First Course on Partial Differential Equations - II
Prof. A.K. Nandakumaran
Department of Mathematics
Indian Institute of Science - Bengaluru
And
Prof. P.S. Datti
Former Faculty, TIFR-CAM - Bengaluru

Lecture-01
Review of PDE-I and Introduction to PDE-II

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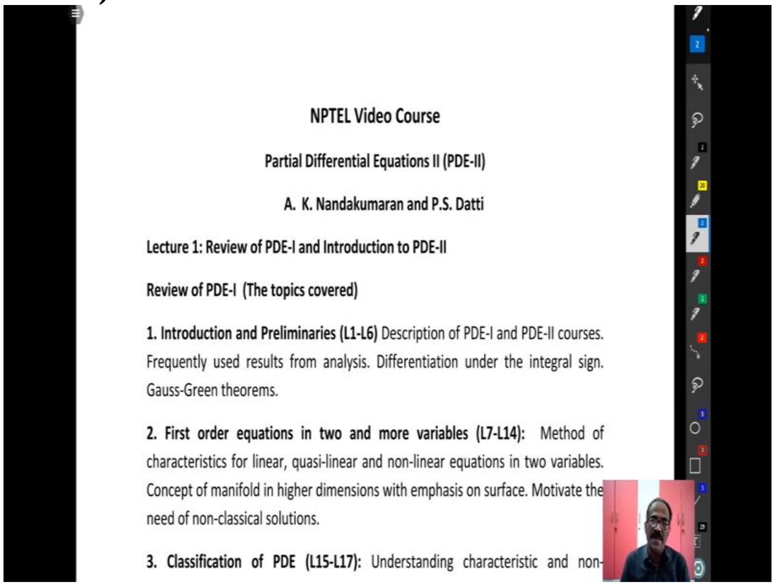

Good morning and welcome to this second course on partial differential equations. And this is a continuous of our first course on partial differential equations. So, before trying to attempt to take this course upon to learn this course we strongly advise you to go through the course PDE 1. On the other hand if you are already familiar with the course PDE 1 you may directly look into this course.

So, it is not a really an advanced course but it is a first course for the PDE 1 and PDE 2 together consists of one full semester course. So, what I am going to give in this introduction basically consists of 2 parts, I will quickly in 10 minutes or so I will try to give a brief overview of the PDE 1 course which we have covered. And then I will give a brief introduction to the course on PDE 2. And so let us begin with what we have done in our NPTEL PDE 1 course. So, let me quickly review through the one.

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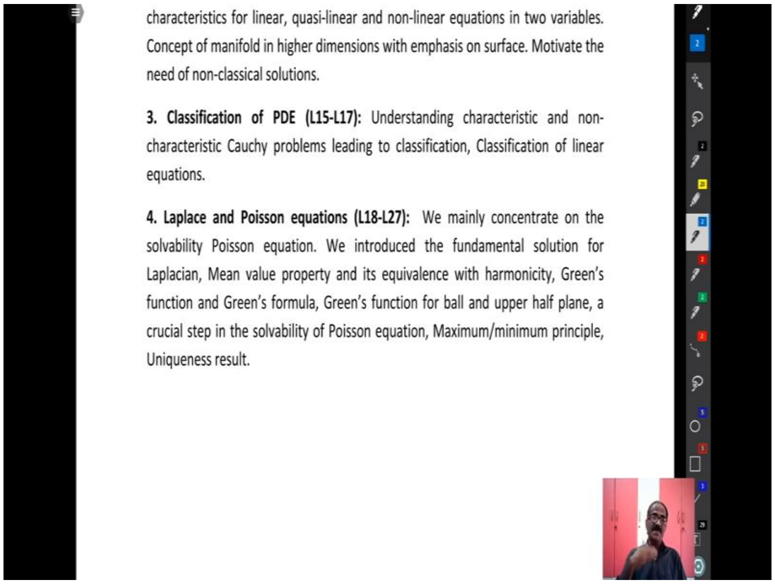

So, we began this course by giving a basic introduction to the PDE and some preliminaries required. So, again we advise the students to go through the preliminaries mentioned in our PDE 1 course, especially the understanding of the domain and its regularity and integration by parts, especially the Gauss Green theorems and stocks theorems and the basic analysis from that convergence uniform convergence and various other parts things involving elementary thing.

It is important to understand the regularity of the domain, the existence of the normal derivative and normal directions and then how do you integrate by parts in higher dimensions in omega. And you know integration by parts in 1 dimension but you also need to know how do you do this one. So, various fundamental theorems like Gauss Green theorem, divergence theorem and all that please get familiar and that is what we have done it in first 6 lectures of our PDE course.

And then we have introduced to what is called the first order equations in 2 and more variables, one of the main thing we concentrated is the notion of characteristic by the time you would have gone through the course PDE 1, you would have seen the importance of the characteristic concept in partial differential equations that even goes to the next section what are called the classifications.

So, initially we are given a detailed analysis of your first order requirements in starting with the 2 dimensions and then later on higher dimensions. And we also classify introduced in a systematic way about the first order requires above method of characteristics. First do we get for the linear equations and then for quasi linear equations and then non linear equations and various geometry concepts including mangier con and mangier strips integral surfaces and all that.

And then in the process you also seen to motivate how the non classical solutions exists, especially when you dealt with the quasi linear equations do you have seen the conservation laws. And this course you will see more about the specialized the first order the equations which we will slowly describe. And based on that we have that unlike ODE, ODE is basically classified into linear and quasi linear, linear and nonlinear.

But then you have seen that is not enough for PDE's the Mathcad using the characteristic and not characteristic all the problem let to various classifications like elliptic, hyperbolic and parabolic even linear equations you will have this type of classifications you have seen it. But then we after that we explained 3 important equations. We are given the motivation why we studied these 3 important equations.

These 3 equations like namely elliptic, parabolic and hyperbolic is a representative candidates a Laplace equation and heat equations and wave equations and the idea all these equations arise in the literature as it is that is one reason why we studied these 3 equations. And we also seen these equations are representative things like equations, because it has all the characteristic of the equations belongs to that particular class.

For example Laplace belongs to the elliptic class heat belongs to the parabolic class and then wave equations belongs to the hyperbolic class but what we have done in Laplace and Poisson equation are very minimal in the first course the main idea was to introduce the fundamental solutions greens function and mean value property, we have not done the solubility much except very little solubility in 1 or 2 specialized cases.

And we will see more in the coming course which we will explain so now. So, the important thing you have to recall all these results go through it even if you have studied the course 1, please go through the Laplace and Poisson equations and especially understand fundamental solutions, the mean value property and greens function via special greens functions we have introduced for upper half plane and a circle or a sphere which plays a crucial role in the solubility of the Poisson equation. And we have seen that under important things called maximum and minimum principles. So, you have to go through all that.

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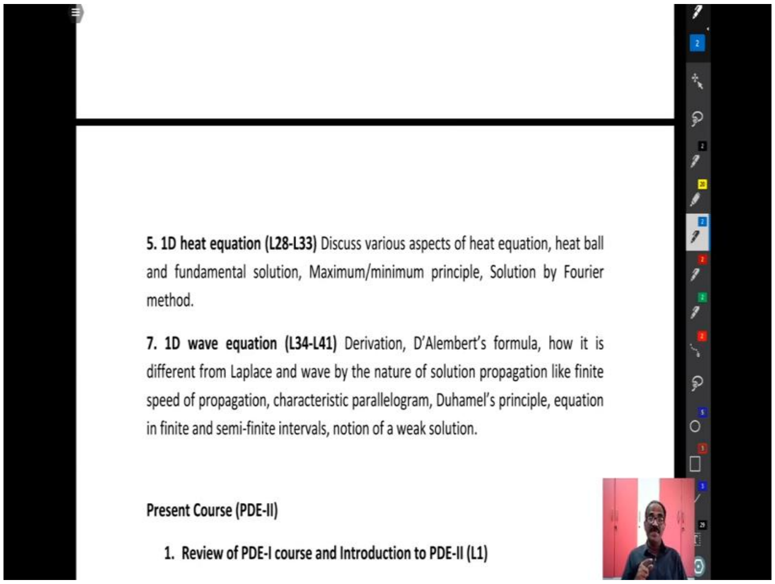

After that we have done 1D heat equation and 1D wave equation. So, in 1D wave equation we have seen and wave equations we have seen some of the physical applications. And we also seen the heat ball and fundamental solution and maximum principles and you also seen in the first course the solution by Fourier method. In 1D wave equations you have your derived using the separation variable formula the D'Alembert's formula you have done.

And how it is the more important thing is that wave equations can play something is so much differ from the Laplace equations and wave nature of the solution like propagation like finite speed of propagation, characteristic parallelogram and you have also seen Duhamel's principal, equation in infinite dimension, in semi interval. So, we you have seen various things so we have only discussed heat equation in the wave equations in 1 dimension.

On the other hand you have seen the Laplace and Poisson equations that we have introduced the fundamental solutions and all these property even n dimension. That was the rough idea discussion about the review of our first order PDE. So, let us now go to what we are going to do in this course.

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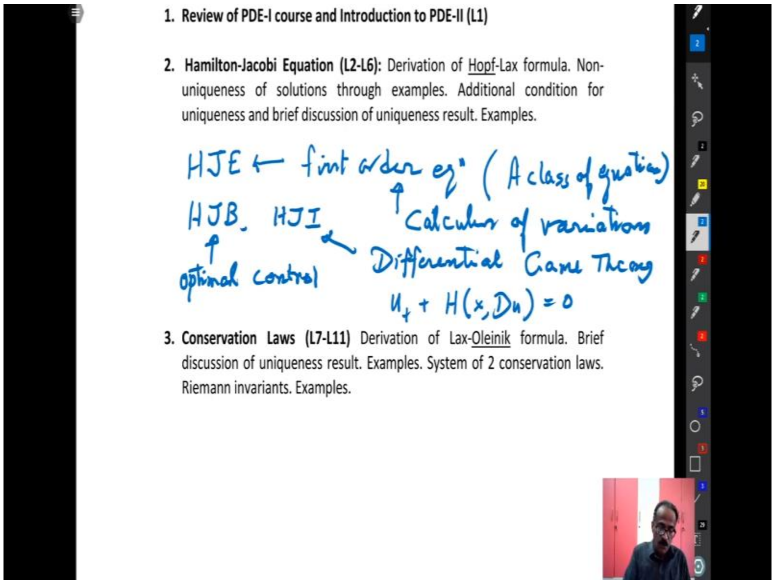

As I said the first lecture is about the review of this course and also an introduction to that one and that we will do it in this half an hour lecture and then we are going to discuss 2 important first order equations. Both these equations are first order equations, this is called the famous Hamilton Jacobi equation and here you will see the conservation laws. This is a first order equations a special first order equation.

In fact this is a family of equations you are there it is not the one equation a class of equations. And this appears from various things. So, we are not going to do Hamilton-Jacobi equations of more generally Hamilton- Jacobi there are other generalization like Hamilton-Jacobi Bellman equation and Hamilton-Jacobi Isaacs equation. We will not be discussing. So, the Hamilton-Jacobi equation is a more of a classical equation coming from calculus of variations.

And on the other hand this comes from optimal control problems and this comes from differential game theory. As I said we will not do it in a very generality. This is a very special type of nonlinear equations which is does not have in general the variation structure. So, our main idea is to derive what are called the Hopf-Lax formula which is a very, very special formula very special case.

So, we will be deriving the Hopf-Lax formula which gives you a formula for the solution of the Hamilton Jacobi equation. Typically the Hamilton-Jacobi equations will have this form where H is called a Hamiltonian we will motivate you through various examples. So, if there are more general, so this is a very special class of equations arising from more general nonlinear equations. And you will see where again here the discussion of the uniqueness.

And we will not be able to prove the unique results which you may refer to the one of the books which we suggest but the important thing is that you get the uniqueness but uniqueness not in general. So, you have to get into the kind of what we call it a weak solution concept. So, you will see here in this case one type of solutions but that is not enough in general when you want to go into more general equations like Hamilton-Jacobi Bellman equation now Hamilton-Jacobi Isaacs equation. So, there is a big general theory what we call it a viscosity solution.

So, we will not touch that concept of week solutions of viscosity solution, this viscosity solution concept is different from your weak solutions which you normally look for using soblers spaces. Similarly this we will do it in a 2 to 3 hours maybe 5 half an hour lecture.

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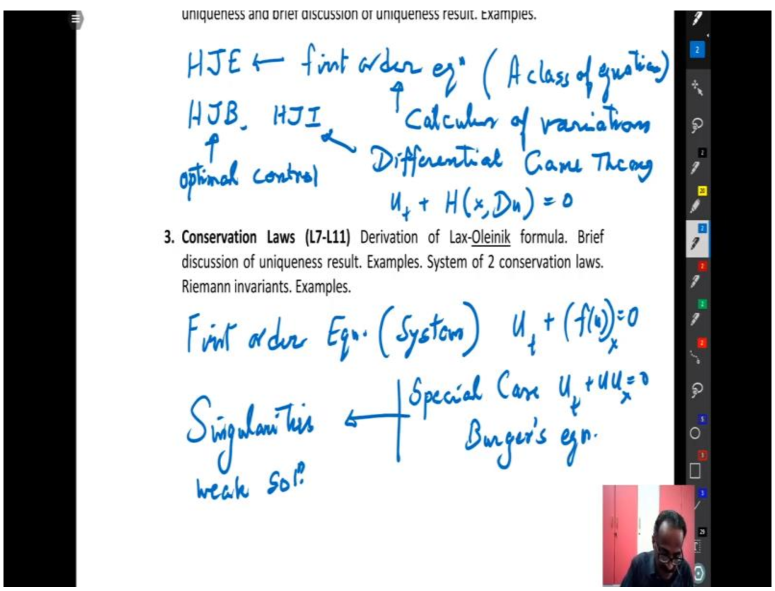

And similarly there is another first order equation this is also first order equation or first order system and we derive again our presentation of this one what are called the Lax-Oleinik formula.

So, you derive and again we will discuss about the you brief discussion about the uniqueness and we also discuss about the system of conservation law and Riemannian invariants. By the way these courses, again is going to give 2 of us my colleague professor Datti and myself.

Just like what we have done it in our first part of the course. So, the conservation laws is an another special class of equations of this form u t so you have a conservative structure or the variation structure and you have already seen these equation special case u t + uu x = 0 this is the Burger's equation and you have seen in our first course how singularities propagates and you have to understand the singularities you need to go through what are called the weak solution concept.

So, we will do in fact what I said what we will do is very, very minimal you can spend all your whole life in concentration of in this case. So, we will also try to do a little bit about the system. Systems are very, very interesting in conservation laws and you also give introduced to what are called Riemannian invariants. So, after this first order equation this is the continuation of our first order thing 2 types of first order equation Hamilton-Jacobi equation and Conservation law.

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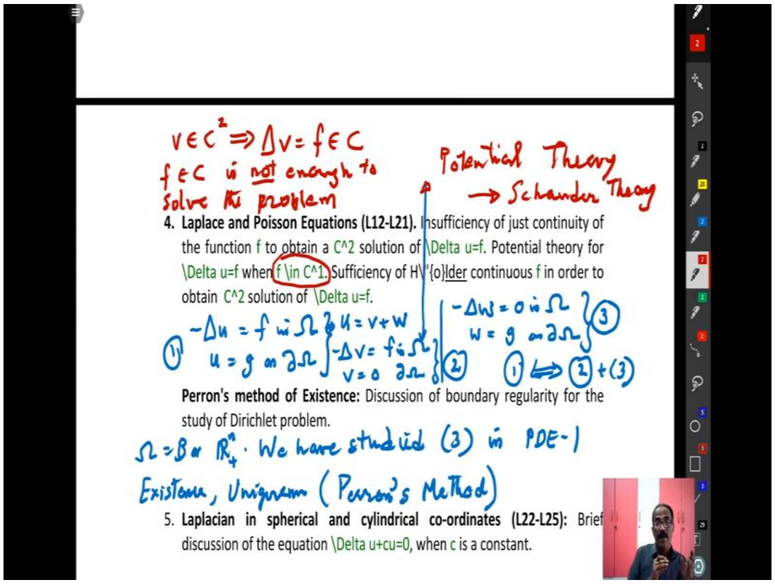

And then we will go on to study the 3 equations again not general. So, let me briefly tell you what we have done in the Laplace equation our aim is to study this is that thing, our aim is to study Laplacian u equal to f in that domain omega and then you want to study u = g on d omega.

And in our first course you have studied that you can represent the solution this you can be return as u = v + w.

And then you can solve these 2 equations minus Laplacian v = f and then v = 0. So, you split it into 2 things, so you have a non homogeneous equation with a homogeneous boundary condition and then you will have the other w will satisfy minus w = 0 in omega. So, this is a homogeneous equation with non homogeneous data this is in omega. So, you can write so instead of studying this equation the equation 1 you can study this equation 2 and you can study these equations 3.

So, one is studying is equivalent to 2 and 3 you can study radicals. So, what we have seen in this first equation and glimpses of this equation 3 and that also in a very special domain when omega is a quasi 1 we have seen when omega is equal a bore over upper half plane we have introduced the Greens function and we have solved omega is equal to one so when omega equal to bore over upper half plane we have studied in 3 in course 1 in PDE 1 course.

So, what do we will do the existence uniqueness not in the full thing discussion, so existence uniqueness etcetera in a more general setup, so outset that and that is why are the Perron’s method. So, we will spend some time and again we will not do it in very detailed way but we will try to introduce what is called the Perron’s method. So, Perron’s method is to the study of the equation 3 with in a non homogeneous equation with in a non homogeneous data on the boundary.

So, you have on the boundary you have a non homogeneous data this you cannot do it you cannot you cannot impose a boundary condition for arbitrary domains. So, there is a notion of called boundary regularity, so we will introduce the concept of boundary regularity and we study the existence and uniqueness for the homogeneous equation and think that. But then we have another part. So, let me discuss this now as you do not want to take another page here.

So, you want to study this equation and this equation we call what is study the potential theory. So, you have to study this, what are called the potential theory. And this is the basis for leading to general second order equation what is called as Schauder theory we will not get into the

Schauder theory but even understanding the potential theory basically minus Laplacian u = f here we want to study. The problem is that when you want to study so normally we look for a solution v in c 2 space.

So, v in c 2 for example you study v in c 2, this typically implies minus Laplacian v = f in a continuous function. So, I it is very natural to look for solutions when f is continuous. But unfortunately just f contiguity is not enough f continuity in c is not enough. You will see this important thing you are not enough to solve the problem. You see so that is something a bit of a surprise.

Because we have v = c 2 Laplacian v = f, so you expect to solve when f is a continuous function but so you need a little extra regularity. And so you of course if f is a c 1 function that is what become mentioned here then f is c 1 function, you will be able to solve it easily. But then that is not enough f is c 1 sorry f is c 1 is good enough but that is a too much regularity. You want to solve it in f is c.

So, you need some sort of an extra not as much as c 1 you will easily be able to solve a but not if continuities not enough. So, this introduced a new concept called a Holder continuous functions. So, we motivate you why Holder continuities required. This is exactly if you look at the first course on our PDE where you introduce to the fundamental solution and in the fundamental solution and its derivatives are locally integrable.

But when you take the second derivative and that is more important because your f is a fundamental solution you are interested in taking Laplacian n of v. And then the Laplacian n of v is the second inverse second derivative and the second derivatives are not locally integrable this causes trouble. So, this extra regularity on the functions is precisely to take care of the not integrability part of your second derivative of fundamental survival. These things will be discussed in the one part of Laplacian Poisson equation.

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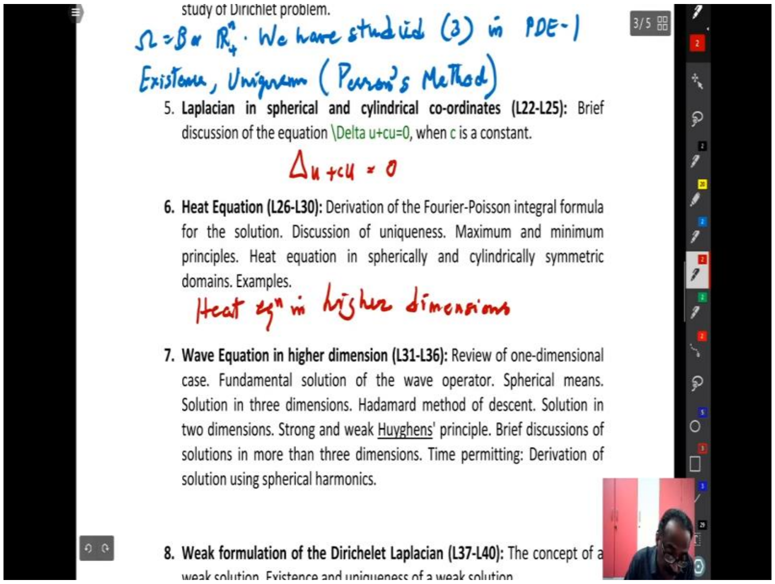

And then we will also give a set, this is equation is nothing but Laplacian of u + cu. So, we also discuss cu for a constant. So, we will discuss these equations and we will discuss especially Laplace equation in spherical and cylindrical coordinate. You know as you said I do want to understand your operate in spherical coordinates cylindrical what we are so far doing in Cartesian coordinates.

So, we will have some set of lectures maybe 4 to 5 lectures or 6 lectures on the Laplacian in spherical. And then we will go on to do the continuation of our heat equation. Basically heat equation in higher dimensions this is somewhat you will do it in a similar fashion as we had done it for the first order equation of the first 1 dimensional equation. So, we will also discuss about the uniqueness maximum and minimum principle. And then you are already set the stage for this spherical coordinates cylindrical coordinate for the Laplacian and we will continue to do understand heat equation in spherically and cylindrically equation.

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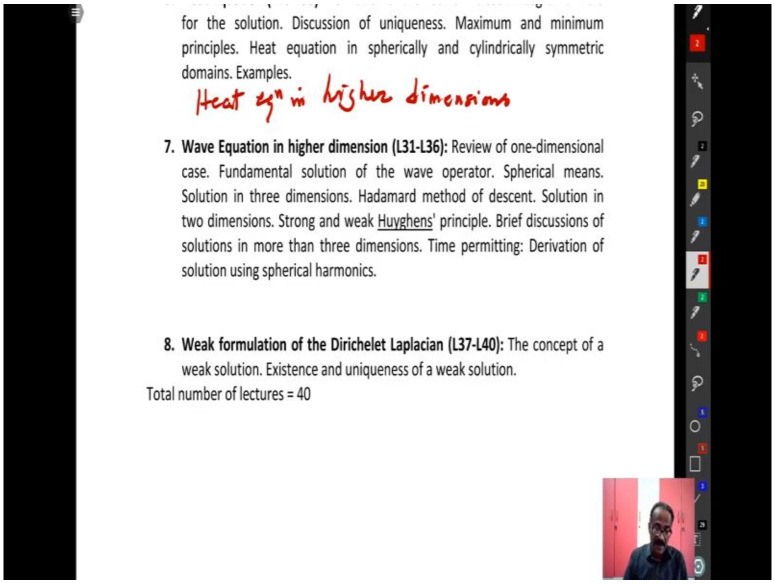

Then we will move on to another important equation what is called the wave question in higher dimensions, there you will see the difference 1 dimension we have studied with the D'Alembert's formula and propagation of the singularities and various other properties but when you go to higher order n greater than or equal to 2 you will see the odd dimension and even dimensions play different ways.

So, what you will basically you will be discussing after the review of 1 dimensional case what you add of course you will see the fundamental solution of the wave equation. So, you will see the spherical means using the spherical means you introduce the concept of solution or you derive the solutions in 3 dimensional space and after doing that one you will what descent that to what is called the using the Hadamard method of descent to study the solution in 2 dimensions.

So, basically you understand the solution in 2 dimensional in the assess equation in 3 dimension write the solution obtained in 3 dimension and using projections you bring it to 2 dimension and you will see one important concept what is called strong and weak Huygens principle. So, the Huygens principle is very very important how the solution is actually depends on the bounty you will see.

And that is not true in thing, so you will see the special Huygens principle in 3 dimensions. The interesting aspect is that this is true for every odd dimension. So, you will be able to write down the formula and solutions for all odd dimensions except the one we have already studied. And that is different but 3 and higher odd dimensions behave differently. So, you can write down the solutions for all odd dimension and using the method of Hadamard's method you will come to the or lower order things.

And as I said you will see we have enough time, you will also see the derivation of solution in spherical can mean. So, it is typically studying wave equation in higher dimensions. And you will do all that equation.

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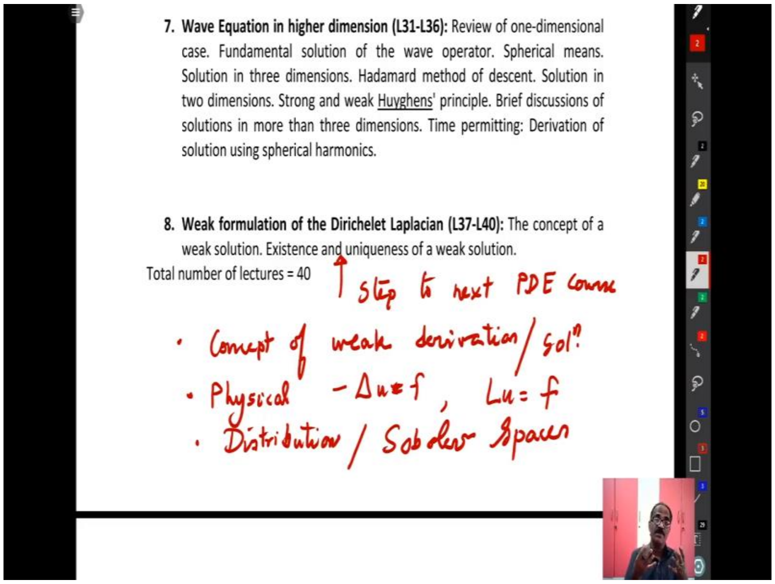

Than this is the section 8 what you have seen is a weak formulation and that is the stepping stone of PDE to next PDE course. So, this is basically step 2 next PDE course. So, we will not be able to do much here but what we want to introduce in this through this course what we call it concept of weak deriving, how do you differentiate functions which are not possible to derive classical derivation suppose you have an alto function how do you really do weak derivatives and weak solutions.

And this is more physical that is what you have to understand. So, we will physical we will give a glimpse of these things. How do you solve these types of equations? Or more generally how do you solve second order or higher order equations and you need the new notions like distribution we will not do this distribution but we will and then Soblers spaces. So, these are all so we will give a brief summary somewhat a brief summary of this notion in maximum 3 to 4 thing a couple of hours of lectures we know it is only very, very minimal.

But we want to introduce that topic in this section. So, you will come to note that there is more to be studied there are more notions are require more functional analysis is required to understand what PDE is and you will also see this is the way you eventually look into differential equations and basically it becomes a hardcore analysis course. So that is a brief summary of this course.

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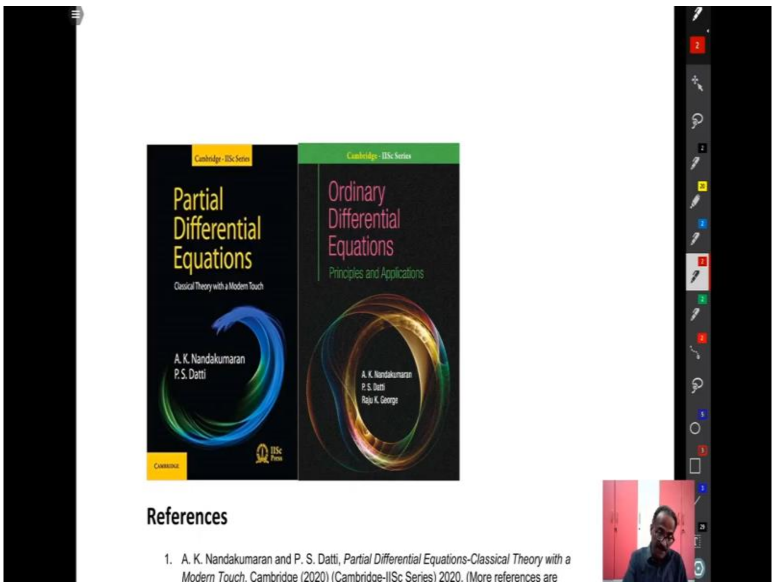

With that let me before stopping, so if we follow the same book on our recent book published in 2020. So, in PDE 1 you have seen some sections of PDE our book PDE book which is published by Cambridge IISc press series. So, we will be covering some topics from that book. So, with this course this book we will cover maybe 75 to 80% of the material of the book will be covered. So, whatever advices or suggestions to all of you please go through that article and then this is an other book we have written and there is a NPTEL course also on ordinary differential equations you can get into that.

Some of the material we have already used from ordinary differential equations in learning this thing learning especially the method of characteristics and how do you use PDE converted into your system of ODE.

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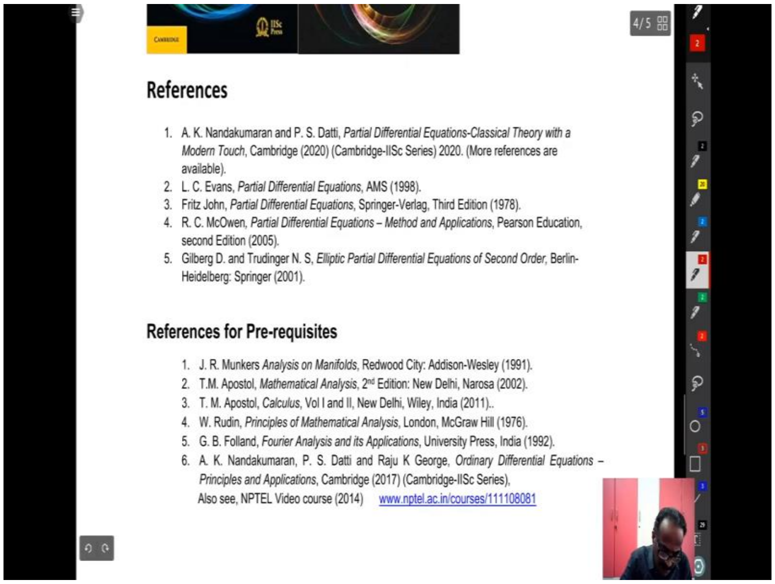

There are many other some reference that there are plenty of references in PDE book you can look into Evans, Gilberg I have mentioned it here Gilberg and Trudinger if you just want to understand a lot about the potential theory and Schauder theory and then many other things says just discussing about the elliptic equation. So, you can look into Gilberg and Trudinger to do that one and other books which you have L.C Evans and there many references given in our book.

There are many other prerequisites is exactly what we have mentioned here. If this is not enough you can look into the Datti. So, this is the brief summary of our course, so as you see if you really look into that one so we are given an overview of our PDE review of before that so we are again repeating please go through these topics of PDE 1, unless you go through this topic and understand the notions defined in the Greens function and other things.

You will not be able to understand this present course. So, we strongly recommend you to go through this course and then we have seen what are the topics; which we are going to discuss in this course. So, I think I will stop here. So, we will start our course with the Hamilton-Jacobi equations from the next lecture onwards. Thank you.