiable almost everywhere.

So, this BV functions are also differentiable almost everywhere and again we can show that if g is absolutely continuous so, then g is a BV function but somewhere is may not be true cantor function is an example again and so if g is Lipschitz function certainly it is absolutely continuous. So, this class of absolutely continuous functions sit between this Lipschitz continuous functions and bounded variation. Bounded variation function need not be continuous but continuous bounded variation.

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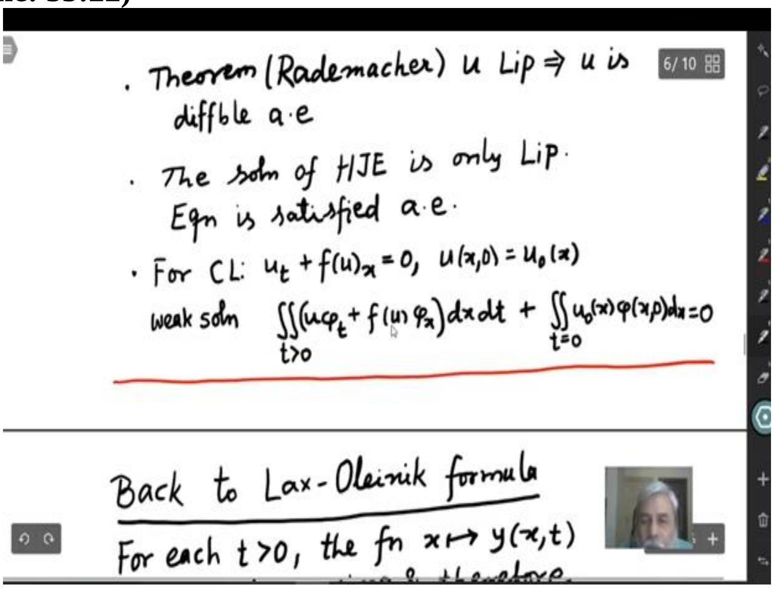

And let me again state a theorem of Rademacher so, this whole C 1 R n so, if u is a Lipschitz function in Rn then u is differentiable almost everywhere and this is what we have used in the study of Hamilton's Jacobi Equation without the solution given by the Hopf-Lax formula is only Lipschitz function. So, the Hamilton Jacobi Equation is satisfied only almost everywhere and again I have written here.

So, for the conservation law, our weak solution satisfied this integral relation and since this is only an integral relation will be satisfied if this u and hence f of u are defined only almost everywhere and that is what we are going to do. So, let me just since it is going to take a little more time than I expected, so from this point onwards, so we have now got result on the minimizer and that I will use to derive the Lax-Oleinik formula and then verify that the function given by the Lax-Oleinik formula is a required weak solution of our conservation law. So, I will take up this in the next class. Thank you.