First Course on Partial Differential Equations - I
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Lecture 8
First Order Equations in Two Variables - 2

Okay welcome back to the First Order Equations and Method of Characteristics. Let me quickly recall what we have done. The initial value problem. We were studying about the linear equations in 2 variables and the equation is a ux plus bu y is equal to cu plus d which is defined in a 2-dimensional domain omega. On that domain, you have an initial curve gamma naught and the values of u is given on that curve gamma naught.

So, the question is to, why do you in omega or if that is not possible, try to find u in a neighbourhood of gamma naught. So what we have introduced in the previous class that you can (())(1:05) the characteristic curves probably filling the whole domain. It may or may not be because we have not proved anything result. Our wish is that you try to fill the whole domain by characteristic curves. On each of these characteristic curves you try to solve an ODE.

So, the message is that you are trying to solve the entire system by solving by solving an ODE. That is what we have seen it in a thing. Now the issue which I want you to tell, some serious issue, even in the first example, which we have done is that there is a process of inversion. So let us give you some hint about it. So you have, let me again write your domain. So you have a domain omega, and then you have an initial curve here.

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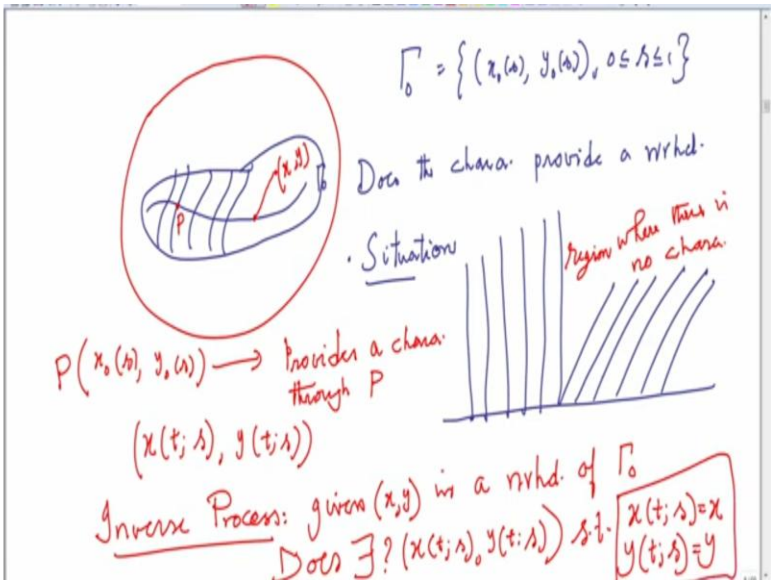

So, this is your initial curve gamma naught and each point is, gamma naught is parameterized by, so let me recall again, gamma naught is equal to x naught of s, y naught of s, 0 less than equal to s, less than equal to y. So on each of this point, it is a x naught. You can solve that 2 by 2 system of ODE so you can find a characteristic curve.

The existence of characteristic is given by the system of ODEs. So in each point you have characteristics. So the question is that, does this characteristic curve provide a domain, provide a neighbourhood of gamma naught? Does it provide, does the characteristic provide a neighbourhood? So look at the situation. So I wanted to show you a situation. May not be in this example, but let me tell you a situation.

So let us, so suppose you have a curve like this and then suppose you have a characteristic like this. This may not be for linear equation and from here, you have characteristic like this. So you see, so you have every point there is characteristic but there is a region, this is a region where no characteristics, where there is no characteristics.

So, how do you get back your values? That is the problem because your value is determined by transporting through your characteristics. So the issue is that, the question is that given a point xt, can you find a characteristic which meets here, can you find a neighbourhood in such a way that every point in that neighbourhood you should be able to find a characteristic passing through that point.

So what is what is, so how do you mathematically form it? So you fix a point x naught of s, y naught of s, x naught of s y naught of s. So that is a point here p. This provides a characteristic curve, provides a characteristic curve through x naught, through p. What is a characteristic curve?

Characteristic curve will be xt. Since it depends on s, I call the characteristic curve x t, s. So you obtain a characteristic curve. This is to indicate that the characteristic curve depend on this point p because if the point p changes along the thing, you will get a different characteristic curves. So each fixed t, you have 1 characteristic curve, but s changes you propagate different characteristics. So what am I trying to tell thing?

So, this is your inverse process. Inverse process. In the inverse process you are given xt, not xt sorry xy. So let me write it, x x okay, your xy xy or xt. Yeah so you are given an xy there. So what I want is that, I want a characteristic given xy in a neighbourhood of gamma, in a neighbourhood of gamma naught.

Thus, there exists, that is the question, x ts, y ts, you want to get a characteristic curve which should meet the initial curve. That means you need to have a characteristic curve passing through xy. It should meet the point x. That means such that x ts should be x and yt s should be y. This is what I am looking at it.

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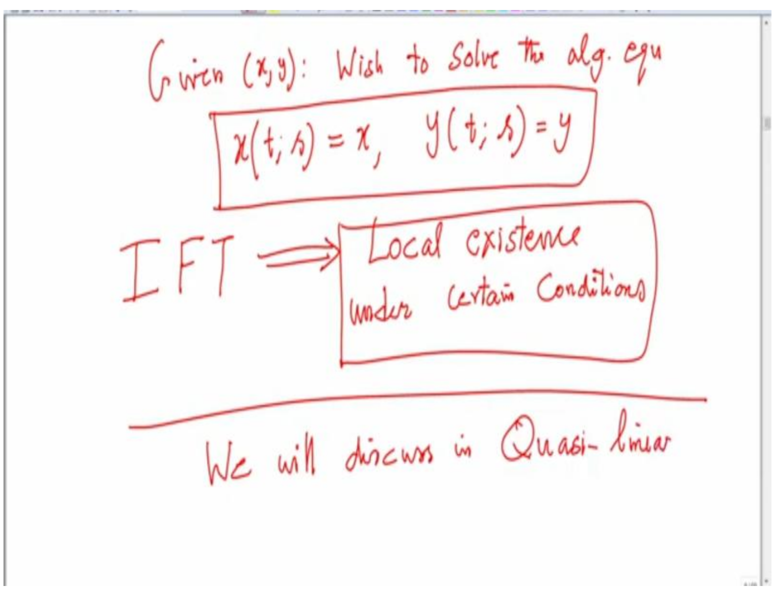

So, given xy, so that means given xy solve the, wish to solve the algebraic equations x ts is equal to x and y ts is equal to y. So this is a, this you know now. So you have 2 equations and 2 unknowns and you want to invert it, you use inverse function theorem. Inverse function theorem implies local existence under the condition. Some conditions are there. Local existence under conditions under certain conditions.

We will discuss these conditions in the context of quasi-linear equations but I, so we will come back to this. We will come back to that. So we will discuss more about it in the context of quasi-linear. We will discuss in quasi-equations. So this also will be covered. What is that condition? So because inverse function theorem, you need quasi-linear equations. So let me give you an example before coming to quasi-linear equations and I want to.

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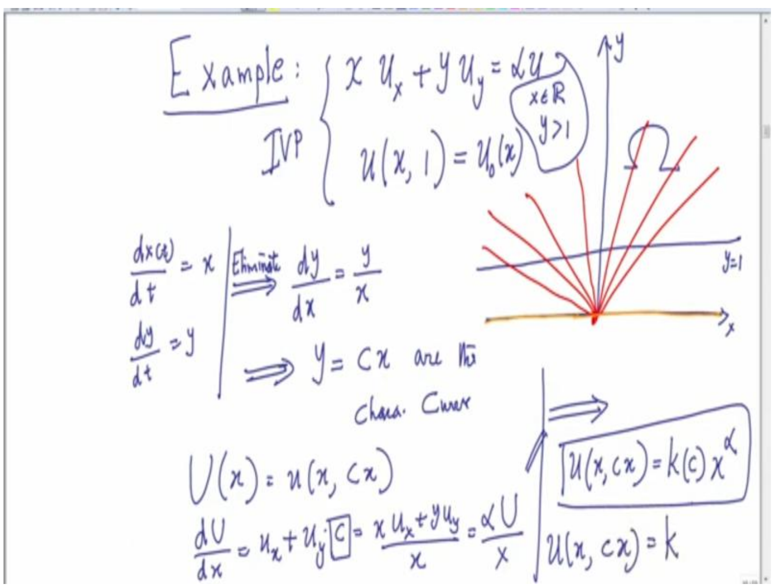

So, an example. Example. Developing theory is one aspect. Solving problems is a different skill. So you have to develop both. You have to understand the theory. Then you have to also develop the skill to solve problems. That is a part of, so both is easy. To understand the theory better, you need to solve problems, to solve problems you need theory and so it goes hand in hand.

So just by theory, learning theory, you may not develop the skill to solve problems. So both are different ball game altogether, but goes hand in hand. So you have to work out as many examples as possible and in the course, it is not possible to do so many problems. We give only some sample problems. So let us do 1 problem.

So x ux plus y uy equal to some alpha u. So this is the case I want to solve it. So what is the initial value problem? This is the my IVP. Please careful about it. I am solving the problem when in the axis u x1 is not u x0. You will see some trouble if there is a u naught of x is given to you. So basically you are solving this problem. This is your x variable, this is your y variable and you are solving this problem with initial values here and with xy axis in r, solving it with y greater than 1. So the equation is in this domain, in this upper half plane. This is the domain, in the upper half, this is y equal to 1.

So, you are solving in the upper part of it. So you are solving, this is your domain omega. So you are solving it there. So let us do it what we have done. So you have your dx t by dt, dx t by dt equation following the method is x here dy by dt is equal to y. So but then it is the very thing you can eliminate.

So characteristic curve, you can also write it as dy by dx equal to yx y by x. You see, so you can eliminate d. If you can do it so that is what the skill you develop. When you do not need it, do not use the parameter and you can solve this. This is an ODE and you can solve that y equal to cx are the characteristic curves. This is the constant of integration. Characteristic curves.

You see, you try to plot it here. So if you try to plot your characteristic curves, you can see this is y equal to cx. So you see, you have all your characteristic curves meet at the origin. So naturally you have trouble here. You see, you have trouble here, right, because if you give the initial values on the line y equal to 0, the characteristics are not meeting so you may not be, because that is due to the singularity appearing in the coefficients of the PDE. These kinds of troubles you can anticipate.

But on the other hand, all the characteristic curves intersect the line y equal to 1 and you can now solve it. So that is what. Now you so you restrict u, now also you define my u of x equal instead of t, so you write your u of x instead of t. T parameter, I removed it, u of x is equal to small u of x cx.

If you do this one, your du by dx is equal to ux plus uy into c but c is equal to y by x, y equal to cx, c is this constant of integration. So y equal to y by x, so you will get x ux plus y uy by you have y by x. There will be an x. So the u will have alpha u, u is equal to capital u. It is on the line u. Not alpha, yeah alpha by you get alpha u alpha by x. So you can solve this equation. Let me now do the computation and you can actually see that your u of x. So you solve this equation because this is a du by d equal to alpha u. So you can solve along a particular characteristic.

So, you are fixing a characteristic and solving that ODE along the characteristic with a constant of integration. This constant will depend on the constant c. This constant of integration will depend on this c this constant. So the c constant is fixes the characteristic and then you have an ODE on the characteristic and that gives you another in the constant of integration. That constant depends on which characteristic curves you are integrating. So, this is kc into x power alpha. You can integrate. That means you have c of x, cx is equal to k of c is y by x. So you have y by x into x power alpha. Of course, that is a thing.

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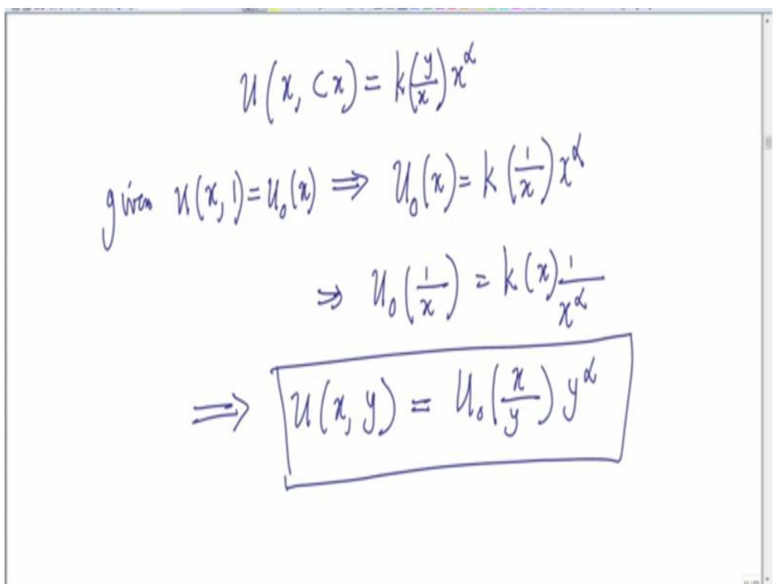

So, let us do the, we have not apply the initial values. So, let me write once again, u of x cx is equal to k of, k is a constant of integration y by x into x power alpha. Now given that, given u x, 1 is equal to given u at x1 is equal to u naught of x. So that implies is equal to 1, so you take cx is equal to 1.

So that means y is equal to, you get u naught of x, so you have your cx is equal to 1. So cx is nothing but y, y is equal to 1. So, you will have that is equal to k of 1 over x. This is on the initial condition. Initial condition is when y equal to 1. When y equal to 1, you have your initial condition into x power alpha.

So, do some little more process so that implies u naught of, u naught of 1 over x is equal to k of x. You are just inverting into 1 by x power alpha. So you substitute here k of x, you get it k of x is and substitute here. Please do this little bit of work. You can finally get it u of x, y, do this analysis a bit, u naught of x by y into y power alpha. So you see you can solve it.

So, what is, you could solve the problem but your initial curve has to be proper. So if this problem if you put your initial curve as the x-axis, you will not be able to solve it in this way because your characteristics are not meeting the entire side of the thing. With that we will go to what is called the quasi-linear equation.

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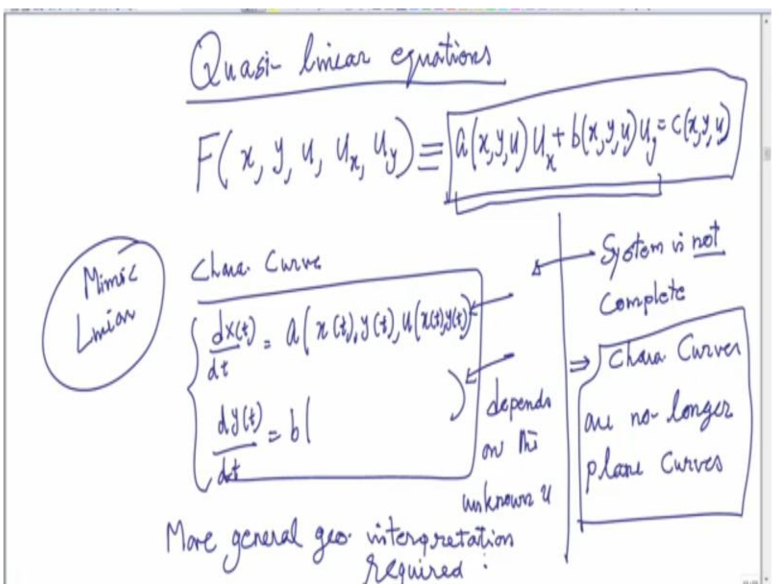

So, now let us go to the quasi-linear equations. Quasi. So we are only making little changes, but then you will see immediately lot of troubles in quasi-linear equations. It is the life is not that easy like linear equations. Even linear equations, you need conditions which we did not state it, which will come in this, but quasi-linear equations possess lot many problems because there is a non-linearity in the highest order derivative. So we are trying to understand the similar example in the quasi-linear set. So let me write the quasi-linear equation. So what is your quasi-linear equation? F of x, y, u, ux, uy equal to, now it is linear only in the last variable.

So, your coefficient will be, co will be depend on u but in the ux variable it is ux. So b of x, y, u, uy, this is equal to c of x, y, u. So this is your PDE. So you have here PDE. It looks like that. So what do we do it, we try to, the first difficulty what you will see that you try to mimic the linear situation.

When you mimic the linear situation, you want to introduce the characteristic curve. So mimic linear case. So what is the characteristic curves in this case? You have the characteristic curves dx t by dt so in such a way that the differential operator is similar. You see, the major issue is that in this case a ux plus bu y is the still the derivative of u along that direction.

The problem is that a and b depends on your unknown. So even the characteristic direction is unknown. In other words, the characteristic direction ab depends on your unknown. In the previous case, the characteristic variables or characteristic direction were independent of the unknown function u so you could determine. So we are not able to determine the characteristic curves right now.

So if you look at it here, it will be a of x t, y t and y t and u of x t y t. So you see, you are restricting your, you are looking for, this is what I am mimicking because I put the coefficient. This is what I wanted, dy by dt is equal to b of similar function. So this is why characteristic and this depends on u. Depends on the unknown u. That is where the problem.

Therefore, this system is not complete because there is an unknown sitting there because only if you have xt and yt are unknown, which is a characteristic curve, so as far as this ODE is concerned xt and yt are the unknowns, but do you have that unknown depends on the unknown function which we are supposed to determine for the PDE.

So, system so you are not complete if there is third unknown coming when you have 2 equations. So that means that implies characteristic curves, characteristic curves are no longer plane curves. So we have just changed a bit. No longer plane curves. That is where the issue. So you need a more general geometric interpretation. More general geometric interpretation required.

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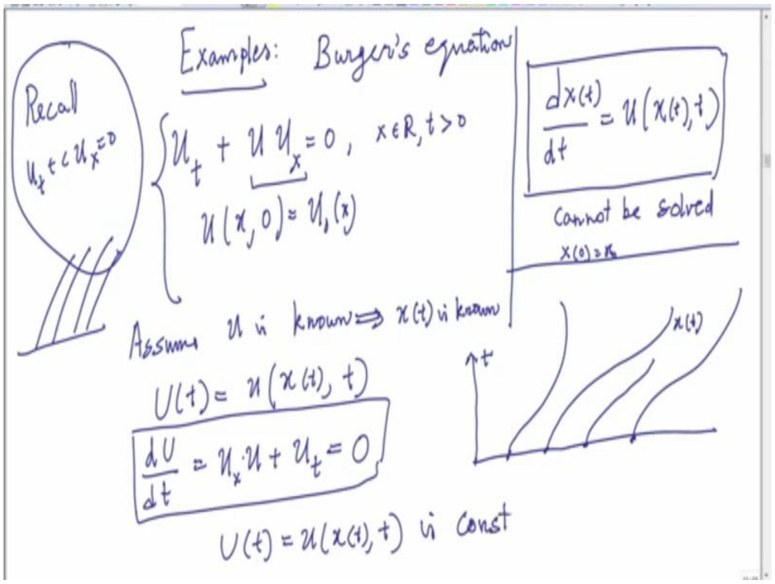

So, let me do a start with an example and then we will go on to do a little more about it and so let us start with an example, similar example. I will not complete it now. Maybe I will complete it in the next class. So let us start with this is called the famous Burger's equation. Burger's equation. I just want to demonstrate just by changing a constant to the, so this is your quasi-linear equation u t plus u ux equal to 0 and u at x 0 is equal to u naught of x, and this is given for x is in r and t positive.

This is exactly the kind of equivalent to recall, if you want ut plus c ux equal to 0. This is your linear case. So you have a linear equation on that corresponding linearity where the constant c is replaced by, this is quasi-linear because it is no longer linear, but it is linear with respect to the x, ux and ut variable and we want to solve this quasi-linear equation.

So, I just mention one thing and then I will continue exactly from here because here I want to tell you something more. So in this hour we will not be able to but let us try to introduce exactly what we have done in the linear case. We have introduced the d xt by dt is equal to u of, instead of c, you have u here.

So, I am trying to derive some property even though I do not know the characteristic curve, cannot be solved. Cannot be solved. So this is a general procedure, we follow always in PDE and you do not know how to solve it xt, so the general procedure the assuming that u is known, you try to do everything.

You just imagine that u is known. Once you know u, I can determine my xt because dx t by dt is equal to u. So question of u is not known, so when you have such a situation, try to assume that u is known, try to proceed as it is and see that whether we can derive any necessary conditions. Is it possible to derive some conditions, some qualitative analysis or whatever way you call it. So I am going to do that one, cannot be solved.

So, assume u is known. Once u is known, this imply xt is known. So you are pretending that u is known and you are pretending that you know, so you can solve with an initial condition with x at t 0 equal to x naught, so if I have an x naught in xy, x at 0 is this thing and then I know that I have a curve here.

So, you have a curve xt, xt. So you have a curve here xt. So knowing just like 1 dimension, in the linear case. Linear case you have solved with c and you obtain the characteristic curves like that. So in this case, you may get a curve like this. You do not know so I am assuming because but assuming that u is known and you may get various curves, various characteristic passing through that equation. So each point I am trying to solve it. This is what I am trying to do it and this is the t variable. So with that, so exactly follow it, so I introduce ut. You introduce ut is equal to u of xt t and then you can compute your du by dt. I am blindly following.

We can compute u into x into dx by dt, dx by dt is my u, so you get u plus u t equal to but then that is a PDE and you have this 0. So you have still a similar or if you know the characteristic, whatever be the characteristic curve, I do not know the characteristic curve because I do not know u but if you know u, you have a characteristic curve along the characteristic curve, my u is constant.

That imply my ut is equal to u of x t t is constant. Do you see something interesting happening here? Think about it. What will happen to that? What is this giving an information? Does it give any special information about the characteristic curve which we will see in the next class. Thank you.