Partial Differential Equations - 1
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Lecture 40

One Dimensional Wave Equation

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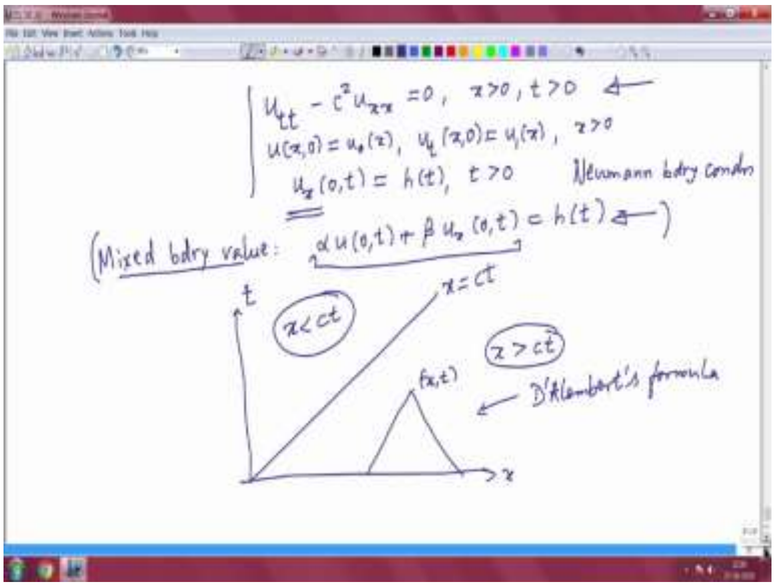

So welcome back. In this lecture, we continue the discussion on the initial boundary value problem and now we take the Neumann boundary condition for the wave equation in the first quadrant. So let me again write the problem. So wave equation, one dimensional wave equation in the first quadrant, initial conditions ut, x0 u1 x, again x on the positorial axis and now we consider the Dirichlet boundary condition.

So u0 u1 and h are given functions with appropriate smoothness. So now let us try to find out a formula for this solution and see how it differs from the earlier formula for the Dirichlet problem where u, the value of the solution on the boundary was provided. So the general, let me just remark before mixed boundary value problem, so if we prescribe a condition of this form, because of the linearity in the equation and because of the linearity in the boundary data, we just concentrate on the Neumann boundary condition.

So this is Neumann boundary condition. So again, the idea is same, so again you just consider the first quadrant and this characteristic x equal to ct. So again, in this region the solution has domain of dependence completely lying on the positive real axis, so the solution again is given by D'Alembert's formula. There is absolutely no problem there.

So again, problem comes here, seeing the previous case of Dirichlet boundary condition, we could use characteristic parallelogram property because the value of the solution was given on this boundary line. But in the present case we are not given the value of the solution, but it is first derivative, so we cannot use CPP here, we cannot use Characteristic Parallelogram Property.

So we have to proceed slightly different way and that I will show you. So again, let us concentrate what happens in this region. So x is less than cricketer, so there absolutely no problem in the region x bigger than ct.

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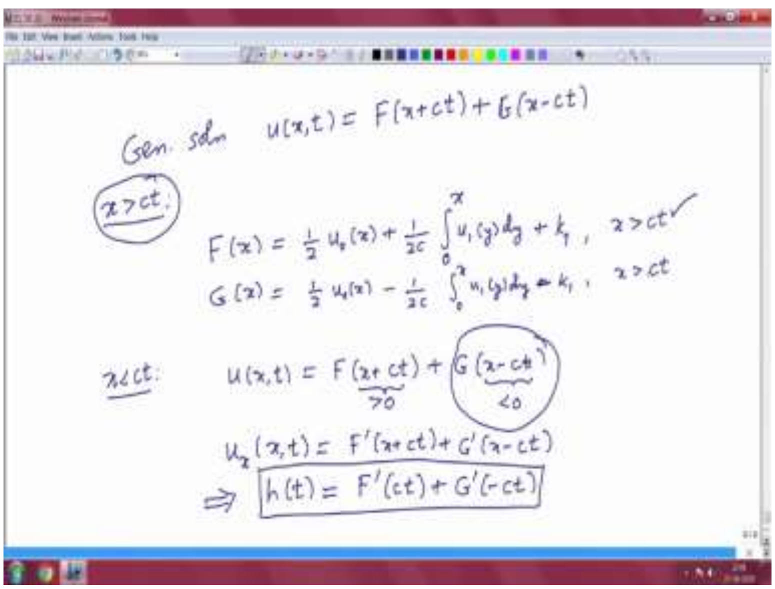

The general solution, again we depend on that now, so general solution, so u xt is given by F of x plus ct plus G of x minus ct. So as we did in the case of D'Alembert's formula, so again in the region x bigger than ct, so we just use the initial conditions. So as before, so let me write it, so you just check, this is what we did earlier, u 0x plus 1 by 2c 0 to x u1 y, so we can always put a constant here, k1, and now this is valid only for x bigger than 0.

And G is also similarly given here, so G of x, so let me write it, minus 1 by 2c minus k. That is how in this region, so x bigger than ct means x minus ct is positive and x plus ct is positive anyhow, so this we can use this F and G in terms of the initial data and compute the solution. So what happens when less than ct? So we still have the general solution, no problem there, F of x plus ct plus G of x minus ct.

So, x plus ct is positive, so there is no problem, so again we can use this formula for F. But when x is less than ct, so this is negative and this formula does not provide us with an expression for G, so we have to now invoke the boundary condition in order to find this G, so that is the only difference between the Dirichlet boundary condition and Neumann condition.

So here we have to do little more work. But we are given ux, not u. So let us calculate ux from this, so ux, so let me calculate at a general point, so this is F prime x plus ct and that is G prime x minus ct, and using the boundary condition we therefore get, so h of t is equal to F prime at ct plus G prime at minus ct.

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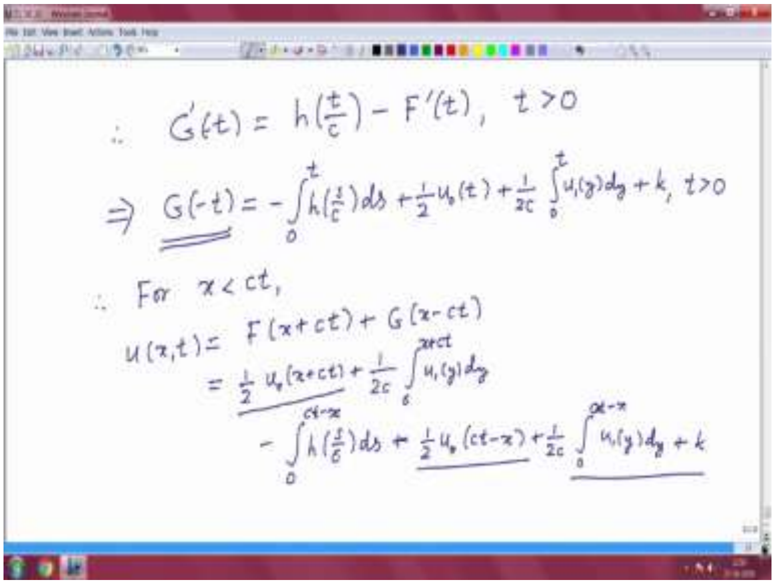

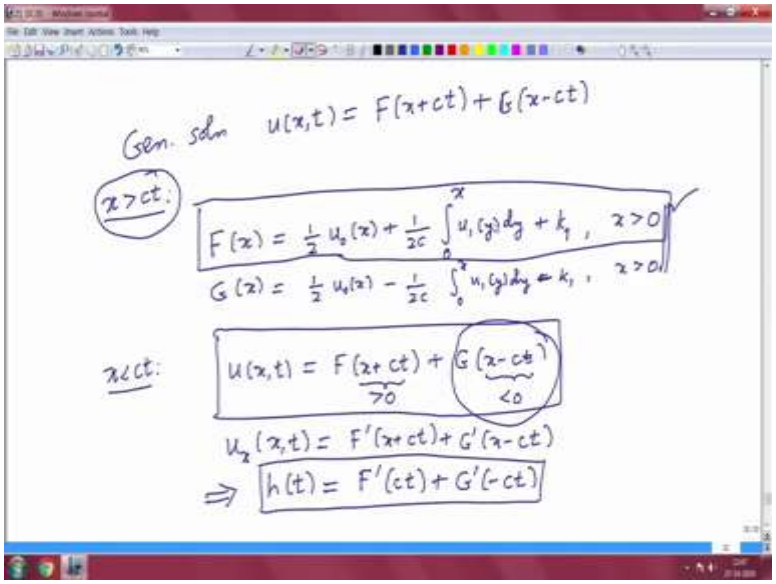

So therefore, G prime of minus t equal to h of t minus c, minus F prime t, [audio gap] and F is already known, so just remember that. So F is already known, because here also it is positive so for F we know, so you just take F prime there and substitute here. And do one more integration so there is a minus sign, so just watch for that thing, so we get G of minus t is equal to, let me write it, minus 0, this minus is coming because of that minus 0 to t, h of s by c discuss.

And this simply get F and I just substitute, again there is plus because of this minus, so that you have to just observe. So the signs are important, so we can always add a constant of integration, so k is just an integration of constant and now this is valid for t positive. So we have obtained now an expression for G of minus t when t is positive. So again, go back, this is our expression, you simply substitute that.

So therefore, for x less than ct, u of xt, so let me just write that, so this is just F of x plus ct plus G of x minus ct and this is just, let me write once again, so this is half of u0 x plus ct plus u0 x, so again go back, so this is very important, this is the expression for F of x, so I am replacing x by x plus ct, so just remember that, plus 1 by 2c, 0 to x plus ct, u1 y dy. This is for F.

And now, so this is minus t, so I write this as minus ct minus x, that is positive, so just to get minus 0 to ct minus x, h of s by c ds plus half of u0 ct minus x plus 1 by 2c, 0 to ct minus x, u1 y dy plus a constant of integration. So let us now there is half u0 x plus ct there, plus half there and now these two integrals, 0 to ct minus x and they cannot be combined, so that is ct, we can still do that, ct minus, but this is a plus there, so there is no cancelation, so that is one important difference, so we will just write the final formula.

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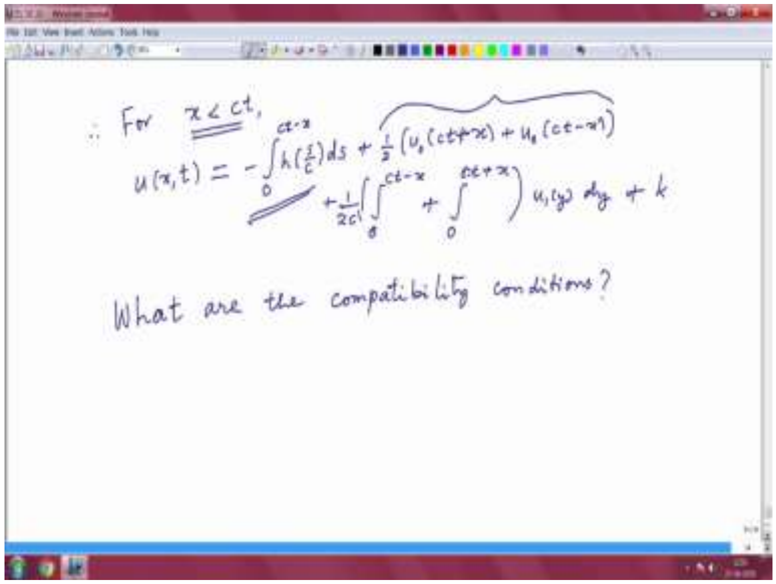

So therefore, for x less than ct, u of xt is equal to, so first let me write the boundary data, so this is ct minus x, h of s by c, ds plus half u0 ct plus x plus u0 ct minus x. And now we have this half 1 by 2c, 0 to, two integrals, ct minus x, so let me write that also here, 0 to ct plus x, there is no cancelation here, so we have two integrals here of the same function, integrate is same, let us briefly write that, and there is always.

So earlier that was not there, so now there is a constant of integration. And this is valid for x less than ct. So just little, this term is similar to one in the D'Alembert's formula, there is no change, but there is slight change in the integrals, so it is more, this part is more like D'Alembert's formula and this is coming from boundary term.

Again, so it is more work and more algebra, so you can find out in this case what are the compatibility conditions. I am not going to do that, so you just, because now have again expression for the solution, one in the region x bigger than ct and another one for x less than ct and now you again take the limits of the solution, its derivatives and try to find out what are the compatibility conditions, some more algebra and just compatibility. That just you work out.

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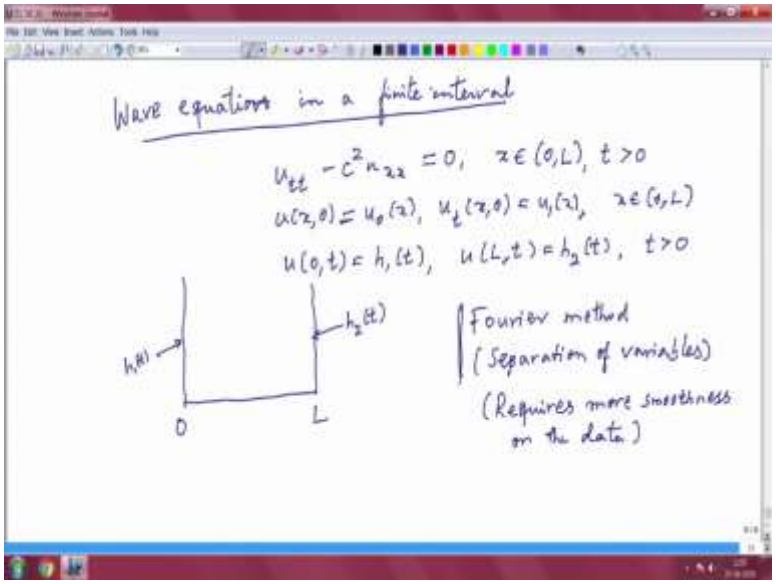

So finally, just to make wave equation in a finite interval. See all these developments eventually lead to wave equation in a finite interval. Let me state the problem again, so now again, the wave equation, and now x lies only in a finite interval, so call it 0L, so we can always bring any finite interval into this form, so let us, it is operating here, so we will just write 0L.

So again, initial conditions, ut x,0 u1 x, x is in the interval 0L. And now there are two boundary parts, so we have to provide 0t, so let me call it h1, t, simplicity again, Dirichlet boundary condition, so we can also take Neumann boundary condition h2, t. So this is case of a finite string which is placed on the interval 0L, and some initial profile is given and initial velocity is given in the string, and now so you provide the data here.

So here it is h1, t, h2, t. Generally, such problems are approached by Fourier method which is same as separation of variables. So you seek a solution in the form of separation of variables and then manipulate and then try to get the Fourier series expansion. That is why it is Fourier. I will show you an example, but let me here describe this method generally requires more smoothness.

I will show you and some nice formulas do not yield c2 solution, requires more smoothness on the data, the initial conditions are boundary conditions, we will see that.

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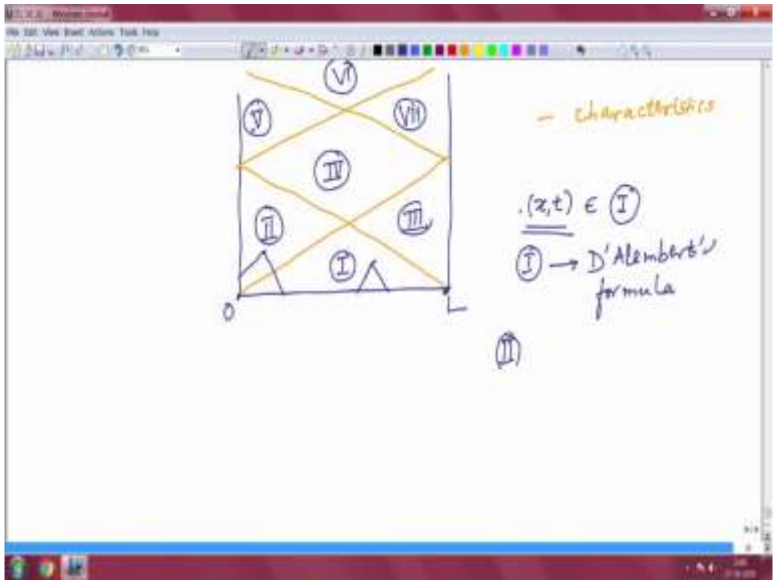

So let me show you here quickly how we approach this problem just using the characteristics and how a solution can easily be obtained, so though no explicit formula can be written in this case, but depending on t we can determine how the solution is determined. So let me write again this 0L and now you draw the characteristics, so let me just use different color through these points and keep on doing that. So these are characteristics.

And you see how the characteristics play an important role in determining the solution of the wave equation in several different problems. So we saw it on the entire real line, first quadrant and now the finite interval. So let me call them 1, 2, so this infinite cylinder, if you want to call this infinite rectangle, so it is divided into several regions.

So now depending on where the point xt lies, the solution is determined in different ways. For example, if this belongs to the region 1, that is the first one, so here is domain of dependence lies on the interval 0L and so in region 1, we can simply use D'Alembert's formula. And in the region 2, now one characteristic hits the boundary line x equal to 0 and another one comes to the first region, so we can repeat the argument in the case Dirichlet problem for the quadrant, so we can obtain a solution in region 2 using the boundary data only on x equal to 0.

And similarly in region 3, and as you go up, so there will be more interaction between both the boundaries and in principle we can obtain a formula at any point xt, though there is no simple formula to write down for any xt, it depends, first we have to find out in which region it belongs to and the we have to work backwards till we get to the region 1. Though we are not able to write explicitly but still there is a procedure how to obtain the formula.

And here, we do not require more smoothness as compared to Fourier series, that I will briefly discuss next time. And again, suitable compatibility conditions, now there are two interaction point, 0 and L, need to be determined in order to show that the solution is a c2 function in the entire region. Otherwise, if compatibility conditions are not satisfied then u, R as derivatives will not be continuous across this characteristic, now there are plenty of characteristics.

So earlier there was only one, now there are many, many characteristics. So with that thing I will come to an end of this talk, so next time I will show you how Fourier method is applied and what are its limitations and also show you some examples. Thank you.