Partial Differential Equations - I
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Lecture - 6
Preliminaries - 4

Yeah, good morning. So, we will complete some preliminaries little more, I do not know, how much we can complete in next half an hour or so, but let us me try to at least recall.

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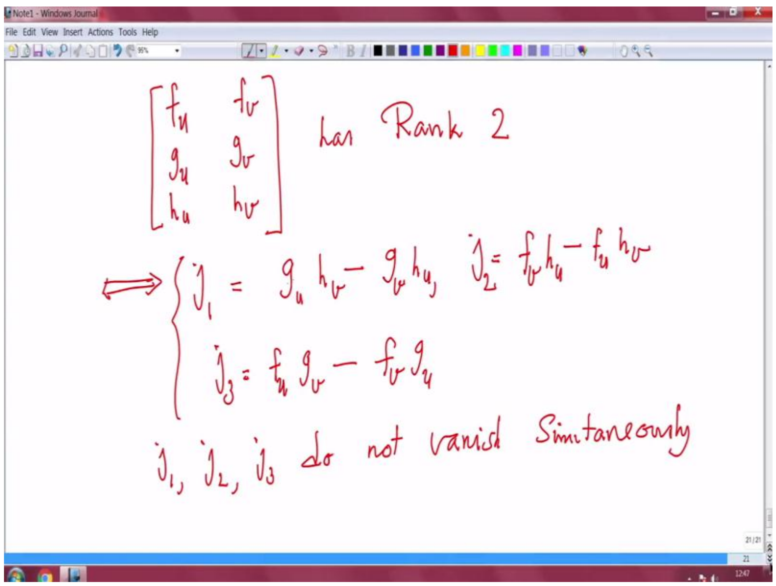

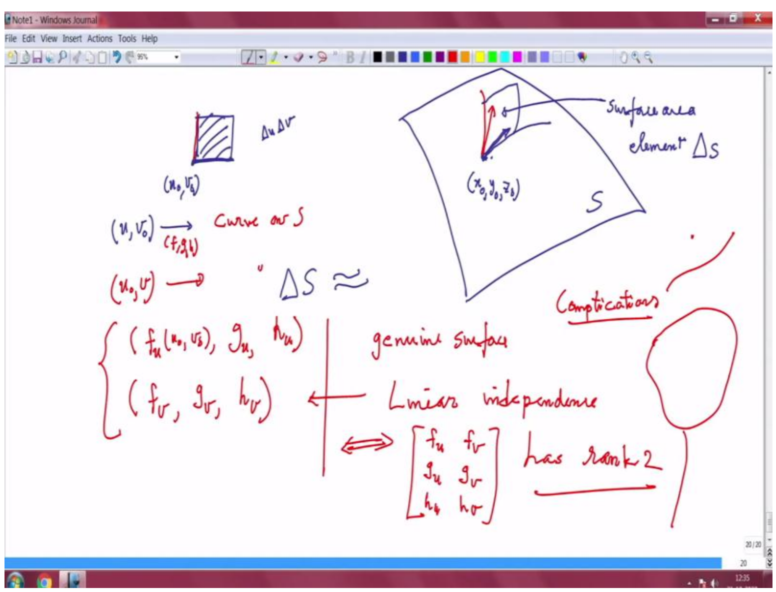

So, last time we trying to introduce the concept of in some sense a genuine surface, that genuine surface as the, you see is the linear independence of these 2 vectors, that is what we have seen or these vectors which you have seen as rank 2; fu, gu, hu and then fv, gv, hv, that is what do we have seen as rank 2.

So, that is equivalent to saying you can see something like that I can introduce what is called a j1, j1 is equal to gu hv minus gv hu; j2 similarly, j2 is equal to fv hu minus fu hv and j3 fu gv minus fv gu. And this equivalent to saying that j1 j2 j3 simultaneously you see, so, you have your simultaneously. So, that is what we will do make use of it.

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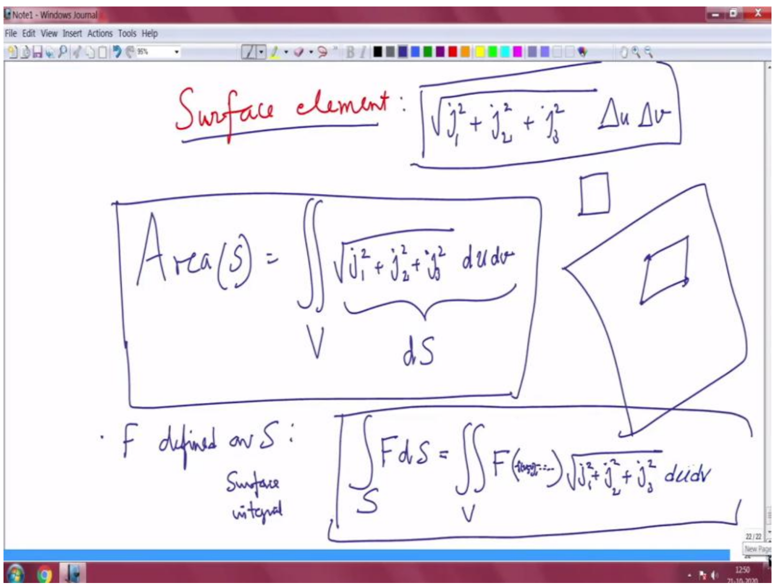

So, so, with this you get the surface element, surface element, let me use the color, so your surface element will be square root of, this is exactly one dimensions we have seen, I am writing it in a very general form. So, the surface element will be j3 square delta u delta v, the delta u delta v, we explained a small thing. So, you because this is for this thing, it is going to be a surface. So, it is so, you get a, for each point you learn you get a tangent plane, vector in the tangent plane.

So, this is approximate the area of that tangent plane parallelogram and you get a tangent element. So, this is the element which we are looking at it. So, using that you can define the area of S, area of a S is nothing but double integral. Now, it is a double integral over the domain v.

So, we have all that you recall what I have defined in this one, so this is root j1 square plus j2 square plus j3 square into du dv.

So, you see, this is a 2 dimensional integration happening on the V because you have a V here. And this is precisely your surface measure dv right dS. So, if you have a function now, F defined on S, suppose you have a function F defined on S, then I can define what is a surface integral. You have your surface integral defined on S F with respect to the surface measure S is nothing but integral over double integral overview that is F of U V.

So, F, F is defined on S, so you will have every point you can view it as a view under that map F of UV gV all that map F of that into square root of j1 square plus j2 square plus j3 square du dv. So, is equivalent to saying that you are integrating, so because the function is defined on S the function is defined on S and but every S can we view it as a points from here. So, you here you have to write a F of UV like that. So, it is a function which you do this is called the Surface Integral. So, you are not surfaces integrate with directly with respect to the Lebesgue measure. So, to understand that properly.

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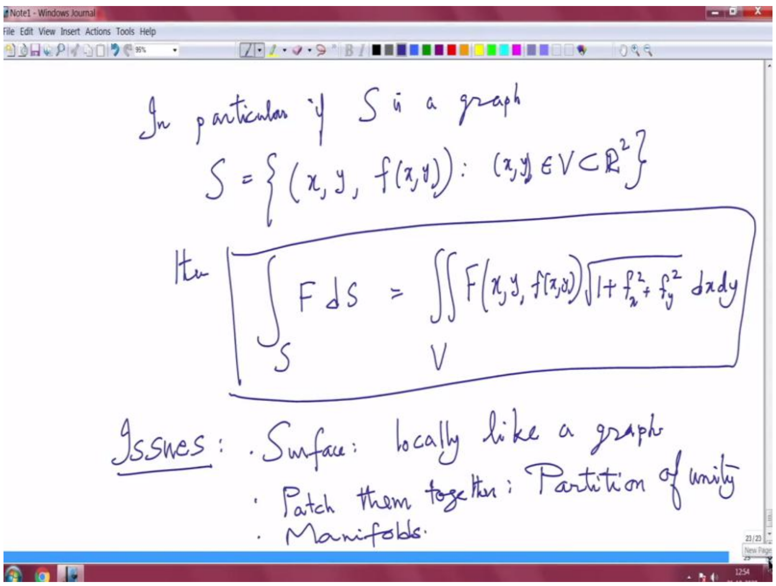

So, in particular, if S is given by a, if S is a graph, this is special case is a graph that means, your S is equal to set of all xy f of xy with xy is in V, xy is in V which is a subset R2 then your integral of F dS over the surface S we will be double integral over the V F of xy, f of xy square root of this is a familiar formula for to you, fx square plus fy square dx dy. So you will see So, you have a very precise definition of surface measure and surface integration.

So, there are many issues which I do not want to know I want do not want to get the, the there will be issues this are prescribed where, so I cannot do it. So, there are many issues to do it. So, the problem that you have to do it for a local surfaces, I told you, you may not have surface given by such a nice form. So, the locally it will not local surfaces, local surface, surface and locally like a graph that is where the results to more difficult to prove it.

So, it will be like a local graph then you how to patch them together. So, there is a concept called partition of unit and there are more general results to manifolds. So, I suggest all of you those who are attending this course, get familiarized with some kind of notions in more. Now, now, I go to the last part of the minimum thing problem what do we call it as integration by parts and theorems.

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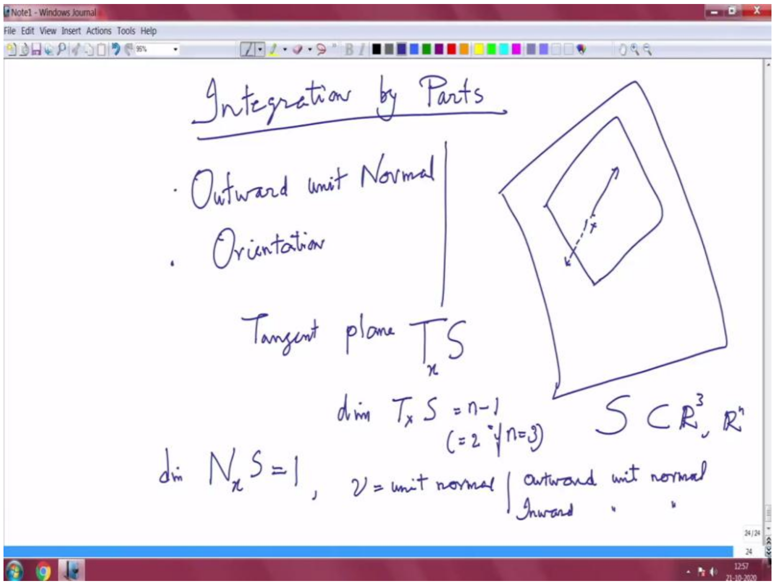

Integration by parts and some related theorem. So, let me generally talk without writing much the reason is that again you have to understand the existence of an outward unit normally, that is very crucial in writing the own thing. So, there are concepts like outward unit normal or you have to be careful. I will and then there is a much deeper concept what is calling orientation. I do not orientation, I do not want to get into these things, but these are the important thing, but outward unit normal I want to tell you something.

So, you have a surface why I am telling you that these are the things you should get familiarized, if you want to study, as I said to PDE, and of course, even other subjects, so if it is surface, if it is a hyper surface, surface in Rn or R3 if you want to familiar or Rn then you will have a tangent plane. So, you have a tangent plane. So, you have a tangent plane at any point tangent plane at a point x.

So, if you have a point x here, you have a tangent plane at that point. So, tangent plane at that point, so you have your tangent plane to that surface and dimension of this one, dimension of tangent plane will be n minus 1 that is equal to 2, if n is 3. You see, so you have a 2 dimensional thing, that means whatever it is, the tangent space will be one dimension less and hence, it will have a normal surface, normal to that and dimension of this one is always 1.

So, it will be spanned by 1 vector. So, there will be a normal thing. And then hence, you have a unique normal you will have a 1 unit normal vector. So you have, so the important thing is how do you understand outside and inside, so what is the outward unit normal and you have to understand what is the inward unit normal.

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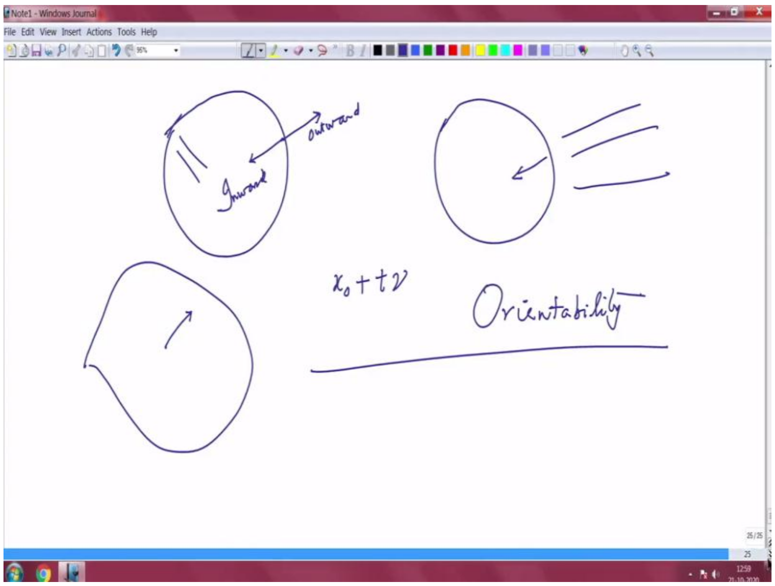

In most of the surfaces, which you are familiar, for example, if you have a sphere and for this region, this is your outward unit normal, for this will be the outward. This is inward. But the problem is, on the other hand, if this is your region, if your region is this one, you have this thing, the issue is that I want to vary these points. So when I vary this point, so if you have a surface here, and then I vary this point, I want to get some unique normal varying, outward unit normal varying in a smooth way and nice way.

So you need to have the existence of a, it is a bit difficult. So if you get the unit to normal at a point x naught. And from there, along that point, I can so it is bit difficult so I do not want to get into this one. And then x naught is varying basically. So, you want to vary the unit normal all around the surface and then I should be able to choose some unit normal in a very neat and uniform fashion.

It should not become an inward unit normal. So there is notions like orientability. So I do not want to get into there are very precise definition of orientability and all that. So, so, we work on domains for which there is a unit normal axis, outward unit normal axis at all points, and you have to study the surfaces more closely. So as I said, since I do not have much time, I want to define a class or domain for which this everything is fine.

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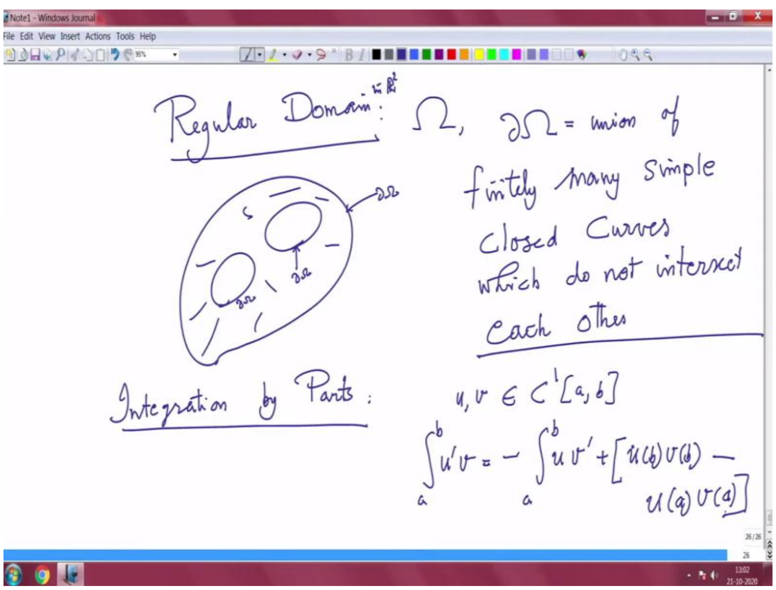

And then that is what we call it again I will regular domain. So let me not give a very precise definition. So given a domain omega, you have your d omega. So you let me write a thing d omega is the union of finitely many simple closed curves, finitely many simple closed curves, which do not intersect each other. For example, if you have a domain, so your domain you can have something like this.

So this is the boundary, the Omega part. So this is the domain. So this is the so this is also a (()) (16:06). So, and when you have a symbol close to curve in R2, so regular domain in R2 that is the easiest way. So, if you have a regular domain in R2 you know that if you have a symbol close to curve, it defines an inside and an outside a boundary, simple close to the defense a bounded inside and unbounded, you can have any inside and outside for a simple closed curve.

So that is a very important to understand that one. So this tell you basically, we want to so with this, I want to go to an integration by parts now. Yeah, I know that this lecture, which I am giving is a bit vague in the sense that as elementary there are too many concepts. And giving that all that in half an hour or 1 hour is not possible. So let us call this one dimension if you have a u v in C 1 of ab you know this one dimension and say a to b u prime v is equal to minus integral of a to b u v prime plus f ub vb minus ua va. You have that one.

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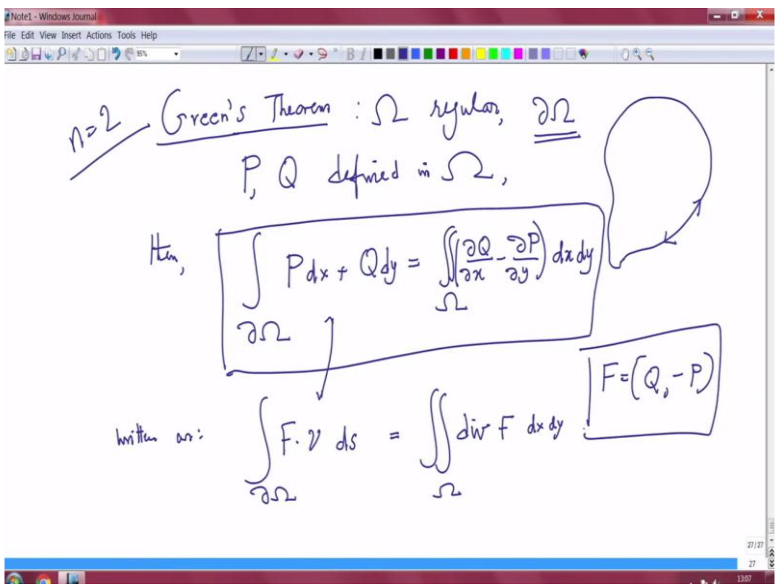

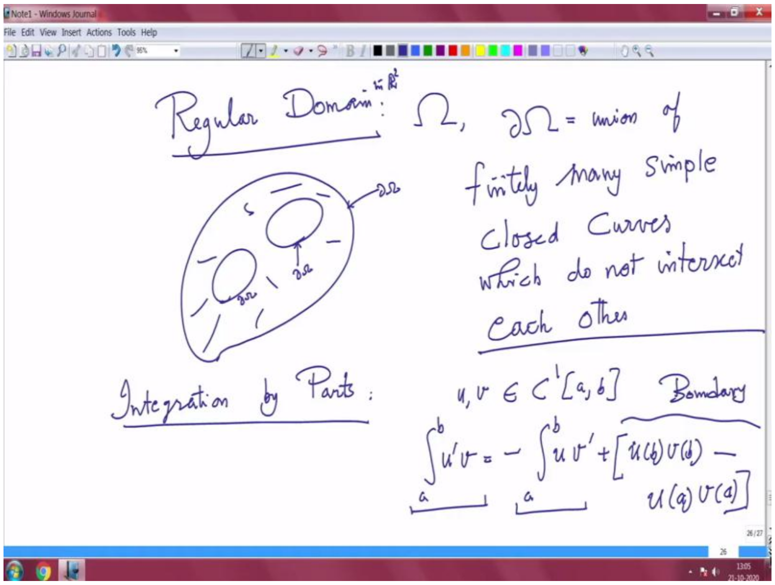

So, I want to state again, as I said, stating itself is a difficult thing. But in two dimension, I understand my left and right very nicely. But in higher dimension, that is difficult, because when I say in 2 time, it is also it is a kind of bit vague, as I said, but you can make things very precise. For example, when I move along a curve, I know that which is my left side and which is my right side. So, with that, so understand my omega regular as above.

This is when n equal to 2, Green's Theorem, regular and d omega when I saw I have 2 directions when the problem is this one, when I have a because the integrals will change here, when I integrate along this direction, and when I integrate along this direction, both will be different, it may change with a minus sign. But when you write a theorem, you have to be precise which direction you are taking.

So for this Green's theorem, I am assuming that I am moving along the curve in such a way that my domain part comes from the left side of it, that is very important. So I am taking in such a way that my domain lies to the left of that, this left is a notion which you well understood for n equal to 2. And you have your P and Q functions defined on in omega, then the theorem tells you and you have your curve gamma.

That is what I said d omega d omega needs that assumption. So, look into the proofs for very precise definition. First try to understand additively, geometrically, and then understand in 2 dimensions properly, and then other dimensions whichever, then the Green's theorem tells you the double integral, integral over, this is called the line integral, this is equivalent to the previous thing, if you look at it, this is an integral, integral interval domain, this is full interval, but this is only the boundary data, boundary values.

So, you have when you are trying to do an integration by parts, you will have some boundary values coming into picture. So, you have your Pdx, so, let me use this notation Pdx plus Qdy equal to this is integral over omega double integral over omega dq by dx minus dp by dy, dx dy you see this is what the Green's theorem, you see as I say even the explaining the theorem stating the theorem itself require appropriate terminologies.

As I said if you look at this d omega Pdx plus Qdy and that d omega, you can move along two directions. So, you have to take the appropriate direction otherwise, there may be changes in design etcetera. Alright, so this is the integral. So quite often you may get how to, how to remember these formulas that is not fine, the term is always Q is dy. So, dq by dx dy, you can just for remembering, you can say that dx dx cancel you get Qdy, that is an easy way to remember.

So, you do not make whether for Q I use dq by dy or dq by dx you do not have to worry about that one. Sometimes this is also written as written as, so these are all different forms f dot of mu this is precisely same, both are same because you are integrating dS over the boundary. So, it is a line integral and this is equal to divergence of F dx dy over the domain omega. What, what is your F? F is equal to Q minus P, if you want it this is another way of writing. Now, let me go to the last part of the theory, these basics what is called a divergence theorem.

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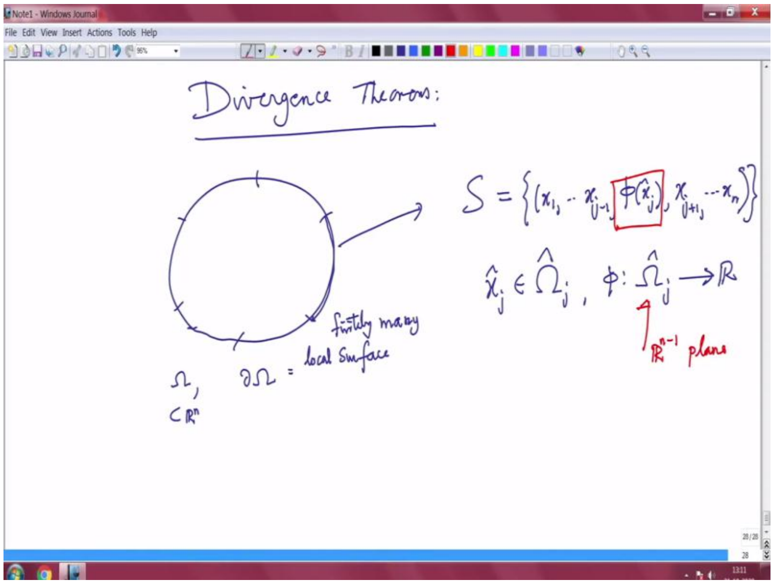

This is the higher dimensional version of the Green's theorem, if you want it. I forgot to tell you these are all, why these are all important? These are all the higher dimensional versions of your fundamental theorem of calculus, you know the importance of Fundamental Theorem of Calculus in one dimensional integration.

So, this Green's theorem, Divergence theorem, Stokes theorem these are all various forms of fundamental theorems in the higher dimensional setup, and you need more complex thing because in higher dimensions if you take a 2 dimensional area, the boundary going to be itself is a one dimensional thing, it is not just 2 points as in, that is why you need to have an measure, you need an integration all that you need orientation, you need the normal existence, all that is required and all that should be defined appropriately.

So, let me go to again you need conditions just like Green's theorem stated in the surface you have a domain omega. So, let me not again getting into it because it is again and again repetition. So, you have a domain omega, so your domain these are all implicit functions will gives you these kinds of assumptions. So, d omega can be divided into finitely many. So, this is n, general n is not a 2 dimensional domain. So, omega in Rn with the boundary thing.

So, we can divide these into thing. So, loosely speaking d omega can be written as a local surfaces finitely many local surfaces, finitely many which you can do it right because each point implicitly some function theorem tells you the existence of the neighborhood which we will cover the whole thing, but then the compactness of d omega will give you the finitely many neighborhoods.

So, you can put conditions on which when this omega satisfies these conditions. So, each omega so you have each one look like a surface of this form given by a graph. So, that is what you advantage. So, globally you cannot write it, even for a circle you can write it that way, but then locally you can write that S, S is of the form for some x1 etcetera, xj minus 1 then you will have a map from a lower dimensional space xj hat something into xj plus 1 etcetera xn.

Where xj hat is in some n minus 1 dimensional space omega j hat. And phi is a mapping from omega j hat to R that is way locally I am able to write this you see, that is the whole thing locally only jth point as like a graph I can write it and omega j hat is a thing lying in a lower than R n minus 1 plane, it is part of an Rn minus 1 plane. The thing is that the which component which plane it.

So, one of the planes of Rn minus 1 so, that way you can view, so your d omega is not globally as cannot be written as, locally with a finitely many d omega with a finitely many pieces of this kind of local graphs. For such a domain, let me define the Divergence theorem.

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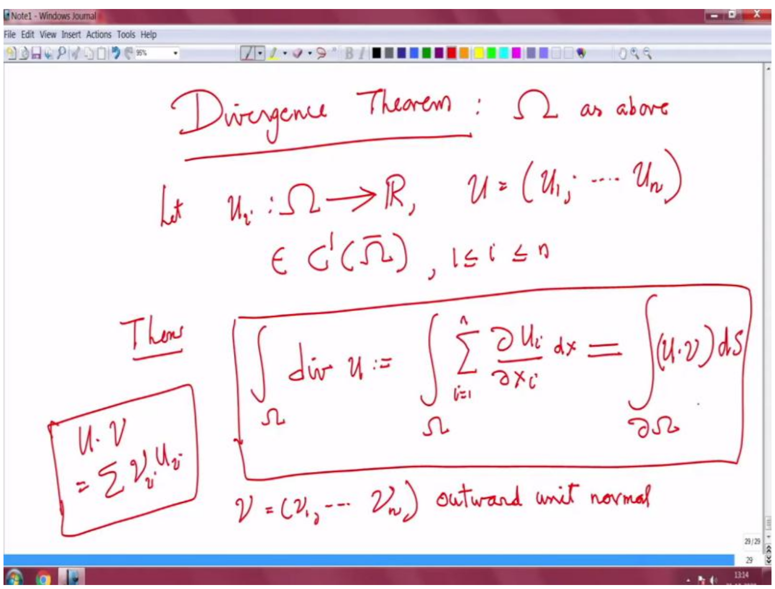

So, writing once you know the terminologies writing divergence theorem is okay. So, you have omega as above and you have your map that ui from omega to Rn, R omega to R and you write u is equal to u1, etcetera, un. And I assume that ui is in each one is in C1 of omega, then 1 less than equal to i less than equal to n, then in another five minutes we will finish, integral over omega divergence of u, what is divergence of u, is nothing, that you know it, you would have studied, but let me this is by definition the dui by dxi i equal to 1 to n dx.

So, this is the integrator over omega that is the meaning of this is in, this is the n, n dimensional, this is the n dimensional integration. So, theorem is this one u dot of nu this is the surface integral, so over the boundary, so this is the surface integral where nu is the nu 1, etcetera, nu n outward unit normal, we are assuming that right we are taking domains which has an outward unit normal as a function of x.

So, vary is nu and nu dot of nu of course, that you know summation nui ui. So, you can take these things. So, you see when you are taking a differentiation, so you have an n dimensional integration which reduced to the n minus 1 surface integral and the last some results.

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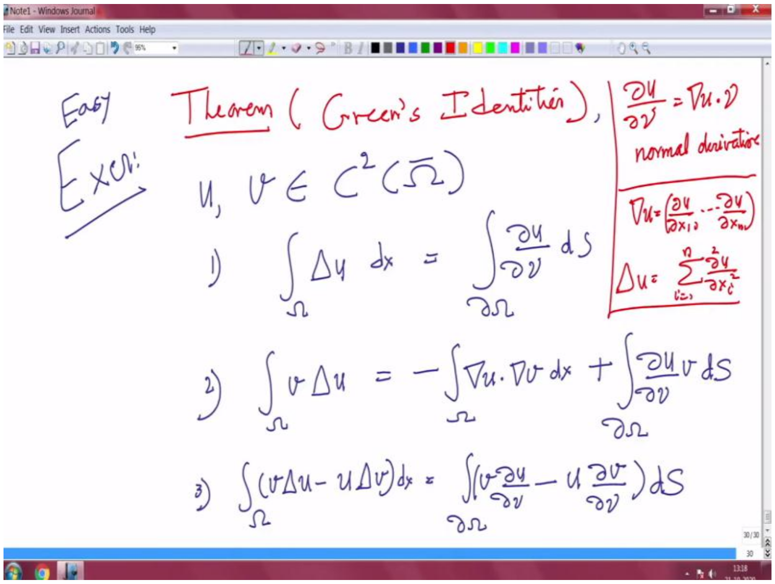

These are the results which we are going to use it theorem Green identities. These we will use throughout the lectures. So, you have to be familiar with that. It is nice to understand everything but then these things you should even remember I think. So, we are assuming u v belongs to and this is an exercise now, assuming the previous results, this is an easy exercise, but so you do it, it is not difficult, u v belongs to apply previous divergence theorem, that is all.

You apply the and this we will be using in plenty all the time throughout this course. So, it is better to learn this one. One integral of Laplacian of u because this is the operator, we are going to study dx is equal to integral over d omega du by d nu dS. And two integral over for omega, these are direct deduction from that divergence theorem is equal to minus integral over omega grade u this dot grade v dx, this is again integration.

Keep the sign properly you should not change it the sign you will get the wrong result and nu has to be outward unit normally, du by d nu v dx and I am just writing because this we will be using again and again v Laplacian of u minus u Laplacian of v dx will be integral over d omega v du by d nu minus u dv by d nu dS, oh sorry, this one is surface integral dS, this is also the surface integral dS.

So, let me complete that by giving what is the definition what is du by d nu? This is called the normal derivative grade u the dot product nu, grade u is a vector is the normal derivative, it is not the normal you are taking the derivative along the normal direction. Now, you know since you know that the derivative from the previous class you can define that one and grade u you already know this let me set it once a then after that we will not use it, recall.

But we will be using this again and again, dxn and your Laplacian of u is equal to summation of the du by d square u by dxi square i equal to 1 to n. So, with these as I said this is a trivial deduction from the divergence theorem. So, remember divergence theorem if not the proof I recommend is to go and see the proof get some important ideas and concepts whatever I, what I defined is not complete, I not defined because we are not, do not have the time.

And anyway, that is not the aim of this course, to get into that basics, but you have to get to know and subtleties behind it and the important concepts. With that these theorems we have to know learn it and derive this equation as a simple exercise. So, I will stop here. After this we will get into the main part of the course. Thank you. So, I will meet later. Next time.