Partial Differential Equations - I
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Lecture - 5
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So, again welcome to the lecture. So, in this two talks, we are going to briefly giving you some motivation behind the surface integrals. And then we want to again state 2 theorems mainly Divergence theorems and the Green's theorem, things are a little more delicate even to state that theorems is not easy, you need some notions, but our course does not now allow to get into all that details, because that is not our aim of this course. So, just to understand.

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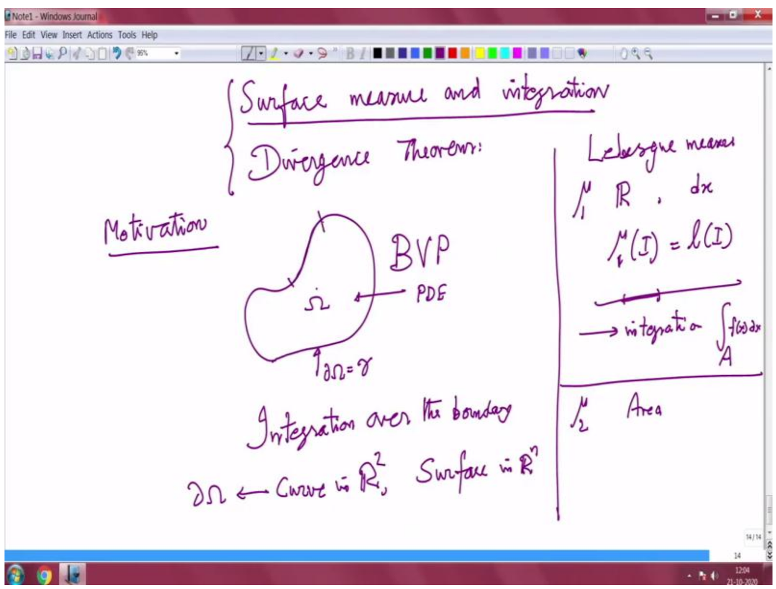

So, we want to tell you the Surface Measuring and Surface integration, Surface Measure and Integration to begin with, then we will these are the 2 topics we want to discuss, and then the Divergence theorems. So, in this class, so you try to understand because we will be trying for the kind of motivation basically. So, when you study a PDE, you will be studying Boundary Value Problems.

PDE, Boundary Value Problems, so, you will have a PDE here and you will have a it is a boundary. So, if this is omega, the boundary is normally denoted by d omega maybe sometimes you can denote by gamma, or sometimes the other gamma. So, you come across all the time you know how to integrate so, if you have a domain here, if you have an so, you it will come across integration our boundary that is where the integration over the boundary which is a lower dimensional subspace.

So, you will have a boundary d omega maybe is a curve in R2 or it will be a surface in Ra when you go higher, that is why need to introduce a measure if you look at the Lebesgue Measure, so, let me quickly call what is Lebesgue measure. You need a measure to integrate. So, how do you compute your mu 1? So, this is the in Rn or this is also denoted sometimes by dx, you get 1 dimension. So, let us start with our R1 measure R. This is based on the concept of mu 1, if you take an interval I it is the length of I.

That was what do you do it in 1 dimension? That is a flat domains, it is a concept of flat and then using this Lebesgue measure is developed to measure the length of arbitrary sets. Of course, you cannot do it for all sets. So, there are some notions like measurability and all that, end of it, you will be able to define what is an integration. So, you can talk about integral of fx dx over measurable sets let me not get into all that measurable sets and all that concepts.

But then you can do it mu 2. So this is a based on the area concept. So this is basically gives you the mu1, the 1 dimensional Lebesgue measure basically gives you the length of the interval and mu 2 gives the area of that, mu 3 gives the volume of that in general mu n gives you the volume that is a very general notion of that one. So, that is a concept you want to do it. So, when you have a domain with a boundary here, so if I take a boundary I want to measure it. So, let me go a little more define.

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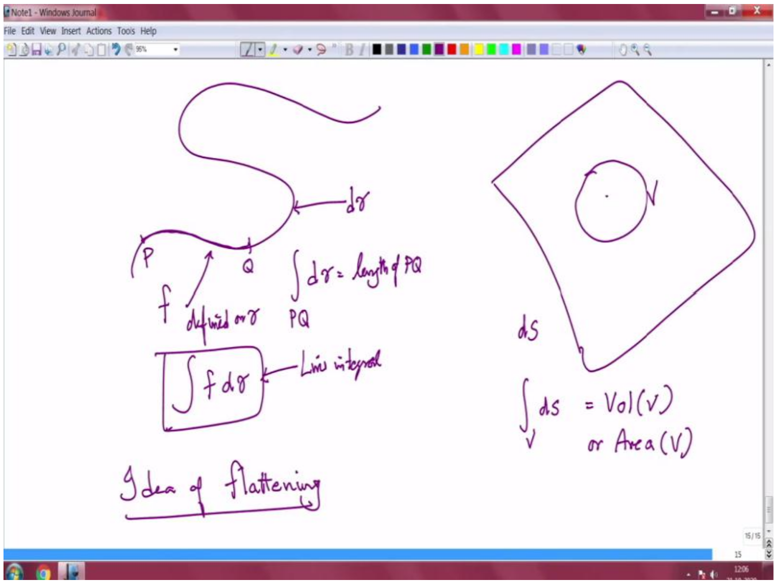

So, if you have a curve, the curve can be here, I want to define a measure of gamma d gamma here. And if I take any say in the along that set if I take an arc or a piece of that, so from P to Q I want if I integrate PQ these are all I want it basically I want the length of PQ. So, you see, I cannot use 1 dimension it looks like 1 dimension, it is 1 dimension in some sense, but then I cannot use it because this is not flat domain which you are doing it, it is not R, R1, R2, etcetera.

It is a curve sitting in R2, or it can be a curve sitting in any Rn, R3 R2, R3, R4, you can see that curve. So by the end of it, the length of the curve, you want to, so you want to and more generally, if there is a function defined here, function f defined on gamma, it is not in R defined on gamma, I want to define what is f d gamma? This in one dimension, you already know that these are all line integrals.

But then, the idea behind that is some sort of a measure, which I am going to tell you so and more generally, if you have a surface I want to define a surface measure and then if I take any region here and if I integrate over that region with respect to dS, I should get my area or volume of V, volume or area depending on there are different notions of that and this is what I want to actually do. So, I need to have a measure I should be able to measure so they get the idea for the Lebesgue measure, but then it is what we call it a in some sense idea of flattening, but then the issues will be enormous. So, let me start with a simple case.

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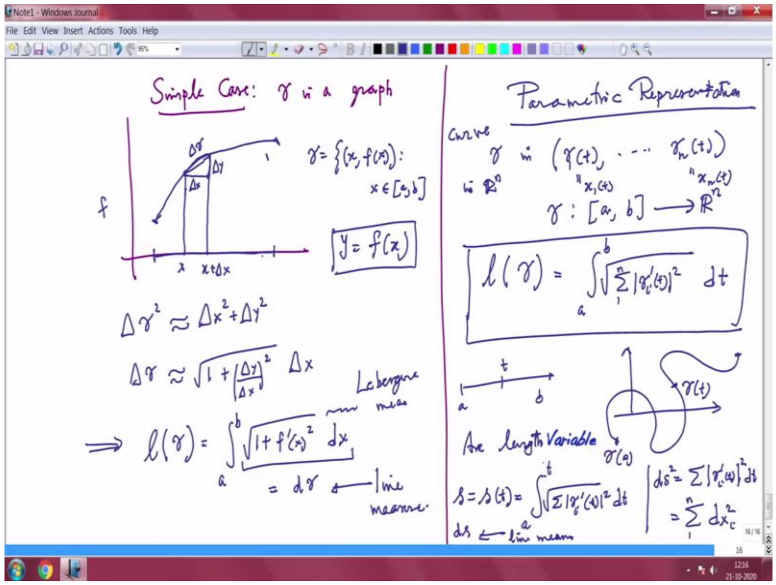

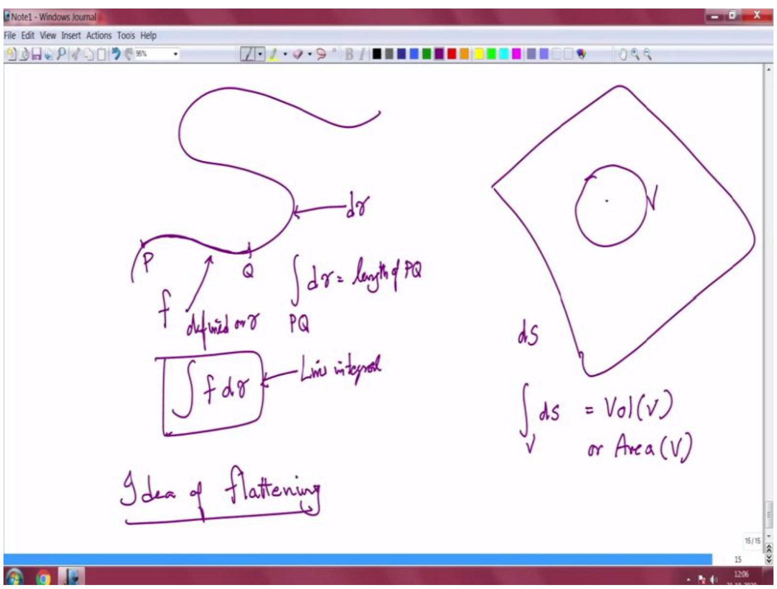

Then we will first we understand the simple case, which you are already familiar, suppose the curve is given by a function, curve gamma is a graph. So, let me do that, gamma is a graph means, I have a function here, here. So, I have a function here that means, the points on the curve gamma is of the form, so curve gamma is of the form set of x, f of x. So, this is a map f you have a set of for points of the form x f of x such that x is in an interval. A curve means a general thing as you see a curve in the previous picture if you look at it, the curve can be any form this is not given by a graph.

But then what you have studied in implicitly function theorem tells you that locally you can make everything as a graph. That is essentially the application of your inverse and implicit function theorem, you can locally you can get a graph, you can write it in this form, even in the general case.

So let us try to satisfy every point on this curve gamma. So I want to measure the length this is easy. So I take a point here, I take a point close by point suppose x plus x delta x I do this one. So let me put the curve. So if I have delta x here, here, this looks like. So I have an x here. I have a delta x here. So I have a see which I can do that.

So, this is delta x, this is delta y, so I have a delta x here, delta y here, see, your delta gamma square is approximately, if you look at it, delta x square, so I am only giving you a very intuitive picture, you have to make it very rigorous, which and there are much more general theory if you want to learn it. So, your delta gamma is approximately square root of, I write it in a specific form delta y by delta x square into delta x.

So, this is in the limiting case. So now you do a limiting analysis, then you can see that this implies, so delta gamma is approximately this number, and then make that your partitions and then to the limiting you. So this is precisely a small element of the it is a, it is actually a small element. So, this is a small element delta gamma, you see and that, hence, the length of the curve gamma will become integral a to b square, if you put it you get y is equal to f of x.

And then square root of this formula is familiar to you already. So, this dx corresponds to the Lebesgue part, Lebesgue measure. So, you are basically integrating here. So, this length is different from this length, but then just this length and you need this factor. So, this is the Lebesgue measure and your d gamma is basically given by this measure, you see, so this is the square root of 1 plus f prime of x square dx is the one you have to multiply with that one.

So you have a multiplying factor together with your Lebesgue measure looking at this here, so that is the kind of intuitive picture you want to do. So this what do we call it a Line measure. You can call that a line measure and then this is a very one way of representing curve. So there is a parametric representation. So let me write that also. I will recall this later parametric representation.

So, this somewhat immediately an idea if you want to find a kind of measure, and I will anyway come back to this one, more about it is, more about it soon. So, let me do that in a parametric. So, what is a curve gamma, so, you can write a curve gamma, in a curve gamma, you can precise definition get it into the literature as I said, I want to motivate you. So, curve gamma can be written as in Rn, curve gamma in Rn you use it to 2 or 3 you can write it as a gamma 1t, etcetera, gamma n t.

So, the curve gamma is a smooth map or continuous map from ab to Rn, you assume it is even also for your understanding. Some people this also represented by sometimes x1t, etcetera, xnt you can do that. So, if you do the same procedure, which I do not want to repeat more about it. So, you can calculate the length of the curve gamma in the parametric representation is nothing but integral a to b square root of this is the same in the previous case, but then gamma i of t square 1 to n dt. You see, so, you have your length of the curve gamma.

So, you have your so, in this case the advantage is that you can use any say the curve will go from somewhere. So, the curve is a mapping, which you have to understand. So, you have your ab here, you have your ab and you have your Rn here. This is your Rn this is whole your Rn and the curve can be anything. So, a may go to here and n will go here. And that is what you are calculating the length of the curve.

So, this gives you what we call it an arc length variable, can define arc length variable. So, you take a point here take any point and calculate the length from this is corresponding to a, this is corresponding to, this will be corresponding to gamma a and take any point corresponding to gamma t at some point t here this is b, this will be your t. So, you can define what is an arc length variable s that will be a function of t is nothing but you integrate a to t and this is one variable integration you see same thing.

So, here in the previous case gamma I, sorry I did not write here is a gamma i prime t, gamma i prime t square dt, so this is the arc length variable. So, ds is your measure, ds is the line measure basically, these are a presentation, these are all symbolic representations. So, you can write your symbolically ds square is equal to summation gamma i prime t square dt, you have to write that. This is also sometimes written as summation the x i square or dt i square whatever it is whichever depending on variable you are using it, so you can have this arc length variable. So, have you studied that once you have that kind of arc length measure

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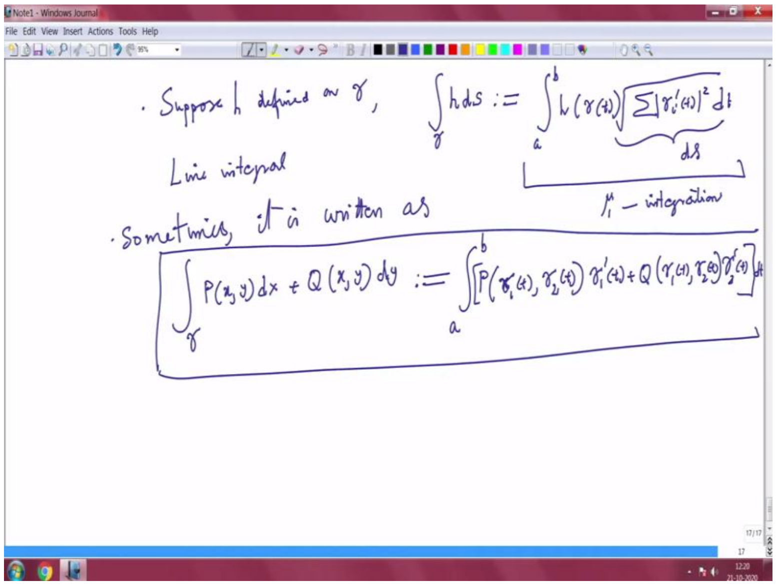

Now suppose you can have the integration immediately, suppose, f is a function f defined on gamma defined let me not use f because it is already there, h is a function defined on gamma then you have your definition you can integrate h with respect to this measure ds. So, this is my definition this is same as integral of it, this is over gamma integration. So, integral of h over ds arc length variable from a to b h of gamma t summation gamma i prime of t square, square root dt.

As I said this is your ds if you want you can write that. So, this is the way, so this is the extra square root of summation modal of gamma i prime square is what you are done, but this is a one variable integration mu1 integration basically. If it is smooth, it is the same as the Riemann integration whatever it is, you know that the that this kind of thing hds comes in higher dimensions of the work done and all kinds of things.

So, you will come across such kind of things. So, there is also an another way of so, this is also called the line integral, this is called line integral and sometimes it is written as, sometimes it is written as integral. So, you have to be careful with this notation P xy dx plus Q xy, I want to write this because this is where I will write my theorem. This is our gamma.

So, be careful the first one is not integrating with respect to x or integrating with respect to y by definition, this is integral of a to b where P and Q are functions defined on that curve or in a domain surrounded by that one integral of a b P of gamma 1 of t, gamma 2 of t into gamma 1 prime of t, now it is a function defined on ab plus Q of gamma 1 of t gamma 2 of t the gamma 2 prime of t dt.

So, this is the definition you are to write it, where line integral is used this way and you have to have this perfect correct definition, keep that the right hand side is an integral of one dimensional integration, the standard one dimensional integration and that is the way you have to interpret and so with that let me go to some surface integrity about more about the surface in R3.

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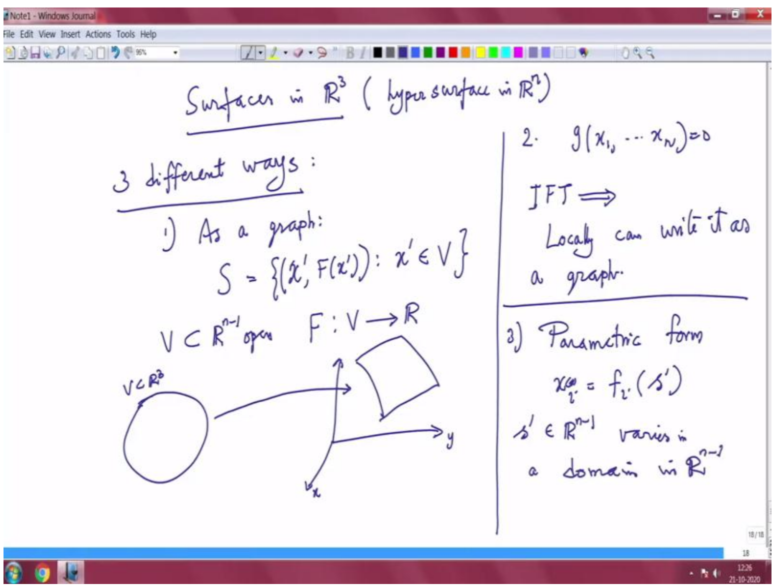

Or more generally, you can have a hyper surface in Ra, hyper surface, hyper surface means, for example, in R3 you have the lines and the normal 2 dimension surfaces. Of course, this 2 dimension if you are not studied, what is the meaning of dimension, which you probably have to understand by some other means. So, a dimension less one dimension less, that is a hyper surface because when you go to Rn and view, you view it Rn as a, you view Rn as a manifold and then you know that you can k dimension and manifolds sitting there and the n minus 1 that suffer, any surface having a dimension n minus 1 as a manifold is called hyper surface.

So, you can have your precise definition on this one. So, before going to that surfaces can understand it three different ways or will be useful at some, you can understand surfaces in three different ways. So, one as a graph. So, I want to understand the function exactly whether as a graph that means your surfaces is S, so, suppose we call it a surface S can be considered as a points of the form x prime some f of x prime.

I will tell you exactly where x prime is in V. What is V? V is a subset of Rn minus 1 open, so it is an open set in Rn minus 1 and f is a mapping from V to R. So, basically, you are having say three dimension when I plot it here you have x1, xy and so you will have a domain V in R2, from that domain R2 you will have some map and then you get your surface, you get so basically looking at a graph globally.

That one of the difficulties that you may not have such a global representation as I said the surfaces can go in a more flat way. So that another second one is considering it in an implicit form. So you can consider surfaces in an implicit form. And then the implicit function theorem will tell you, implicit function theorem will give locally, you can write it as a graph, locally you can write it as a graph. That is one thing.

The third one, so, as I said this is globally writing the first one is a global writing and then the second one is in an implicit form, when it is an implicit one depending on the derivative, one of the derivative, you will be able to solve in terms of the other variables. So, you can write them in a one way locally. And this of course creates trouble, when you introduce a surface measure, you will be able to introduce a surface measure locally, but then you have to, you have to develop that in a global, global way for that you may require partition of unity and other kinds of things.

As I said, these are a little more delicate, but please go through it. So, the third one is a parametric form that means, you have a surface it is given by your xi is equal to, you will have a family of functions, xi is equal to fi of sum s prime that means you again have a where s prime varies over Rn minus 1.

Exactly, when you have a represented a curve in R2, you have gamma 1 t and gamma 2 t, where t is in ab. Here, s prime is in Rn minus 1 varies, you get points. So x prime you view it as an s prime varies in Rn. Some set in Rn minus 1, not varies in Rn minus 1. So, that is the varies mean varies in a domain in Rn minus 1. So let us now try to motivate the Surface measure in R3.

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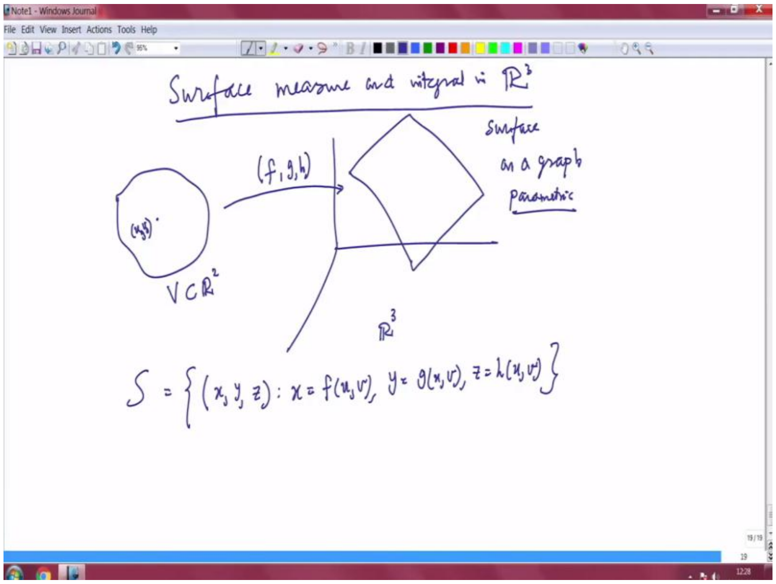

Surface measure. So let me first little bit motivation, surface measure and iintegral in R3. Motivation is fine. So you have a R3 here. And you have a surface there, assume it is a globally defined surface. So let us say it is a globally defined surface graph as a graph, so you view it as a graph surface as a graph. I said you can do it locally. So what is the meaning of that you have an, you have a region here, V, this is subset of R2, this is your R3 totally.

So you have R2 R3 here. So you fix a point here, u naught, v naught. And when you fix a point here, so this is your function f. So, f is a, f maps from V to R2, so your surface S is given by let me write it in a simple, this is the easiest way you can understand set of all points x, y, z such that x is equal to all graph for a parametric form, maybe parametric. That is also fine, but some little more general.

So x is equal to f of u, v y is equal to gf gh. So you have three maps g u, v and z is equal to h u, v. So this is your surface. So you have set of all point. So this u, v maps to this surface here. So let us try to understand this picture in one more nice way. So maybe go to the next page. With that we will do more later.

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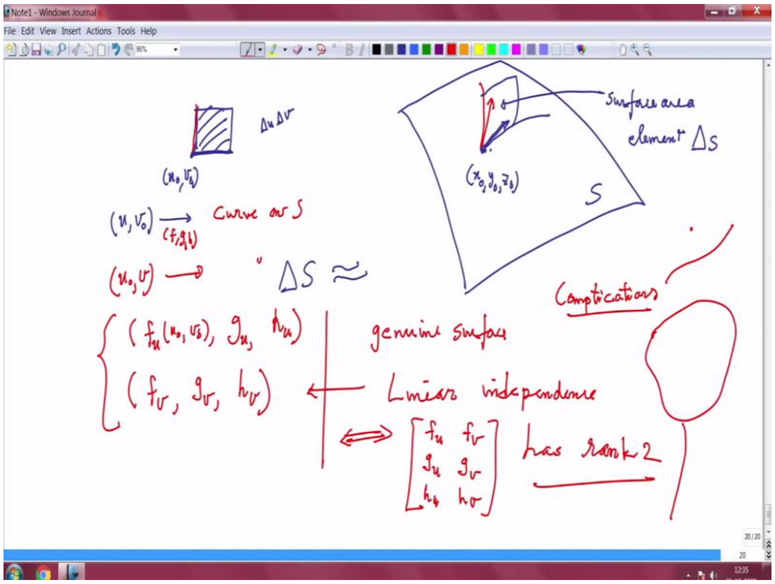

So, you have a region here, so, let me note that thing. So, you have a look a small region. So, you have a surface here, what is our whole understanding you have a surface here, I want to understand a small region here, how to measure a small region on the surface. So, you look at a small point, point here. So, this is your u naught, v naught and then well you look at the map, u, v not going. So, I am varying u, when I vary u, so corresponding to u naught I will have an x naught, y naught, z naught I will have a x naught, y naught, z naught.

So, when I vary u, along this line, it will move along the curve, so you see. Similarly, if I move along this line, if I move along this line, I will have another curve. So, u will not under f, g, h will produce a curve on S you see curve on S. Similarly, if I fix u naught and vary v I will produce under same thing a curve on S. So, what do I want to understand? So, if you have a small region here, so, if I have a small region here, it will move corresponding to that a small region here.

So, this is the exactly this region is the surface area element, I want to know approximately surface area element, if I call this 1 is delta u delta v and this is my if the surface is S that is my delta S. So, I want to approximately understand I want to have an approximation delta S approximately. If I know that one exactly what you have calculated d gamma square root of 1 plus f prime of x square you have to do.

So, for that I have to, what we have done in one dimension I have to understand f prime of x. So, I want to understand this is my tangent here. Similarly, I have to understand my tangent here. So, these 2 tangents are here. So, when I move this one the corresponding tangents will be fu at u naught, v naught I will get my gu and then I will have h that is one tangent and the other tangent is fv, same point gv and h, I have my hv.

So, I have 2 tangents, but there are, will be complications. Surface is defined as a map very generally; the whole idea can go to some points. So, the surface in general can go to a point, the entire surface can reduce to a cramp to a single point or it can go to a line, it is all possible under mapping, I can define the entire mapping from a region to a point, I will be able to define a mapping to a curve, otherwise, I may be able to define a curve like that.

But these are not genuine surfaces. So, what is the meaning of genuine surface? Genuine surface is the one, which we look for these 2 tangential directions. So, basically these 2 tangential directions, which you give here will define a tangent plane, I have to get a tangent plane. That is equivalent to saying that the linear independence of these 2, that is what you have to understand. So you have the linear independence of these 2 vectors will tell you the genuineness of the surface.

Otherwise, there will be degenerate cases. I can define a map from an R2 region, in R3, everything going to a point or a line or a combination all kinds of things can happen, degeneration can happen, that happens, so there may not be a surface all these can vanish. So you can have these 2 vectors can vanish if there is a constant function, so everything can happen. So, you have to avoid all that and you have to see that these 2 vectors obtain are linearly independent, which is equivalent to saying that, linearly independent equivalent to saying that you have your fu, gu, ghu, fv, gv, hv has rank 2.

So, I will continue this little more, I do not know how much time I need to finish. So we conclude for this (()) (36:08), so you get to know this kind of idea about the surfaces and the tangent plane and the normal plane which we are all required. So we will continue the picture in the next class. Thank you.