First Course on Partial Differential Equations - I

Professor A.K. Nandakumaran
Department of Mathematics
Indian Institute of Science, Bengaluru
&
Professor P.S. Datti
Former Faculty, TIFR-CAM, Bengaluru

Lecture 7

First Order Equations in Two Variables-1

So, now we start the main course on PDE. So next 4 hours, that means 8 lectures, 8 half an hour lecture, we will be discussing about the first order equations and the main idea of first order equation is to introduce the concept of method of characteristics and give the idea of characteristic what the method of characteristic and the characteristic plays a crucial role in understanding the geometry and other issues related to PDE. So we will have a detailed discussion about the method of characteristics restricted to first order equation. So we will be doing first order equations.

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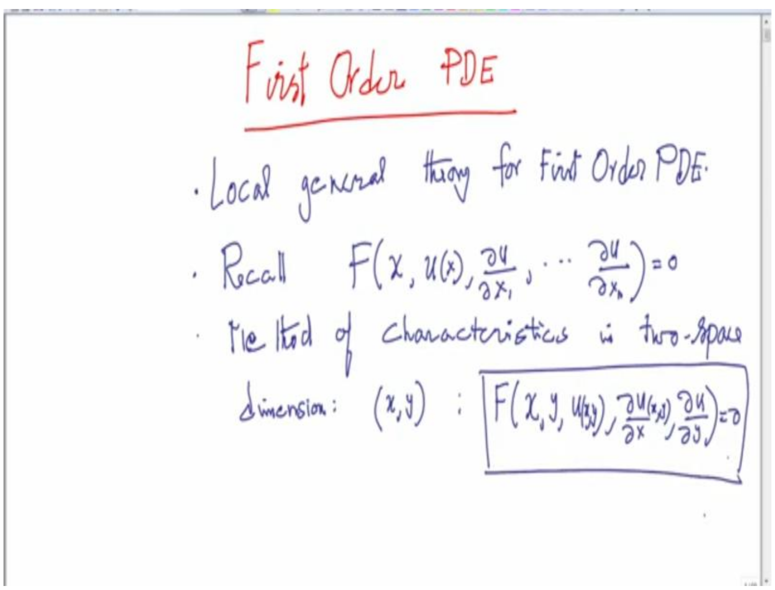

First order PDE. So as we say, developing general theory as general as possible is the aim of this program. You will see that but at the same time at the introductory talks I told developing a general theory even for linear equations is not feasible, but then there is a local general theory for first order PDE and that is the aim of this first course on first part of this first order. So let us recall the general equation, recall the first order PDE. So you will have an unknown x. That is the independent variable.

Then you have an unknown function and then you have all your first order derivatives dx 1, etc., dx n. Sorry, du by dx n equal to 0 where x is 1, xn, etc. That is the general form of your first order equations but as I said, this will be covered in 4 hours, but to make the ideas clear, we will spend the first 2 hours describing and analysing, introducing rather the method of characteristics in 2 space dimension.

Method of characteristic I told you in 2 space dimension. So in 2 space dimension, we use the notation x, y instead of x1, y1. So your equation is f of, general equation, f of x, y, u of x, du by dx, u of xy. So you have your u of x,y and du by dx of x,y and then du by dy equal to 0. So this is the equation initially we are going to discuss. After discussing the 2 variable case, we will go to the invariable case. So we will start with an example to begin with. That is better.

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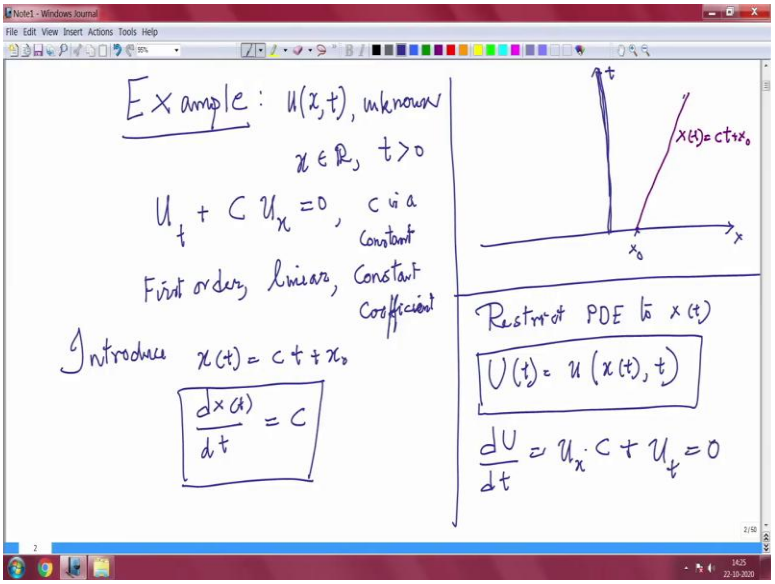

So you have an example. Example to motivate you what is this characteristic about it. So let us consider this particular case where instead of y I am trying to use for the time being u x, t. The reason using is \ that t something like a time variable. This is only for this example. Later we will come back to the notation, use x, y. So the PDE here, u x, t is your unknown with x is in r and t positive. So we are doing it in the upper half. So the PDE is ut plus cu x equal to 0 for x in r and let us look at this one where c is a constant. This is one of the simplest PDE, very easy PDE first order. This is a linear PDE. So the life will be much more simpler. First order linear with constant coefficients. You see it is also a constant coefficient PDE.

So, I am going to solve this problem. You may have many questions after solving this problem. So the problem is in the upper half plane. So this is your x, this is t. What I am going to do is I am introducing a curve. So I take a point here, x naught. On that point, I will draw a line. That is a line.

That line is nothing but xt is equal to ct plus x naught. This is the thing which I am going to do it so that I am introducing a curve here. So introduce xt is equal to ct plus x naught. Here, one point I want to bring. This this curve xt is viewed with respect to t. So you have to view along this direction. So you have to view this as a functioning t variable.

Hence, this slope with respect to t is nothing but 1 over c. So you have to understand. So you are not viewing this xt as a line with respect to x. You are viewing this line xt with respect to the t variable, with respect to t direction you are viewing it and that is thing. So that is the way, you have to understand the slope of this one.

So that means you are looking for a curve dx xt for which dx t by dt equal to c. So basically you are actually looking for a (())(8:10). You may ask a question why I considered that equation. That precisely you are going to understand now here. So once I do that one, I restrict my PDE to this line, to xt.

That means I will introduce a function as a function of T. My ut is equal to u of xt t. So I restricted my t. So this is a 1 variable function. So I have a function u, defined on the line ct. Once you fix x naught, it is a 1 variable 1 parameterised curve. So I want to understand the derivative of u. So I differentiate du by dt. That is nothing but differentiate with respect to ux but dx by dt equal to c and differentiate with respect to u and that is equal to 0.

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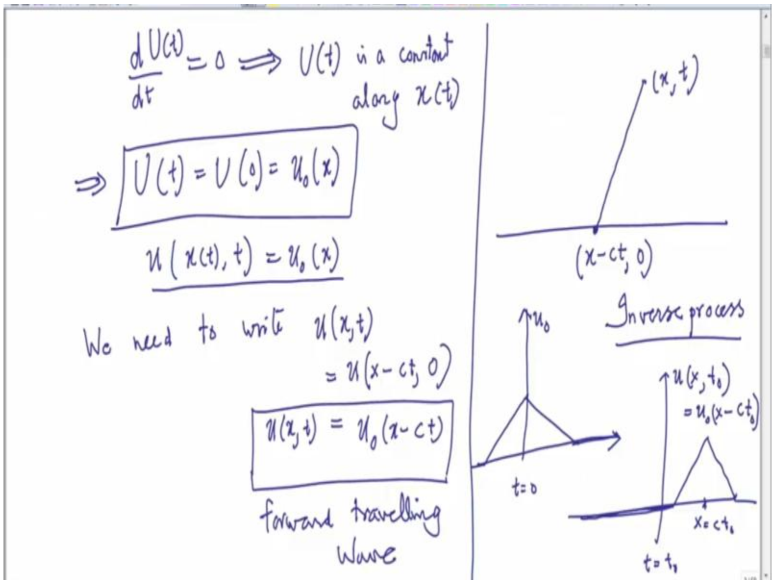

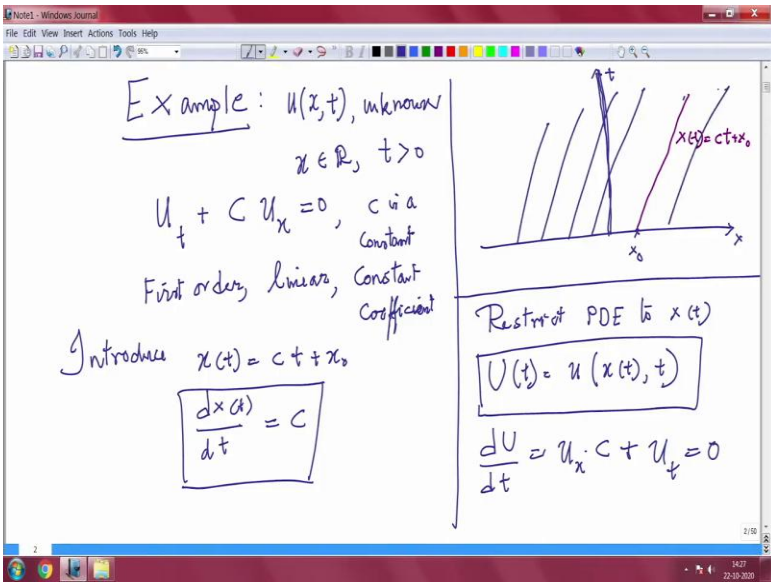

That means what it gives you that that implies du by dt equal to 0, that implies ut is a constant along the line. Ut is a constant along xt. That is what you got it. Along xt. So what does that imply? Ut is a constant along that imply so ut will be that implies u at t will be u at 0 but u at 0 is nothing but u naught of x.

So you prove that basically u of xt, t will be this is what you get it for each of these line but I want to write, we need to write u xt generally. We need to write u of xt where x and t not just along the line. This representation of u of xt t is along the line but we want to write u of xt for arbitrary points.

So, there is some minor issue. This is a very not minor, this is a very important issue. I will come back to that one. So you have an xt here and then you draw this characteristic along that line with slope. So what would be your x naught here? This point will be x minus ct 0 but then you know that in the previous case what you are doing is that, in the previous one, so you see you have straight lines along all the point.

So you have straight line, you are drawing lines along all these points. That is what you are doing it but what we are doing in the next line, we are taking an xt and then drawing backward basically. So it is a kind of inverse process. That inverse process is what makes our method of characters, not complicated.

That is an important point to understand. We will tell about it more later. So there is an inverse process going on. Given point on the initial thing. You can draw characteristics on one direction, but given a point there you want to make sure that there is a characteristic passing through that. These curves are called characteristic. We will come to that. There is a line passing through and then meet the x axis at that point.

So that way you can write that u xt is nothing but equal to u of x minus ct 0 but so because along this curve xt, that curve, it is constant. So, u xt will be u of x minus et 0 so you get u xt finally is equal to u naught of x minus ct. So that is a very nice example, which you immediately see that. So if u naught once differentiable, you can verify that u xt is indeed a solution. So what is happening here? So this is called the wave traveling forward.

Forward traveling way. So that is what happening. So if you look at it at the various plots, so if I plot it here so this is I am plotting when t is equal to 0. So suppose my initial profile is something like that. This is my u naught given. Then after some time it will move forward so with that time t equal to t naught if I plot my u xt naught, I will get this. The curve will be like this. This will be moved forward little bit. That line it moves forward.

So this is what u of x minus ct naught. That is nothing but u naught of x minus ct naught. So this is nothing but the point x equal to ct. So you see, it moved. So as t moves so you have an initial profile and as t moves, the graph moves to the forward direction and you can also draw a 3-dimensional picture of the same.

So, that is what we want to have this example. So basically you solve the problem, du by dt. If you look at it, you got a problem ut plus cu x and you have solved the problem u naught t. So you can ask many questions now. Why did I choose that curve xt? Is there a motivation behind choosing that kind of curve? So you can ask many questions.

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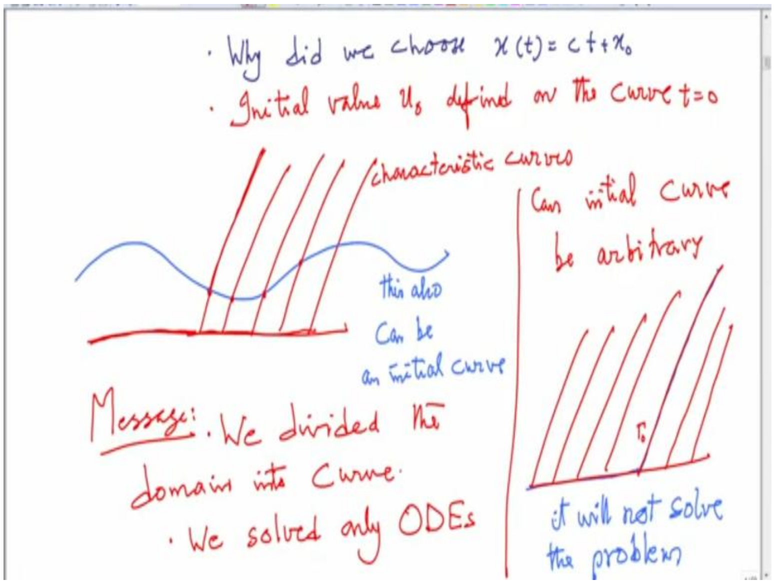

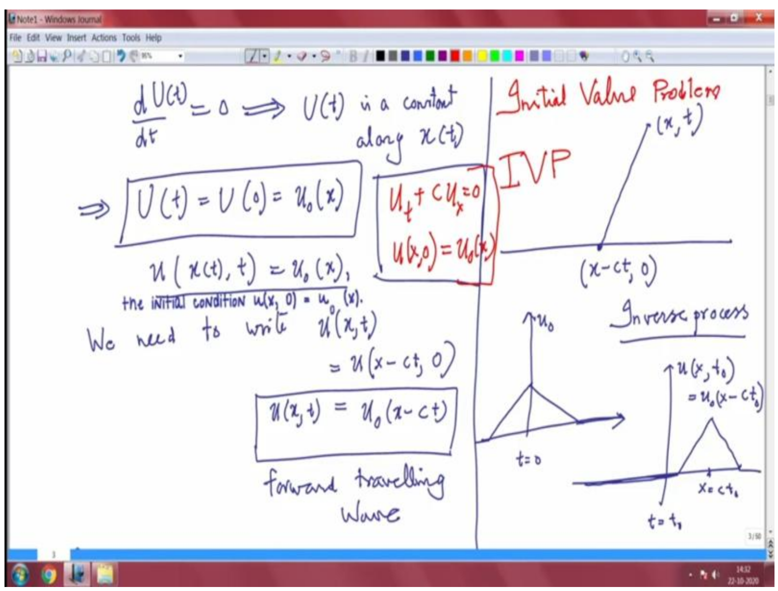

So, you can ask question why did we choose xt equal to ct plus x naught. Now let us go back to one more thing. So the one which we considered is called an initial value problem. So you have a problem ut plus cu x and your values of u naught is given here. That is what we have used it. So I have not mentioned here. So let me mention it here. This is my initial value. So I am solving a problem ut plus cu x equal to 0 and then u at x is equal to u naught of x. So the initial value problem is given. So this is a initial value problem.

So, what we have studied is an initial value problem and so the initial value, so let me write down this, initial value u naught defined on the curve t equal to 0. So the x axis. So you have defined the initial values on the x axis. So why did I choose? So the question is that so you have a line here and you have all your initial curves here. This is your characteristic curves. This we call it the characteristic curves and then in the previous problem I had defined initial values along this but that is not necessary. If you look at it, I can define another initial curve.

So, I can define an initial curve like that. Then again, so this also can be an initial curve because you know that u is constant along all these lines and so if you know the value of initial value of this point, I can determine initial value along that curve. Similarly, if I know initial values at that point, so you may ask a question, can I have you my initial curves arbitrarily? That is not possible. Can initial curve be anything, can initial curve, so you have initially curve be arbitrary. So, for example, I take this one and these are your curves characteristics and now suppose I choose an initial curve like this.

If I choose an initial curve and my initial curve happened to be like this. You see this is not possible. It will not solve the problem. Why? Because you have initial curves here and this initial curve, the values on that initial curve cannot be determined by this initial curve gamma naught. You see so that actually indicates that there has to be a, your initial curve should not be a kind of characteristic curve at any point of time. It should not be parallel. These are the conditions geometry which you need to understand. So if the initial curve is happened to be a kind of characteristic curve or a portion of a characteristic curve, then it is possible that many of the characteristic will not intersect your initial curve.

So, to solve the problem you have to see that very nicely every characteristic curves intersect your initial curve. So initial curve eventually need to be a what is called a non-characteristic condition. This condition we need to understand so you can have some sort of, it is not necessary that initial values are pescribed in that exercises or something. So we are trying to understand these things in a very mathematical way, but then the message is that, what is the message, the message is that we divided the domain into characteristic curves.

That is the one message divided the domain into curves and then that is one message, second message we solved only ODEs, we solved only ODEs. We did not solve PDE. By solving your ODE, we got your solution. So look at here. If you look at it here, you see you have solved this equation du by dt equal to 0.

This is an equation you have solved. Of course, it is an easy equation. It will become more complicated but you look at here, this is the second equation we have solved. So you have basically solved 2 ODEs, to solve your PDE. Once ODE is solved to get the characteristic curves, another ODE is solved to get the solution, the property of the solution, which you got that the solution is constant along the characteristic. So let us go to the next one.

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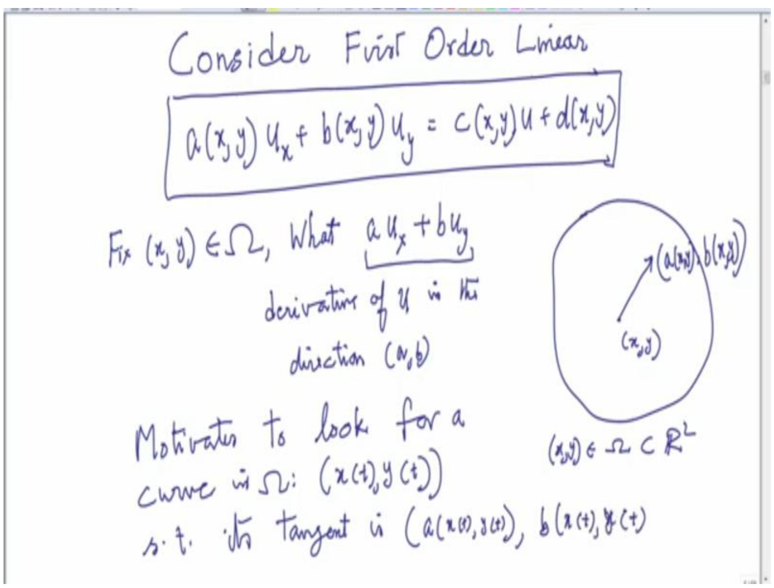

So, now we want to consider the general first order equation. We go step-by-step. So we will go, first we consider first order linear equation. So what is the first order linear equation? Your equation is a xy ux plus b xy uy equal to cx y u plus dx y. So, this is your first order equation. Now I want to give you the concept what did we do and then you have a domain may be in r2, you can have a domain in r2. So you have a domain omega in r2. We will come back to that and xy belongs to that. So you have to PDE is defined in omega. Later, we will come back to the initial value problem. So first try to understand how we introduce the characteristic curves.

So, for that, understand this equation. So if you look at fix xy points here in omega. So you are fixing a point xy in omega and for that x y, what does this represent, what is a ux plus b uy. If you recall my preliminary lecture, you will see that this is nothing but the derivative of u in the direction ab. That is what in the previous lecture you have seen, direction ab. When x y is fixed, you will have a vector direction.

So, this is given by a of xy b of xy. So ab, which is a data given to you. So a b produces a vector field in r2 or a vector field in omega, for a xy bx produces a vector. That means in omega, you have a vector field. So basically, your differential operator part is nothing but the derivative of u along that direction. So that motivates us to look, this motivates to look for a curve in omega, curve in omega. So the curve will look like sum of with here t is a parameter xt yt such that its tangent is a evaluated at each point a of xt, yt b of xt yt. So let me go to the next page and explain xt yt.

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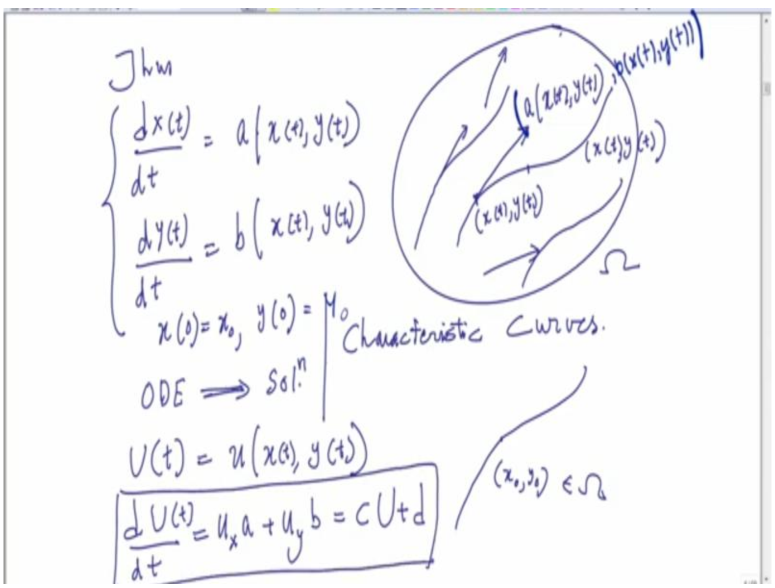

So, you want to look for a curve? So what do you happens that so you have, so you want to look for a curve in your domain omega or r2 wherever the PDE is defined on which at the each points this is the curve xt yt. I am again telling you t is a parameter here. It is not the variable of each point you have the curve. So if this is xt yt for sum t. At that point, I need my tangential direction to be a of xt yt and that is what you are looking at it. So every point has tangential direction a. So everywhere there is a vector a xt, need not be parallel. So what you are looking for is that you are trying to divide your domain omega with the curve has the property that its tangents are xt yt.

So, you are looking for various curve, for that xt yt. So you are trying to fill up the region xt yt. So what happens when you look at it? So what is the equation of the curve equation because its tangent is y of xt yt thus dx t by dt should be a of xt by t. That is the equation because you need tangent to be ab.

So the equation of the curve should be like this, dy by dt is equal to b of xt by t. So if I define, so your curve is such that you have defined. So once you know that, so you fix a point, if you fix a point in the same domain, I am giving different picture so fix a point x naught y naught and look for a curve here whose satisfy.

So, x at 0 is equal to x naught, so you looking for a curve x naught, y naught is in omega. It is a different picture y at 0 and your ODE, will give conditions on ab so that it gives you solution. So let me not get into that. So you learn your ODE system. What are the conditions on a and b so that there is a local solution.

That means when you have a local solution, every point on your domain omega, there is a curve. So you can fill your region because every point, if a and b are satisfied you will have a thing so you will (())(29:25) up like that and these are called the characteristic curves. What happened to u?

So, again define my along. So once you have an obtained a curve xt yt in omega, you restrict my u to the curve. So ut is equal to u of xt yt. So I restricting my solution to that curve. So you will get a one variable function with respect to the parameter t and now compute du by dt. If I have my du by dt compute, I will have my ux and dx by dt, that is a and u with respect to y, uy with respect to b and this is nothing but cu, u of xt yt will be u plus d. So you see you have a ODE again here. So, what is done here?

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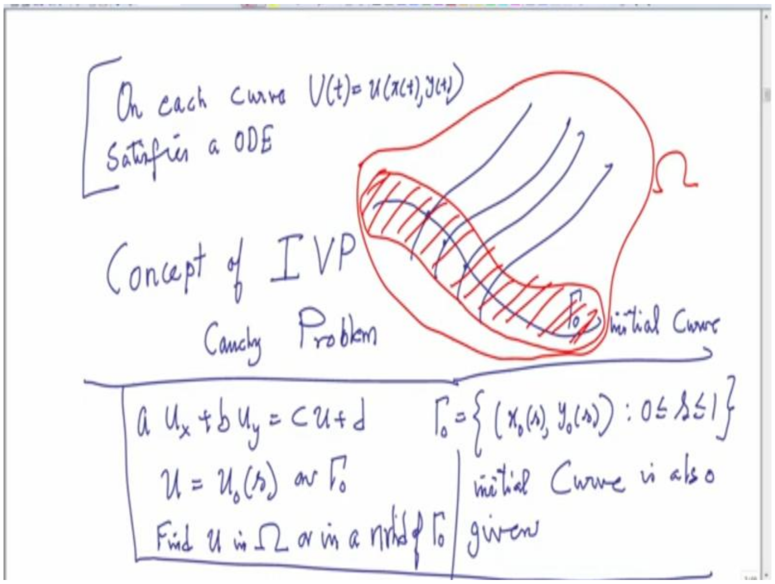

So, you have your domain omega. On that domain omega, you have your curves and these curves are obtained by solving a 2 by 2 system. On each of that curve, you have your u, which is a for, u is given on each curve,u is given by u, ut equal to u of xt yt satisfies ODE. That is what you got it.

So, if you know a function, if you have your curve here and if you know the value of u at one point because it is an ODE, it is a first-order ODE, so along each curve, so the curves are introduced in such a way that when your PDE is restricted to your that curve, it becomes an ODE.

So that can be solved if you know the value of u. So you have to know what we call it an initial curve. So if you have a curve which intersect your characteristic curves, which we call it an initial curve, and on each of that initial curve if the value of u is known to you, then you can determine u effectively. So that leads to what we call the concept of an initial value problem. This is also called Cauchy problem.

So, let me describe the Cauchy problem and then after that I will stop this lecture and then we will continue in the next lecture. So what is a Cauchy problem? You have your a ux, this is your b uy is equal to cu plus d and then there is an initial curve. Gamma naught is an initial curve. So I will give a parametric representation of a curve. So gamma naught is equal to sum x naught of s, y naught of s where 0 less than equal to s less than equal to 1. So that is your initial curve. This is called the gamma naught. So this we are going to use it. So understanding this characteristic curve is quite important. On gamma naught, so x naught s, y naught s is given to you means the gamma naught initial curve is also given to you.

So, you should understand what is given. Initial curve is also given and that initial curve, the value of u, u equal to u naught of s on gamma naught. So this is your initial value problem and find u the problem is that, find u in omega or in a neighbourhood of that initial curve. So this is the global solution existing find u in entire domain omega or find u in a neighbourhood of omega, neighbourhood of gamma naught. That means so you have a domain omega here. So you have your domain omega here. This is your omega. So your characteristics here and then you have an initial curve here and the value of u is given along this initial curve.

So, try to find u in the entire domain omega or if not at least in the neighbourhood of omega. So you want to determine u. This is what is called the locally determining, this is called the existence of local solutions. So you want to do that one. So we will continue this process again and I will tell you a little more about it and then we will go to what is called a quasi-linear equation and in the quasi-linear equation, we will also state the theorem what is the theorem what is required by you.

So, we will eventually write the general theorem, both linear equations and quasi-linear equations together but what we have to understand is the concept characteristic and the meaning of initial value problem. Thank you.