First Course on Partial Differential Equations - II
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Lecture - 10
Conservation Law

Hello everyone welcome back, we will continue the discussion on the conservation law in today's lecture we discuss interaction of waves. So, as we go along I will explain what does it means.

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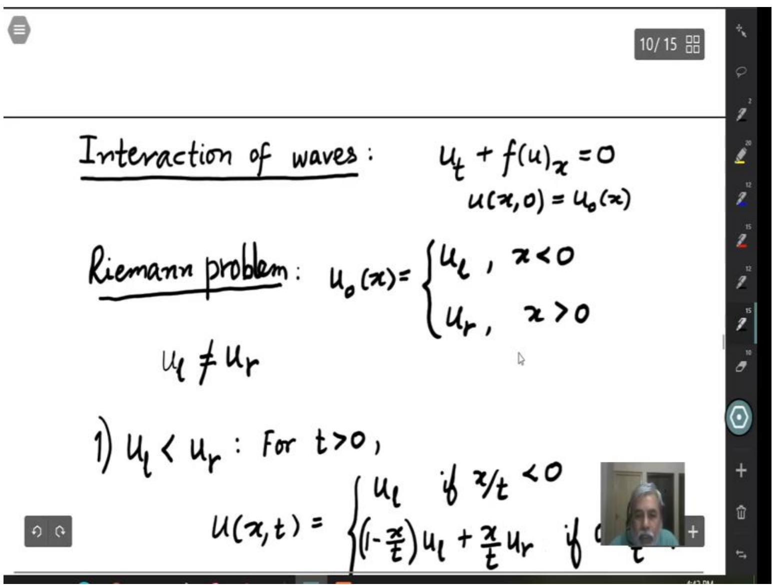

So, again consider this conservation law u t + f of u x = 0 with initial condition u x 0 = u 0 x. So, we will be introducing some new terminology and many of you might be listening to this topic of conservation laws. So, you should study carefully and digest all this new terminology and also the results. So, if the initial condition is given by just 2 constants u l for x less than 0 and u r for x greater than 0.

Such a problem is called a Riemann problem obviously u l not equal to u r, so if u l = u r then it is just a constant the constant is thing only solution. So, in general if we take this bounded u 0 it is possible to break the x axis into small intervals where we can approximate u 0 by some constants. For example taking either maximum in that small interval or minimum and these are the building blocks for a general initial value problem.

And a numerical scheme called Gordon or scheme is based on the solution of the Riemann problem and that is Gordon or scheme is used both for computation as well as for theoretical purposes. So, we start with this discussion of Riemann problem. So, we take this initial function consisting of 2 constant takes. So, we refer to them as constant case. So, yesterday for example we in the previous lecture we discussed by taking u 0 = 0 for x less than 0 and 1 for x bigger than 0 and we obtain a solution for that problem and in fact there were many solutions. So, let me recall that.

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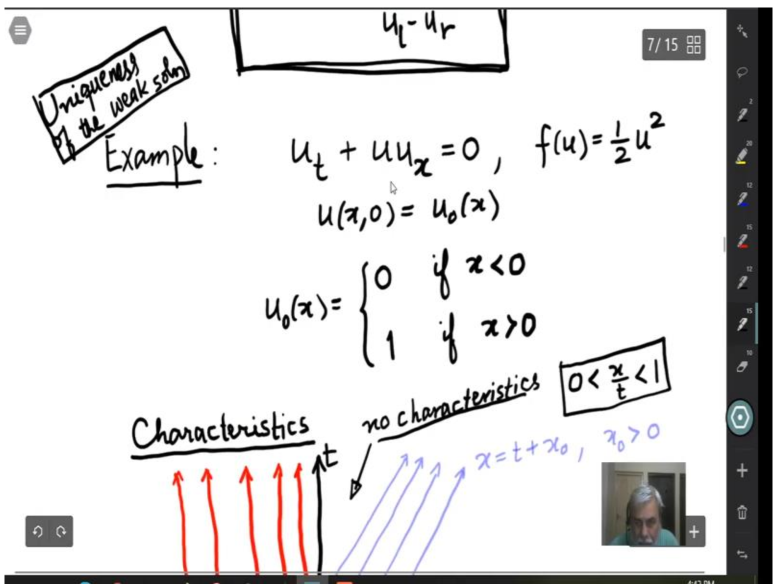

So, this was the burger sequence for Burgers equation we took it so same thing holds for if you take any general f.

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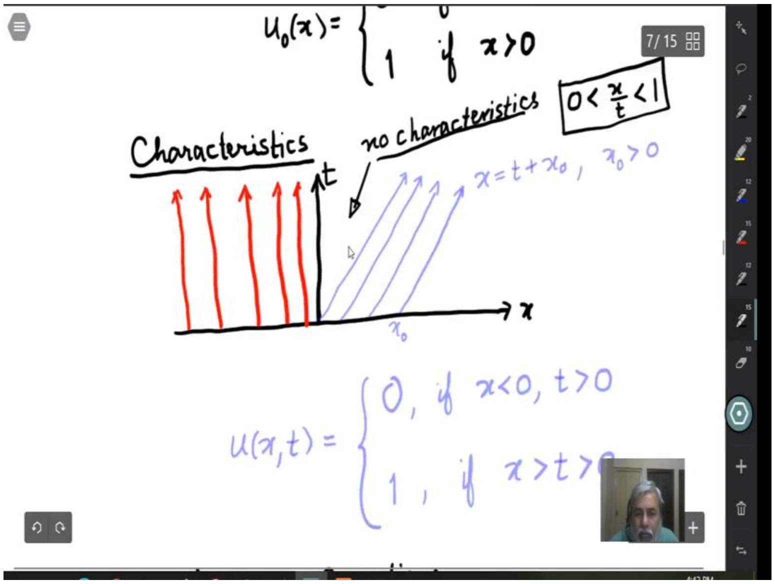

So, characteristics the method of characteristics is solution everywhere except in this region namely 0 x less than t, so that region there are no characteristics.

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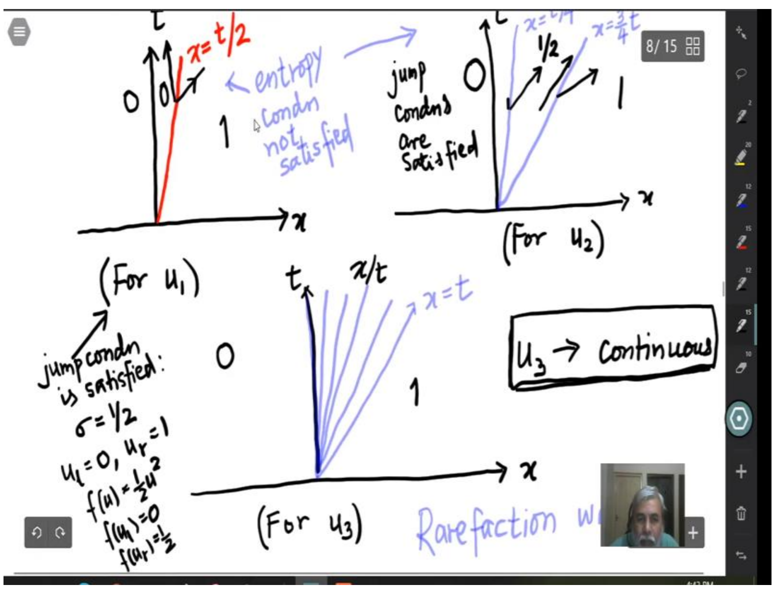

So, then we obtain 3 solutions but 2 were rejected because they did not satisfy the entropy condition. So, the entropy condition we will see later that can be very neatly stated as an inequality. So, again let me write that solution.

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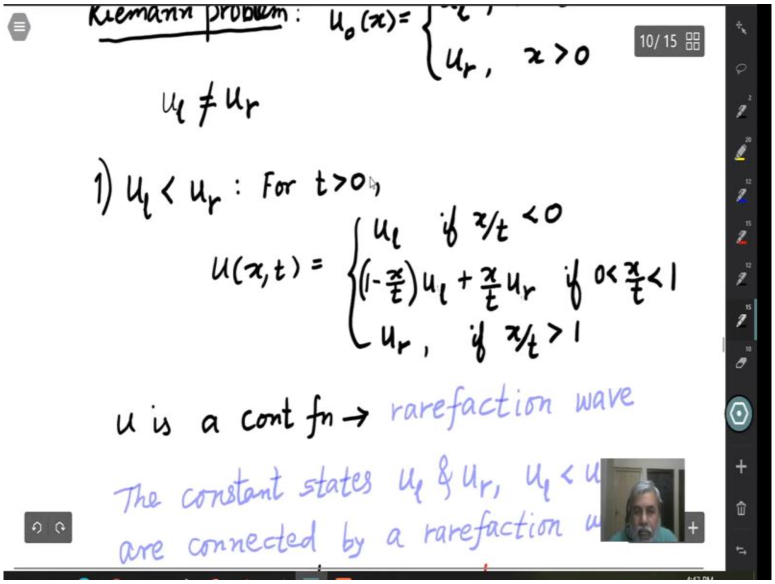

So, if now I take any u l and u r with u l less than u r so you can easily said that for t positive if I define this function u by this relation. So, it is u l if x / t less than 0 and this converse combination if 0 less than x / t less than 1 and again this constant state u r if f x / t is bigger than 1. And we observe that u is a continuous function so it is at x = t is at x = 0 it is continuous and x = t also it is continuous. And this solution is referred to as a rarefaction wave so I was talking about terminologies.

So, this is one terminology, so such a solution is referred to as rarefaction wave and we say that the constant states u l and u r when u l is less than u r are connected by a rarefaction wave.

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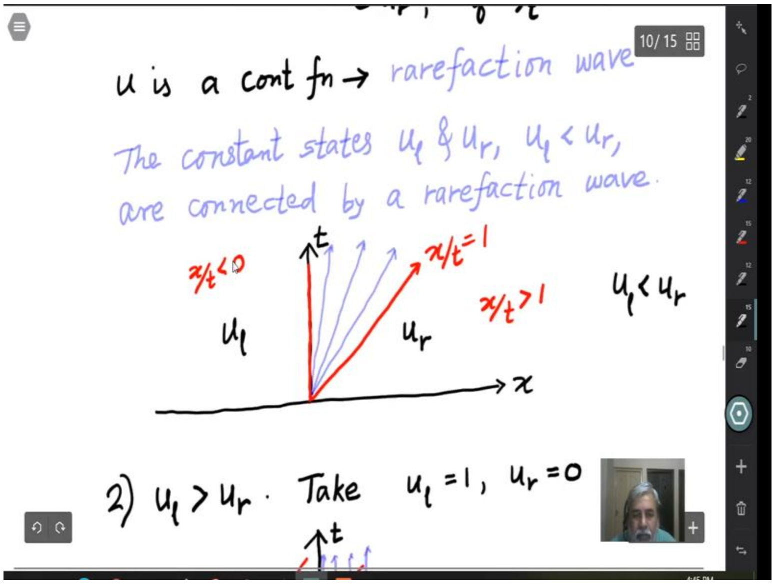

So, here the x t plane depict the characteristics and this is also called a fan wave so it looks like a fan here so on the left it is again the constant at u l and again on the right most it is concentrate u r and in between we have this fan wave so enter things refer to rarefaction wave. So, and u l and u r are connected by this rarefaction so what happens if we take u l bigger than u r that is another case.

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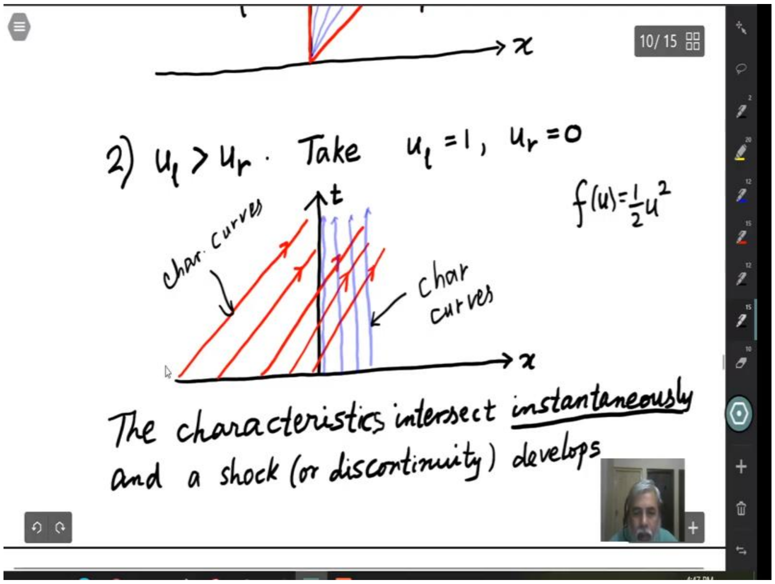

For example let us take a simple case u l = 1 and u r = 0 and again it will go back so for most of the time I am assuming that f of u is so it is a Burger’s equation it can also f of u equal to so this is the Burger’s equation and so again you see if you draw the characteristics in this case when I take that initial condition with the left state as 1 and right state as 0 we immediately see that the characteristics intersect instantaneously.

So, characteristics coming from x less than 0 instantaneously intersect to it characteristics, so these blue lines are the characteristics both are this is for characteristic. So, even this they are straight lines and these are also characteristics. So, how to fit in a discontinuous curve? So, that we obtain a solution of the given initial value problem.

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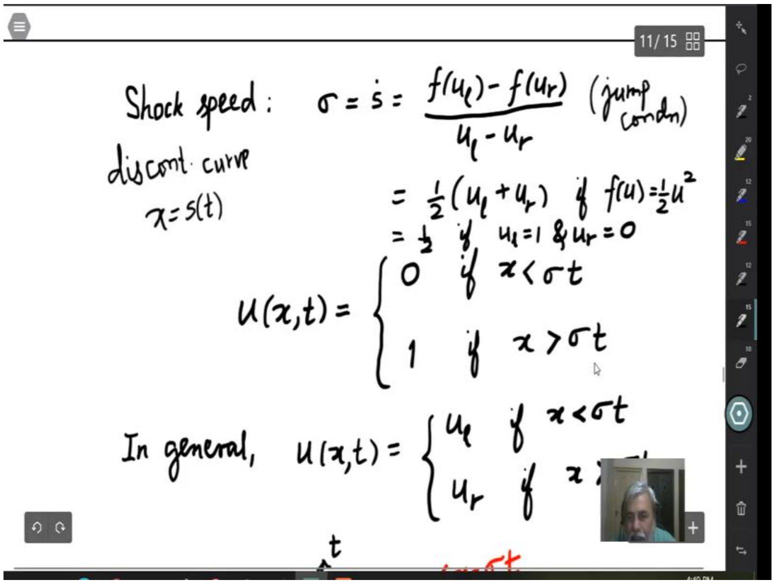

And for that so what should be the discontinuity the line of discontinuity or curve of discontinuity. So, we have already seen that it has to satisfy the jump condition so this sigma the shock speed which is also denoted by so this is the discount continuous curve we are using x = st and sigma is speed and it has to satisfy this jump condition. And for the Burger's equation it is precisely the average of the left state and the right state.

And this is of course equal to half if I take u l = 1 and u r = 0 otherwise also it is just a constant because our initial condition consists of only 2 constant states. So, in this case the solution is given by u of x t = 0 if x is less than sigma t and 1 if x is bigger than sigma t and sigma is the shock speed. So, in general if you take u l and u r are the constant states then we get the solution as u x t = u l f x less than sigma t and u r is x is bigger than sigma t.

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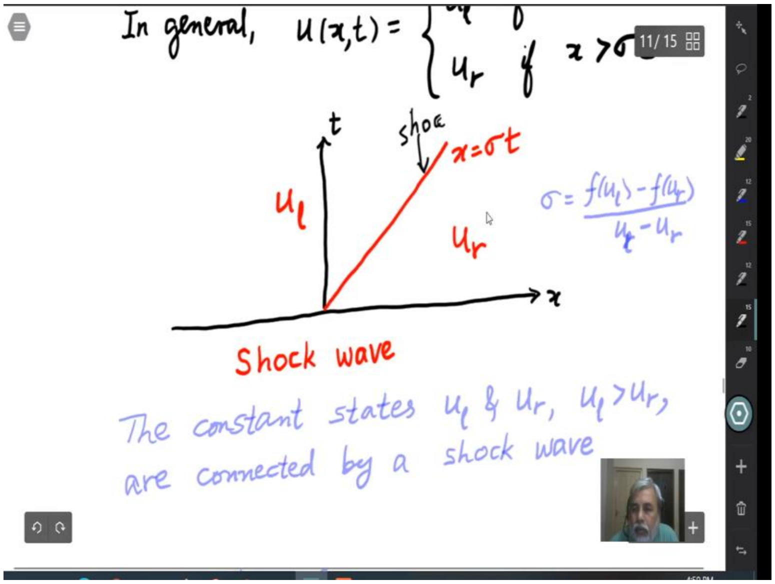

And sigma is given by this ratio f of u l - f of u r divided by u l - u r. So, again in the x t plane we see that so this is x = sigma t this is the shock or discontinuous curve in this case it is straight line and in this case, so we have this we say that the constant states u l and u r where u l is bigger than u r are connected by a shockwave and shockwave this is the solution is referred to as shockwave.

So, it has again 2 constant states now separated by a line of discontinuity, in general this may not be a straight line as we see in that example. So, this is shock, so this is fine, so Riemann problem we can simply solve in either case so either we will get a rarefaction wave connecting the 2 given constant states or they can be connected by a shock wave.

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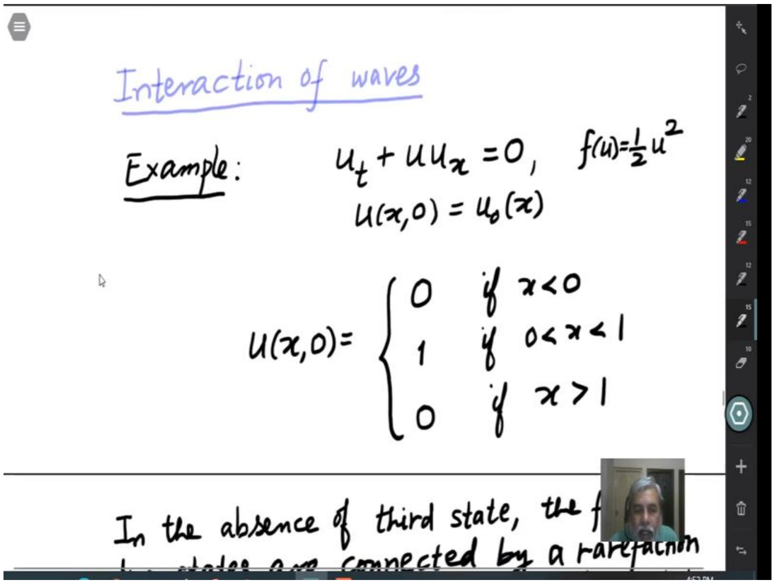

Our next question is what happens if there are interactions of waves, so there is a rarefaction wave and then there is a shockwave support they interact there is some interaction. So, here is a simple example, so quite illustrating so, you should study carefully.

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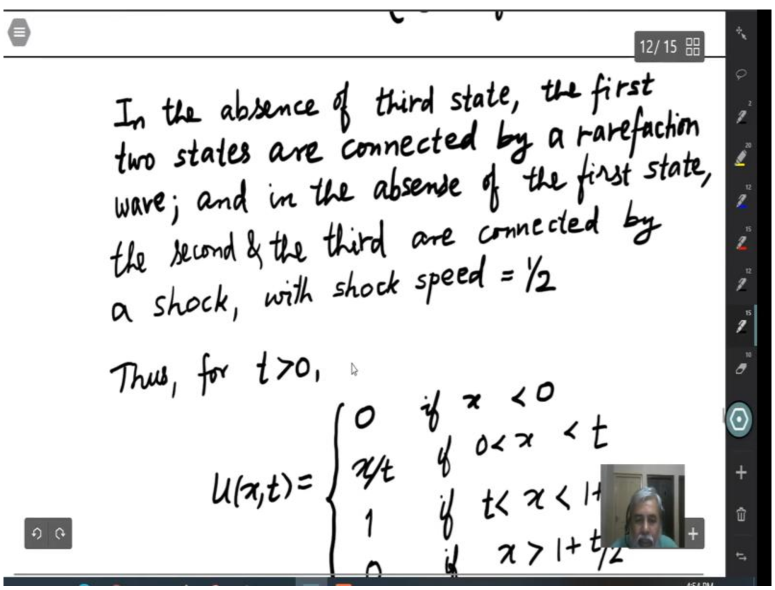

And you see there are some complications here and the more we appreciate such examples we better understand the Lax-Oleinik formula which hides all these computations, so Lax-Oleinik formula which we are going to derive in to course hides all these computations, so it is a very neat formula but if you want to do something by hand we see the loss of computations and also some analysis.

So, now we consider this example again Burger's equation, so u t + u x = 0 and u x 0 = u 0 x. And now so u 0 x consists of 3 constant states for the illustration so this is x axis, so this is 0, this is 1, so this is 0, 1, 0 that is the initial condition consisting of 3 constant states. Of course in the absence of this third constant state if we just consider the first 2 and since 0 is less than 1 those 2 states are connected by a rarefaction wave that we saw just now.

And again in the options of the first state namely this 0 x less than 0, so you make it 1 throughout x less than 1, then those 2 states will be connected by a shockwave. So, we expect so the solution to have a combination of rarefaction wave and a shock. So, indeed that is happening so that is for t positive, so we obtained this solution, so the first 2 states are connected by rarefaction and just for I am writing here.

So, you u x t 0 if x is less than 0 x / t 0 is less than x less than t and 1 if t is bigger than x. So, in the options of third one that would have been the solution. But since now there is another third state and that has to be connected to the middle state by a shock and again you see the shock speed has to be since we are in Burger's equation. So, the average of these 2 and that is precisely half and that is what we get. So, 1 if t less than x less than 1 + t / 2 and 0 if we x less than greater than 1 + t / 2

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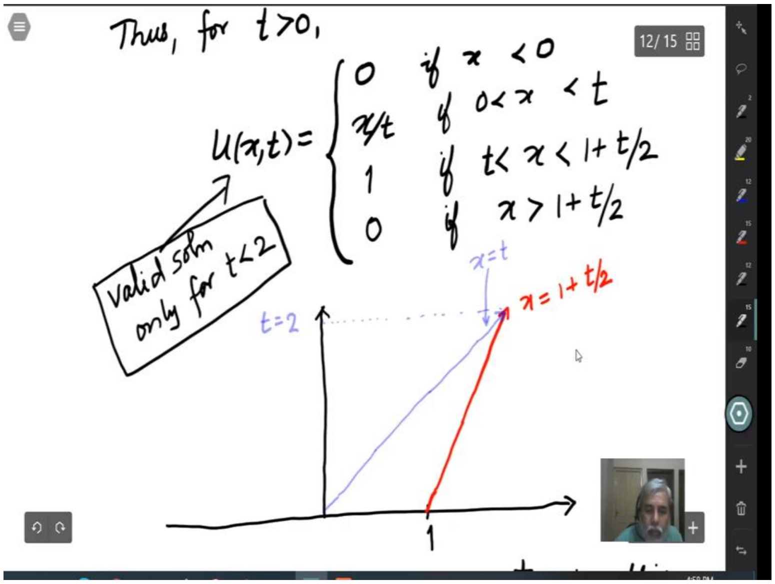

That shock speed is again half so this would have been alright but to see some difficulty. So, if you consider this sharp line in this case it is straight line. So that is the x = 1 + t / 2 and this characteristic x = t coming from the rarefaction region, they to intersect at t = 2. And again it is not possible to continue the solution keeping this x = 1 + t / 2 line I just shock.

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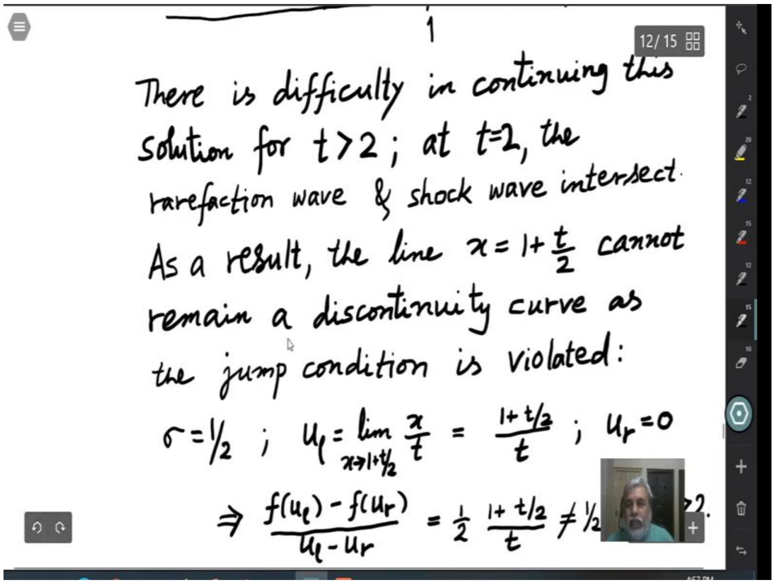

Because that will violated the jump condition so why it is violating jump condition, so that is a simple computation here. So, in case this again this line has to be continued as line of this continuity is speed will be half but if we take the left limit that will be 1 + t / 2 / t and again u r is 0. And if you calculate this ratio again f is burgers equation here so it is half u square. So, we get that this ratio is not half for t bigger than.

So, this line cannot be a discontinuity curve as the jump condition is violated, so this is a valid solution only for t less than 2. So, beyond 2 we have to again now construct a line of discontinuity or curve of discontinuity in order to complete the solution for t bigger than 2. And again the jump condition is our guiding principle, so we will try to find what should be the speed of that discontinuity curve beyond t = 2.

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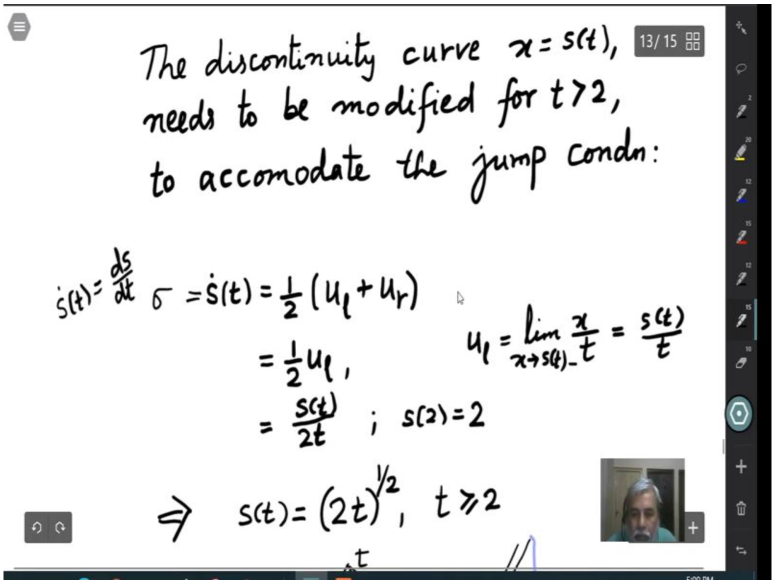

So, we have to now find that curve x = st which will accommodate the jump condition. So, again remember we are in the Burger's case so this s dot t so if you did not also denoted by sigma that is half of u 1 + u r this u r is 0 that third state is 0 just remember that third state is 0. And you will the limit of the solution as we approach the curve of discontinuity from the left. So that is x should also say that and that gives you st / t.

So, this curve of discontinuities should satisfy this ordinary differential equation namely s dot t remember that s dot t is ds / dt. So, solving this ODE and giving this initial condition s2 t= 2 because they made their t = 2, so should have 2 there.

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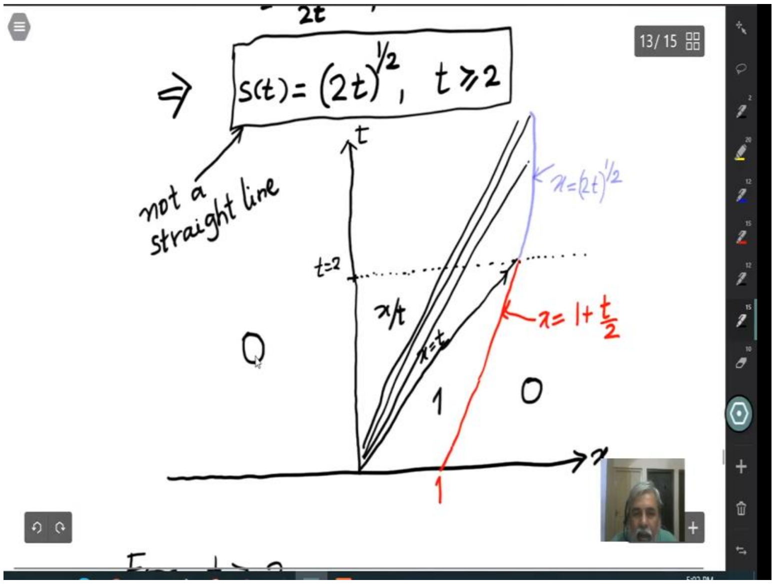

And one is resolved that ODE to get the solution. And first thing you should observe you thought this is not a straight line. So, here is the picture in the x t plane, characteristics, shock curves etcetera. So, up to t = 2. So, there is only this rarefaction wave, so this is 0 here and this is expand t solution again here this is 1and again 0 and beyond t = 2 is now given by again 0, 0 for x less than 0 and in this region it is simply x / t 0 x less than 2t to the half.

Then that is our new shocker and beyond that again it is 0. So, since this shock curve is constructed to satisfy the jump condition so there is no need to further check that so it is automatically satisfied and we have this solution. So, now let us analyze.

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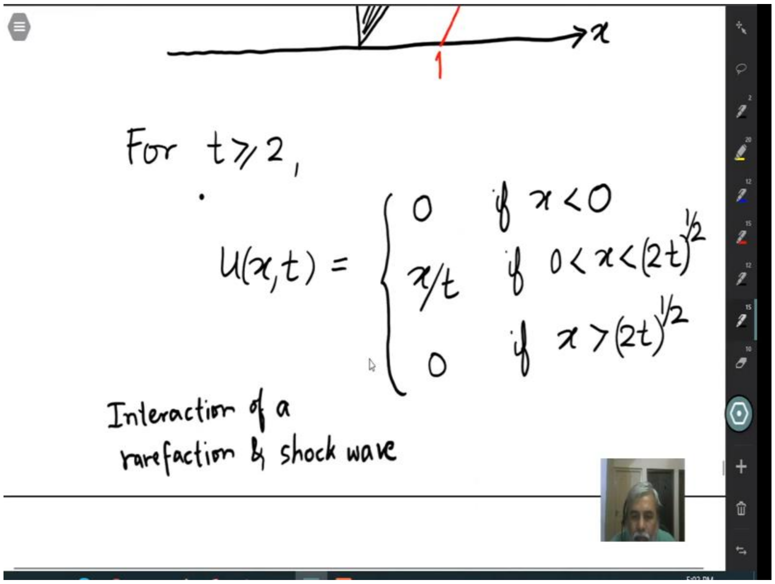

So, this is the interaction of so this is the result of interaction of a rarefaction wave and a shock wave. So, you observe certain things here. So, this after t bigger than 2 this constant state has just vanished. So, only the first and last date of the initial data namely 0 and 0 they are retained and this one vanishes and that is the region because of non linearity. So, we will not observe this in a linear problem.

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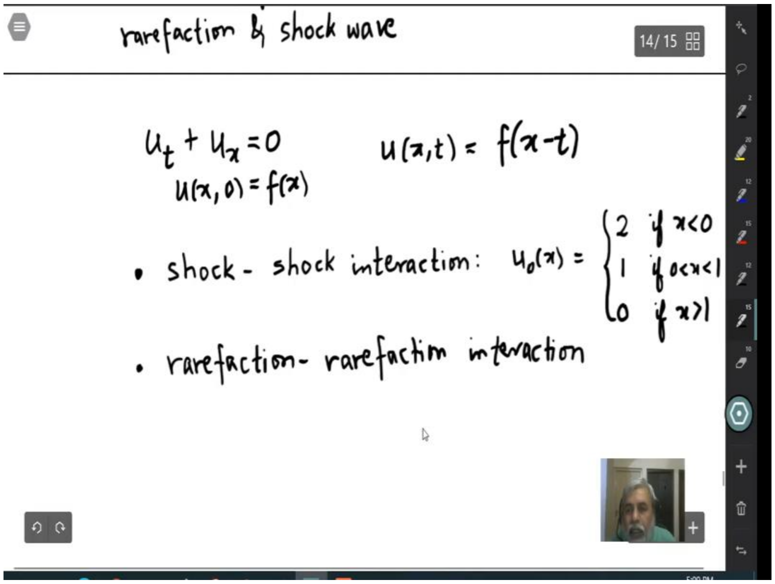

For example if we have this u t + u x = 0 u x 0 is f of x, so at any later time t this just so whatever the initial profile it will be retain for all t positive where in this case it is a nonlinear problem. So, it changes the initial profile in fact there are results showing that as t goes to infinity the solution contains just anyway which I am not going to discuss. So, whatever initially we have in this u 0 more or less everything will be lost.

Because of this nonlinear phenomena and another thing we observe so initially this line x = 1 + t / 2 shown in red here. So, the shock curve was a straight line but after this interaction of rarefaction wave and the shock wave that shock is made to bend. So, now it is a curve it is 2t to the half, so this because of that interaction this shape of the shock changes from t = 2 onwards. So, these are some of the observations we see here in the interaction of 2 waves.

After this experience so we can now try other interactions for example you can try that shock interaction what happens and so for example here we can take this u 0 x how do we obtain such a thing? So, just one example I am just giving so this is 2 if x less than 0, 1 if 0 x less than 1 and again 0 if x greater than 1. So, the constant states 2 and 1 are connected by shockwave and again the constant states 1 and 0 are connected by shockwave and if they interact then what will happen? So that we can again study but competitions are not easy all the time.

So, we can also analyze what happens 2 rarefaction waves interact and we can also examine. So, here the rarefaction was on the left and the shock was on the right initially and that remain later also. So, this rarefaction remains and that shock also remains though with some changes. Certainly changes are there and what happens if I start a shock on the left and rarefaction on the right that also one can right.

So, even whether it makes a difference to have on the left side or right side, whether the interaction remains the same or is it difference all those things one can do with some simple examples. So, when we discuss this Lax Oleinik formula maybe in the next class. So, you will not see any of these interactions in that solution. So, it is for a general bounded initial data, so we appreciate that formula because it hides all these things and for just general bounded initial data.

So, it does not assume any of the Riemann problems and other things but certainly using that formula all these things can be again update. So, I will stop here and next time we will take Lax Oleinik formula. Thank you.