First Course on Partial Differential Equations - I

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Lecture 9

First Order Equations in Two Variables-3

Good morning, and welcome back again. We were discussing about the Burger's equation to understand the difficulties where it is an example looks similar to our linear equation where the linear equation c is constant is replaced by unknown self that immediately see the difficulty that characteristic curves are, a priori cannot be defined because characteristic curves depends on your unknown.

That is the nature of the quasi-linear equations but what we have done even though we do not know the characteristic, assuming the solution exists, assuming that characteristic curves exists, along the characteristic curves your u will be constant. So that is a extra information derived from the equation. If the characteristic exists, characteristic will be from that I want to derive one more beautiful thing which you would have already thought about it, u is constant along the characteristic but look at the characteristic curves now.

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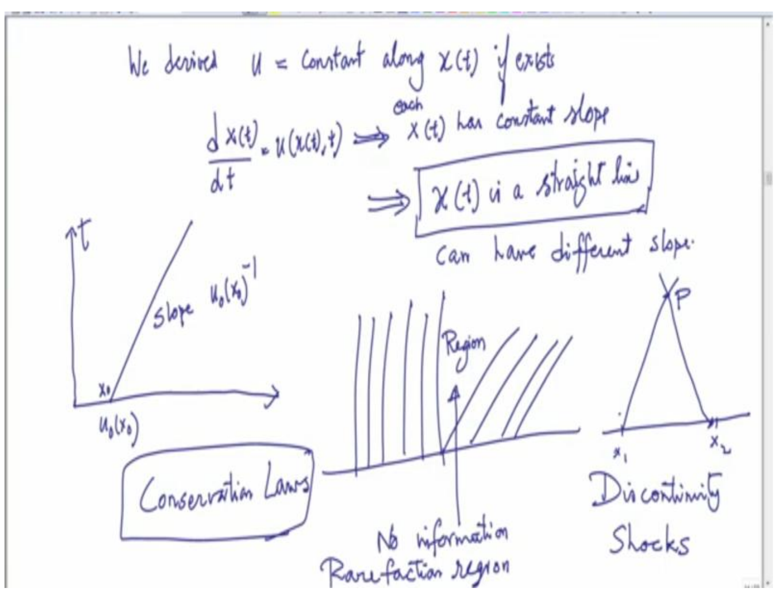

So, what you see is that u is constant. We derived, we derived u is constant along xt if exits. That is a kind of because if exists but what is xt, xt has dx by dt is equal to u itself. That means your curve is constant slope that implies xt has constant slope. Each xt, if xt changes, each xt has constant slope.

What do you mean by a curve with constant slope? That curve is a straight line. You see, so this is an additional information, xt is a straight line. That is a beautiful deduction without knowing if the characteristic curve exists, that characteristic curves are the Burger's equation is a straight line. So what is this different from the linear equation?

Linear equation, you got all the characteristic have the same slope c but then here the, it is a straight line but different characteristic curves, different straight lines can have different slopes leading to difficulties, can have different slopes, can have, that is a major issue, different slopes. So look at that one. So let me draw some figures. That is what you want geometrically to understand.

I said always you understand characteristic curves along the direction t. So what it says is that of you have an initial value here. So if you have an initial value here and if you have a characteristic here, that characteristic slope of the characteristic will be by looking at that if you see the slope is dx by dt is equal to u is constant and u is constant along this curve.

So, the value of u is nothing but so everything sorry, so you have and you have your curve here. So you have your sorry, and you have your here, you have your value of u. This is the point x naught, the value of u at x naught is u naught of x naught, and the slope is nothing but you have to understand t in this direction. Okay, you have to understand t in this direction. So the slope is because u is constant along this one so the value of u along that direction is u naught of x naught, that means the slope is u naught of x naught minus 1. You are viewing this way, you are viewing in the t direction. The slope is u naught of x naught inverse.

The problem is that different points so each point if I take a point x1 here so you can have different slopes, so you can have a situation like this. You can have slopes like this which I already indicated, from here you can have slopes like that and there is a region here where there are no slopes.

Such regions are called rare fraction regions. Region without any information the characteristic curve, the ones which is transporting your information from the initial values. This initial value here transported the values here u naught of x naught is transported along the characteristic. So if there are no characteristic there is no information. No information. This is called rare fraction region, rare fraction region and you want, so this where you need to understand PDE in a more advanced way.

So, the classical theory, in such cases feels, you need to go to non-classical theory and non-smooth solutions. So you have to see even this simple equation obtaining classical solution sometimes if you have your characteristic in the second figure, you will have an issue. There is a more dangerous thing.

Suppose you have 2 points here: x1 and x2 and one characteristic may come this way and another characteristic may come this way and then you have a point. This x1 point will carry along one characteristic, the first characteristic it will carry one value from x2, it will carry another value giving a discontinuity at p. That is a more dangerous situation but these are all will be physical. It is something like a wave phenomenon, waves will faster, waves will come from behind and hit the smaller waves leading to sometimes discontinuities and in the literature, these are called shocks.

These are called shocks discontinuity. So that is a enormous literature in this kind of non-classical study and this comes under the class of conservation loss. This is a very special type of equation and there are plenty of research going on in this area of conservation loss. So you see the kind of difficulty even with simple quasi-linear equations. So we are not trying to get into all that and that is not easy to get into all these picture right now. So let me try to do something more here before going that. So the problem here is that you may not have a characteristic here.

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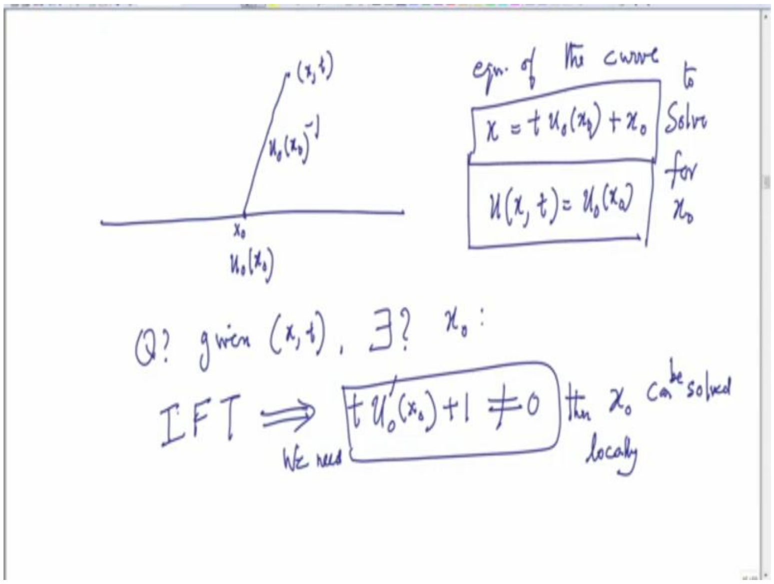

So, the inverse problem. So what do you want to do it basically. So let me do something more in the same example. So you start with the point xt here, you want to find a characteristic here, so I am trying to derive some condition whether it is possible or not. In that previous example, shows that there are situations for which there will not be a characteristic curve.

So, if it reaches this point x naught, the value of that point is initial value of u naught of x naught and you have your u naught of x naught inverse. This is what its slope. So let me write down the equation of that. So what is the equation of this curve? Equation of the curve, equation of the curve is x equal to t u naught of x naught because u naught of x naught is the inverse. So it comes here and it passes through x naught.

So, this is the equation of the curve and along the curve, we already proved that it is a constant and it takes the value u naught of x naught. So these are the 2 implicit form. So the question is that given xt, the same inverse question, given xt does there exists x naught. You want to solve these equations, want to solve, to solve for, to solve for x naught for given xt.

So you have to appeal to implicit function theorem, then your implicit function theorem will give, if you want to solve for x naught in terms of x and t you have to take the differentiation with respect to x naught variable. So that implies t, we need, this implies we need t u naught of x naught, its prime plus 1 should be not equal to 0.

So, this is the derivative of t u naught of x naught plus x naught. So if you have the differentiation, you want to derive this is naught 0. That implies then x naught can be solved uniquely, can be solved, can solved, can be solved locally in a unique fashion and then once you solve it, so the given xt you have an x naught which passed the end characteristic.

Once you get an x naught since the characteristic is a straight line, you connect xt and x and that should be your characteristic and along that characteristic, your value of u xt is equal to u naught of x naught. So if this is true t u naught plus x naught plus 1 is not equal to 0 then you have a unique solution. So actually what we have done, we have essentially proved a theorem.

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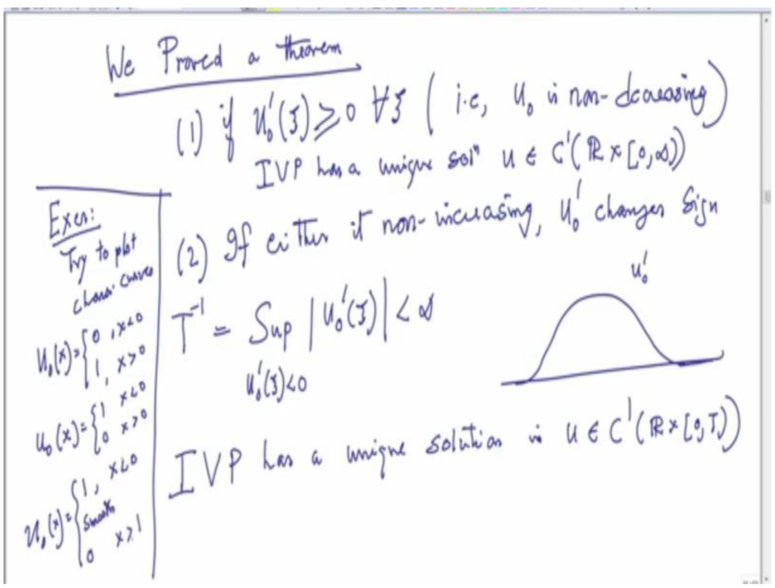

We proved a theorem basically. We proved a theorem. What is the theorem we proved? You need this condition. If t u naught prime plus x naught plus 1 is not equal to 0, this definitely will happen if u naught prime x naught is positive. That u naught prime is a non-decreasing function.

The case one if u naught prime of psi is greater than equal to 0 for all psi than implies that is u naught is non decreasing. That means that the inverse function theorem condition is satisfied and you can solve it and you can get the unique solution, local solution you can immediately get it. There is no issue. You can solve it along the characteristic but what happens in the other situation. If this is not true, if either it is non increasing this case.

So, what happens is that we will show you some examples immediately. If either increasing or u naught prime changes sign because you will have cases like that, your u naught prime will be like this. You will see it is u naught prime, it changes sign. So if your initial value is something like this so if you have your u naught is something like this, u will have, u naught is something, it will change sign, in that case, you look at this maximum or supermom of when it is negative that is a problem. So you look at its maximum. This is what essentially we proved. You can interpret it and read. I am not going to explain more as we have to, where u naught prime of psi less than equal to 0 and I call this to be t inverse 1 over t otherwise you take the other way.

Then the initial value problem for this case has a unique solution up to t, has a unique solution in u in c1 of r up to t you will get it. After that it will have some 0 to t, by that time you start and here you will get the initial value problem as a solution as a unique solution. One simple case you proved u in c1 of r up to infinity you get it. You get everywhere. So that is the situation, so you can have many more example and maybe such exercises you try to exercise, try to plot the, we will have provide various examples so try to plot the characteristic curves.

Say, for example, my u naught of x is something like 0 1, this is x less than equal to 0, this is x greater than 0, you also plot the curves u naught of x is equal to 1 0 for x less than 0, x greater than 0 and if you have another u naught, I will put it you will have 1 if x less than 0 and 0 if x greater than 1. Here it is smooth.

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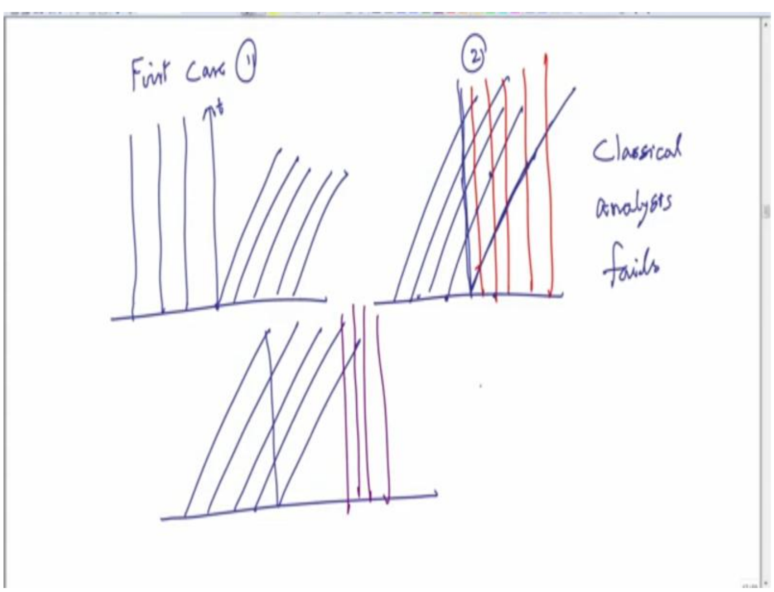

So, the first case, so maybe we will go to the thing, in the first case, your characteristics are, you look at the first case, case 1, 1, 2, 3. In the first case 1, you will have something like this characteristic because it is 1. After that, let me look at it, yeah 0 and 1. So it is 1 here. After that 1, this is 0. I told you have to view it in t variable case.

In the second situation case 2, you will have the other way. The characteristic will be like this for x less than 0, for x greater than 0 the characteristics are like this, you see. So that means the characteristics are interchanging, it takes the values, it brings, it takes the value 1 from here, it takes the value 0 from here and wherever intersecting, there is an issue.

So, there is an issue along this region. So you will see an issue along this region here. This left region and right region, there is no problem. The region problem will be here. So how do you, so your classical analysis, classical analysis, so we are not going to treat your fails and creating shocks, but these are all practical situation. In the third case, so you work out this example. In the third case up to 1, what is the third case?

Third case, it is 1 here. So you will have after that it is 0 so from here, sorry that is not it is 1 here, so again a situation, so you have here, so from here you have something, from 1 you have something like that and here it is slowly going. The other way it will go. So you will, it will have a, it is a 1 to 0 it is a decreasing function. Accordingly, it will move like this. So you have again difficulties of solving this problem. This is the case 3. So you see, even for a simple quasi-linear equation, you have some trouble.

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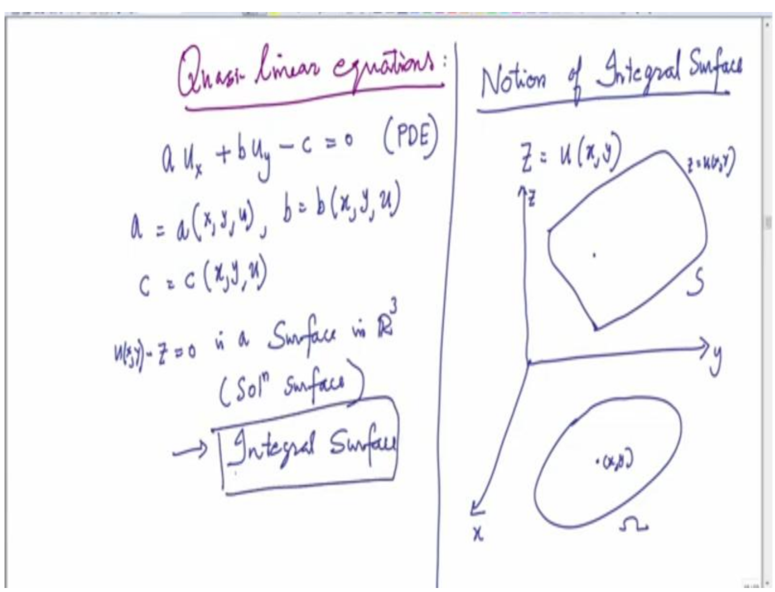

So, let me go to now general study, what we can do in this situation. So quasi-linear equation. So only we have so far we discussed only the example. Quasi-linear equations. So let me recall what is my quasi-linear equations. You have your quasi-linear equation a ux plus b uy equal to or minus c where a, b, c are functions of, a, b, c are functions of x, y, u. It depends on u, a of x yu, a is, a is equal to a of x yu, b is equal to b of x yu and c is equal to c of x, y, u. So I want to understand, I want to introduce a notion of integral surface. So that is a geometry you have already seen from this example that you cannot have characteristic curves as plane curves.

So, you will have to look into not only the characteristic x and y and u because when you are having equations dx by dt and dy by dt, you need to have an equation for u also as an ODE there. So you want to solve not only the characteristic curve together with unknown value. In the earlier situation, what we have seen is that we have the plane characteristic curves, on that plane characteristic curve, there was another ODE satisfying on that curve given by the unknown and this were on a separated way but now what we are going to have is that the characteristic curves together with the unknown combine to form a system and hence, your characteristic curves are going to be the space curves.

So, you have to work instead of the plane, we have to work in the 3 plane. That is what the difference in geometry. When you go to non-linear situation, you will face much more difficult situation. The geometry is much more. So it will not be that easy. So we need to understand that in a better way. So what we want to do is that notion of integral surface. Integral surface. I will first give this notion and then I will do it more later. So and then I will give you a physical interpretation of your PDE. This is your PDE. So look at this z equal to, so I will introduce a notation u x, y. So this is the unknown. So you have an x variable here.

So, now we are doing a picture always x variable here, y variable here and then you have the u variable here. This is your z variable, z is u xy. So your domain is in x y variable. For every xy, you associate a u xy here and then this plane domain omega, will become a surface. So this is my z equal u xy. So in the previous class we have seen the surface introduction.

So this is the surface sitting in that one. So z equal to u xy, or u xy minus z equal to 0, both anyway you like it, you can take it, z equal to 0 is a surface in r3. Basically, this is the solution surface. So when you are trying to solve, so problem of PDE, this quasi-linear PDE, you are trying to find the integral surfaces. This surface is called the integral surface or solution surface.

Now, try to understand my PDE with these concepts. So you have a concept of integrals surface. Integral surface is nothing but z xy, z is equal to u xy, so you have the function PDE is defined in a 2-dimensional domain omega. On each point, you associate your u xy, so you get a surface and that surface is called, so there will be different integral surfaces for a given quasi-linear PDE but the initial value problem is that, you are giving some initial values. Then the initial value problem will become a finding a particular integral surface.

So, when your initial conditions are not given, there will be possibly integral surfaces. So initial value problem is to trying to find a integral surface satisfying some initial conditions and that you looking for whether there is a unique integral surface, etc. We will be setting a theorem and we will eventually prove it. So let us try to understand my PDE now.

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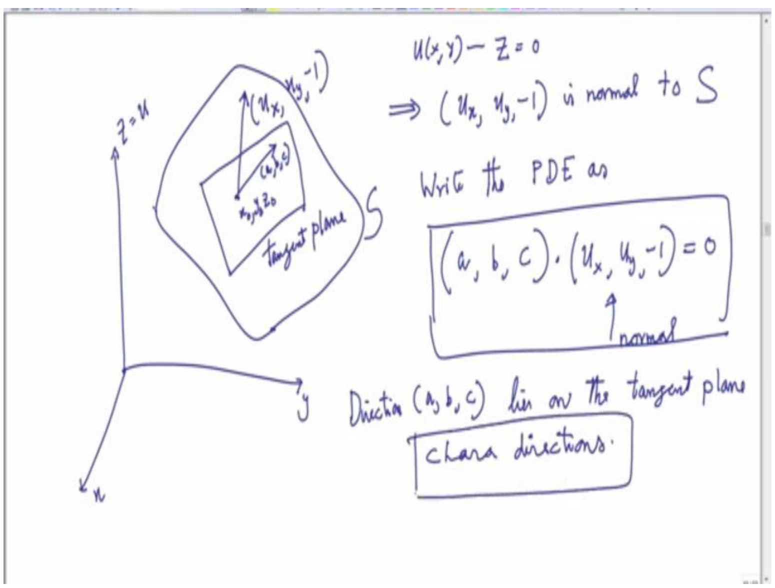

So, you have a, so have this picture. Picture will help you all the time, x, y, u, z or u, whatever it is and you have an integral surface. You may have an integral surface. So what is this equation tells you, what is, so z, so u xy minus z equal to 0. That implies your ux, this we will use throughout in our first order equation minus 1 is normal at the point, is normal to, let me call it the integral surface is s, normal to s. So if I fix a point x naught, y naught, z naught on the surface and this gives you your normal. That is ux, uy, minus 1. So, you see. Now what is now my PDE.

I am going to my write my PDE, write my PDE as, write my PDE, write the PDE, it is also your PDE, write the PDE as a, b, c. I am not writing the arguments but keep it in mind all the times that a, b, c are not constant, a, b, c depends on xy and u dot product my ux uy minus 1. This is equal to 0. So ux, uy this is normal. So whenever there is a normal, we have already studied. It is a surface and then there will be a plane which we call it a tangent plane.

So there will be a tangent plane. So this condition tells us that a, b, c, the direction a, b, c which we call it characteristic direction, eventually direction a, b, c. Whenever you fix a point xy and z, a xy z, b xyz and c xyz at this point or x naught, y naught, z naught provides a direction and this PDE, PDE tells you that a, b, c lies on the tangent plane, lies on the tangent plane.

That means a, b, c lies here, a, b, c. That is the picture. So the PDE just tell that the surface where a, b, c lies on the tangent plane. So this gives a very interesting thing now and this is called the characteristic direction. Characteristic direction. So this is called a, b, c characteristic curve.

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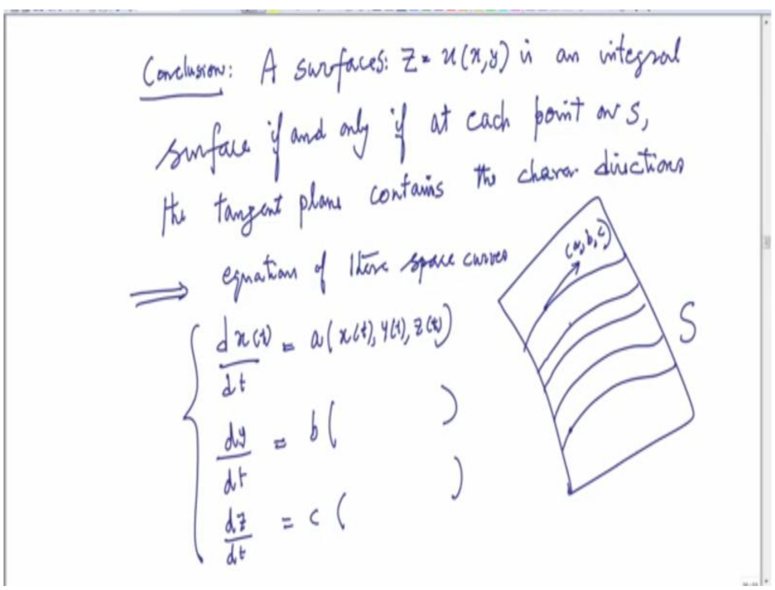

So, what we are going to see so this immediately, a surface is an integral surface. So you have a conclusion now which I am going to, conclusion, a surface xyz, a surface z equal to u xy is an integral surface. So you are characterizing the PDE integral surface if and only if at each point, at each point, on s a surface s which is given by this each point on s the tangent plane, the tangent plane.

I am re-explaining my PDE, contains the characteristic direction. Contains the characteristic direction, characteristic directions. So what I am going to do with this? So this is what motivates us to do something. So you have a surface here s. This is an integral surface which we want to find out. For each point, it will be a characteristic, you have your a, b, c lies on the characteristic direction.

So, what I will be doing is that I want to decompose my surface just like I have done in a linear case my plane is divided into characteristic curves. I am going to divide my s into a characteristic curve. I want to divide my characteristic surface into curves in such a way that on each point my a, b, c is tangential. That means you are looking for a curve on the surface, not on the plane now. Looking for a curve on that the tangent is a, b, c. Once the tangent to the curve is a, b, c that a, b, c will lie on the tangent plane.

So, each curve you have an a, b, c as my tangent to the curve since everything is such that curves are called the my characteristic curves and I try to do it my entire s as union of all the characteristic curves. I am trying to see my s is made up of such curves and that curves has a property that each point on that a, b, c is tangent to the curve. So it will lie on the tangent plane of s making that s is a integral surface. So I want to write down the equations of the curve. Now it is easy. So the equation are then because I want to write the equations of the curve. It is the equations of these space curves, these space curves are given by d xt by dt.

You have to write it in dy by dt and then I will stop here and I will continue this one because so this will be a of xt, yt, zt, z is u basically, but I am not bothered now about that. Xt, yt, zt and this is also equal to c of xt yt curve and this is the curve which I will, so this will be a curve so if I fix a point on the surface, I will get a curve and what I am going to do soon maybe in the next class is that it is, it is a for given a characteristic curve, I can decompose into the, given a characteristic surface s I can decompose into the characteristic, conversely I can make a surface using my characteristics curve.

That surface become a characteristic surface or the integral surface not characteristic is called an integral surface. So it is both ways you can do it. You can either give it a surface if it is can decompose as characteristic curves, it will be an integral surface or if you have a surface is formed using the characteristic curves that surface will become integral surface. I will do this in the next class. Thank you.