Partial Differential Equations - I
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Lecture - 04
Preliminaries - 2

Welcome back again. So, let me continue what I have stopped in the last class, we have introduced the concept of a Frechit derivative or a total derivative. A function from R to R is said to be differentiable at x naught if there is a linear transformation satisfying the certain existence of a limit. And if not that every function is differentiable just like in Rn you do not get it, but once a derivative exists as a linear transformation that linear transformation is unique.

So, let me just state one theorem, so you to understand these results not just by reading the theorem you work out examples this is why it is preliminary, we are not getting into the details work out more details about it. So, let me state one important theorem here what is called the before the main theorem, we will call it a Chain Rule.

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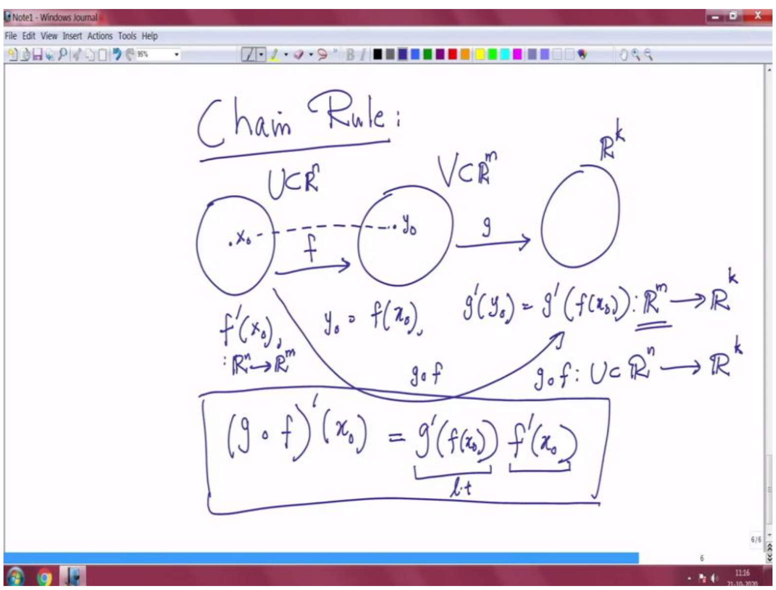

I will not, you will write a Chain Rule let me make it bigger. So, you have Chain rule. So, I do not want to give a explicit assumptions etc. Please go and look into the literature, before which the references are already given. So, you suppose you have a 2 set, 3 sets here. So, you have your

U here, V here and U is a subset of Rn and this is a subset of Rm and this is some Rk in either full set here.

And from here to here, you have a map f and you here you have to have a map g. So, f is a map from u to Rn to Rm, so, you can talk about suppose x naught is a point here. So, you want to talk about f prime of x naught and this x naught going to y naught here. So, so, what is y naught? y naught is equal to f of x naught. So, you can talk about g prime at y naught. So, that is nothing but g prime at f of x naught. So, I can compose it so, I can compose the map from here to here that is nothing but g composition f. So, g composition f is a mapping from U subset of Rn to Rk.

So, I can talk about g composition f its x naught what it would be? So, the chain rule tells you that under conditions of course, you have to make it as a theorem which I will not do it here. So, you need that f is totally derivative at x naught, g is derivative at x existence of derivative then g composition f is differentiable. So, you take your g prime at f of x naught and then you compose it with the f prime at x naught.

So, this formula of course, this is a linear transformation keep this understand that this is a linear transformation from Rm to Rk, you have to understand that. And this is a linear transformation from Rn to Rm, that is fine. So, you see this will be a linear transformation from Rn. So, the x naught this will act on points on Rm Rn to produce elements in Rm. So, this when you act on y, it will give us an element here and that then g prime of f of x naught will act on element on Rn. So, it will give you an element. So, they a relation this chain rule is extremely useful.

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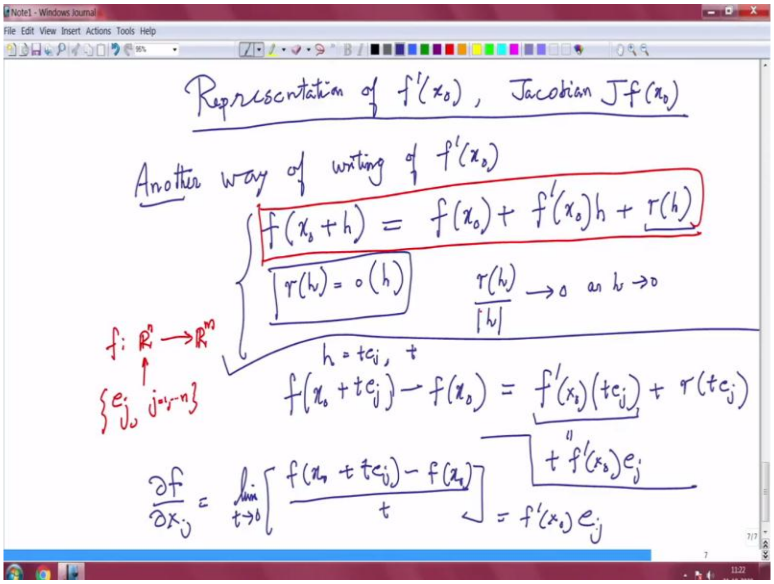

Now, I will want to give some representation quickly before coming to the theorem. Representation of f prime of x naught and also is called the Jacobian which you use it. So, I want to make a distinction between something like that Jacobian of J, this is Jacobian is nothing but the telling you the Jacobian of f at x naught, I will call it. Here before going to the, I am going to that only before that I want to reduce one more thing I want another way of writing the same thing you can rewrite in this form, another way of writing of f prime of x naught.

Here there is a some important subtle point which you have to understand, it says that you look for points nearby f of x naught as an a representation with f of x naught and plus this is the Rm value f prime of x naught acting on h, because this is a linear transformation plus I write a term rh this is just not enough, quite often, we may try to write this f of x naught here and they say that this is small, that is not enough, you need a another condition rh should be sufficiently small.

Just rh small, it is not enough you need to have rh we call it as small order of h. What is the meaning of this? Small order of h means, rh should be in some sense smaller than h this should go to 0 as h tends to 0. So, if you get an expansion of this form with rh is equal to something like rooted something smaller than h or h is not enough you have to write these should be in order smaller.

So, then only you will see that you may be able to write f prime of its matrix form which I am going to represent but that is not enough as you said existence of partial derivative or directional derivatives may not be good enough to do that one. So, this, this is another way of writing here from here I can immediately see this.

So, suppose I take ej, where j is equal to f is from Rn to Rn. In fact, I can do something more general I can write from Rn to Rm even for that you can write your matrix, it need not be from Rn to Rm still f prime of x naught is a linear transformation, though I have written initially for linear transformation from Rn to Rm, but I am saying that you can take linear transformation Rn to Rm or from other only non linear spaces and so, suppose j equal to 1 to n is a basis of Rn, similarly, you can write down that one here.

So, look at this f of x naught h is a vector here. So, you have to rh by mod h you have to write. So, this is a vector, h is a vector in this higher dimension, h is no longer a scalar but here I am writing h equal to t ej, take h equal to t ej, then apply in this equation, we apply in the above equation you can get it f of x naught, f of x naught plus t ej by that formula it is f prime of x naught plus acting on t ej plus of r of t ej where t tends to 0, t tends to, sorry, I wrote it t, it is h I am taking it as h. So, you have r of t ej.

Now look at this one, this is linear. So, I can write this equal to t into f prime of x naught acting at ej. Now, you take limit by dividing by t tends to 0, if I take these f of x naught plus t ej minus f of x naught by t. What will happen to this term? t I have divided, if I divide that one I get exactly f prime of x naught ej and this r t ej by t tends to 0 by my second condition. So, you see, so, this is nothing but my partial derivative. So, I can require all my partial derivatives by the action of my linear operator on the basis element, standard basis element.

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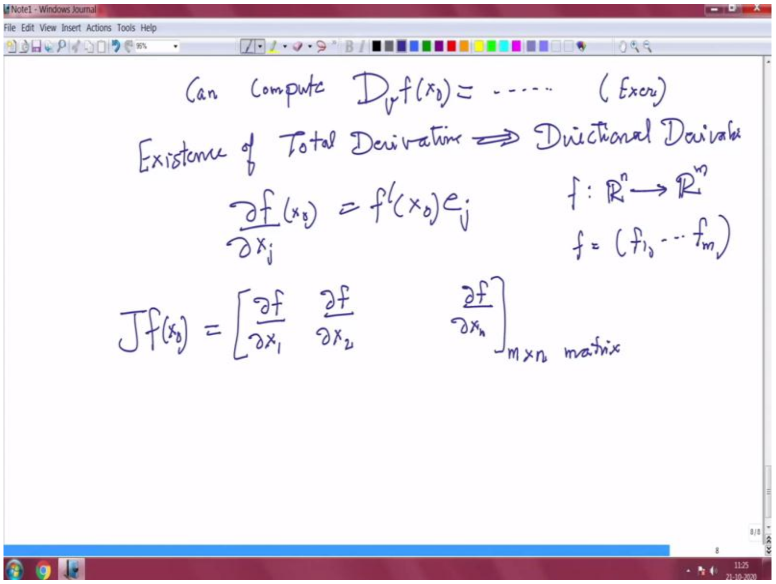

You can actually compute, can compute this I will leave it as an exercise, can compute not only the partial derivative you can also compute in any direction of f so let me write it, write it here. So, you write here precisely what is that, that mean the existence of total derivatives implied existence of partial derivatives that is what I write, existence of total derivative, derivative imply directional derivative. So that is the thing, but converse you already seen that it is not possible to do that.

So, let us so, the existence or direction derivative is not enough to obtain the existence of total derivative, you need some more assumptions on the continuity please refer the books to precisely understand. So, what I want to do is that let me write down this df by dx j I at x naught I have f prime of x naught, I want to give a representation now, so as a linear operator, but we call f is a mapping from Rn to Rm, So, that I can write my f is equal to f1 etcetera fm.

So, when you look into equations like fx equal to 0, it basically gives you n equations in n unknowns, that what you will get it eventually. So, now I arrange this, this is actually it is a vector m vector because m has a component I arrange it as a column vector. So, I arrange df by dx 1 this is a column vector of df by dx 1 is a column vector with n vectors I do it df by dx 2 here up to df by the variable is xn.

So, this column you have m this will be m by n matrix and this is my matrix representation of f prime of x naught. So, I also use my f prime of x naught, which we call it the Jacobian, so we call it I will not do that, so this I call it the Jacobian of f at x naught. So, essentially all information are stored. So, this is where you have to be certainly careful, as I said all df by dx j will exists, but it will not give you the linear transformation because that equation with error term which goes to a small order of h may not be satisfied.

So, this point you have to be limited it says this subtle point you should be very careful in dealing with it, verifying the continuity the additional conditions additional procurement for the existence of total derivative you have to be careful and you have to check it. So, this is how you get, these are all you would have learned it in linear algebra whenever you have a linear transformation from finite dimension to another finite dimension based on it basis, you can represent it as a matrix.

So the Jacobian representation Jacobian is a matrix representation of your linear transformation f prime of x naught. So the total derivative or Frechit derivative is the linear transformation, it is realization through basis, is its Jacobian. So, that well is Jacobian is a some sort of a realization of your linear transformation. So, with that, we will go to the Inverse function theorem. So, let me go to the Inverse function theorem.

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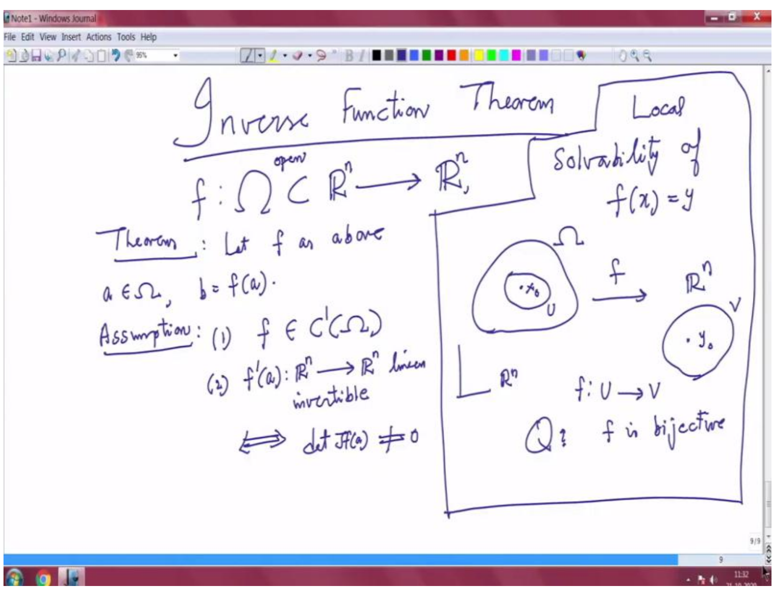

Inverse as I said I do not venture into the proof part I in this other 10-15 minutes, there are 2 theorems which I want to state I do not know let me possible to complete, but then let me do it. So, let me give you an indeed new idea and then probably state that theory. So, you have your function f which is in a neighborhood omega of Rn to Rn, so the both are Rn, be careful what we are writing.

And you want to know the solvability that is whole issue solvability, in fact, local solvability, solvability of fx equal to 1. So, you have a domain Rn here, this is in Rn and you have a map here to this is in Rn. So, you may have a point x naught this is what we are looking at x naught and that goes to a point y naught here. So, the idea is that you want to so this is omega. So you want to look for a neighborhood U and you want to look obtain a neighborhood V.

This is our aim, I am looking for a aim and this is my V. So I restricted my f from U to V. So the my question is that, question, can I have such a thing, question f is bijective? I want to do such things. So, this is my objective solvability of local solvability. So, this is what I am trying to do a theorem here. So, I am going to state a theorem now, this is the I hope you understood this is what exactly we have done it in one dimension when f prime of a not equal to 0 and f of a equal to b and then you have neighborhood of a in neighborhood of b which are intervals and then f is a bijective mapping.

The inverse theorem tells you and I put condition here and inverse theorem, function theorem tells you the inverse map is also continuous which I will state it as a theorem now. So, please understand the theorem well, and then try to understand the proof. So, let f as above, let the f as above and a belongs to omega, omega is open, open all the boundaries set and all that is not necessary, but open is important and b is equal to f of a instead of x naught y naught I am using a and b.

So, I have two assumptions understanding assumptions are very, very important. I will explain to you one of the assumption without theorem without the proof assumption one; f is a C1 function you have to be careful what is the meaning of the C1 now? I will tell you, so, existence derivative is a local point, so, when I say that function is differentiable at each point and that is one thing that would not give you C1.

So, f of a is defined in an open interval, when I say f is C1 of ab, I need some continuity of the derivative. So what how do you put that continuity here that I will explain to you the term. The second assumption that is crucial look at the derivative only at the point a, exactly y so, it looks like similar, but now, f prime of a is a mapping, linear mapping, linear and that you already know that since f prime of a is linear and I want to see that this mapping is invertible.

In terms of matrix, this is equivalent to determinant of this Jacobian that J fa which you write and is not equal to 0. So, f prime of a is the linear operator from Rn to Rn, I want the invertibility of that 1-1 (())(20:46) and all that and also telling that determinant of that meant these are the 2 assumptions. One is even a, then the conclusion, let me tell you.

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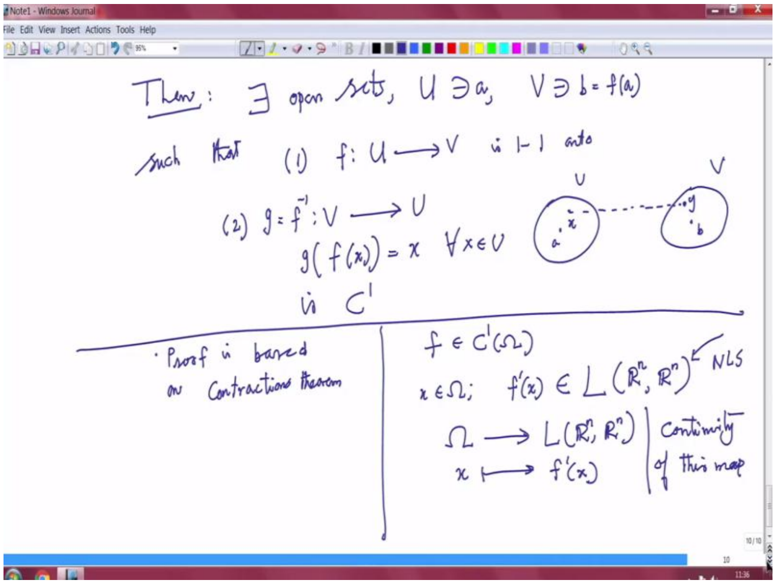

Then and then I will come back to that, then there exists open sets U that is containing a, V, that is containing b, which is f of b, a, b is equal to f of a, such that, such that exactly what I stated which I am writing, once again, there are two things one; f from U to V is 1-1 onto, that means the problem, there is a for every y in every y in V, which is a neighborhood of P, that is what you have saying.

So, you have U here, a here, you have V here, you have b here, this is a, this is b, what it says that for every y here, there is an x here, unique x, every y there is a unique x here. Let me put it in another color. So you have a y here, you have a point y here. You have a point x here, this maps to this part your see, so you have a unique, what it says that a second thing, since it is a 1-1 onto I can define my g equal to that is my f inverse V to U. How do I define my g of y?

g of, so if I take a point x in U, it will go to a f of x, and then that g will send that f of x to x back. That is the meaning of the inverse, g of f of x equal to x for all x in U. That is for every x for every V in, every point in V is obtained from U by the f, that is what it says it will be f of x, I am assuming that, so what is the condition this is only the definition it says that is C1 map. So that is a theorem.

So, I will not as I say the proof is based on what we call it contraction principle, of course, it involves some amount of work is not easy directly to prove it but contraction theorem, contraction fixed point theorem. It may be you all of you know that. But I want to tell you what is the meaning of f is in C1 that I just tell you and then I will go to the Implicit function theorem. f is in C1 means, so every point x in omega, I can talk about f prime of x.

f prime of x which is a, where is f prime of x? f prime of x is a linear transformation from Rn to Rn or linear operator form Rn to Rn please understand that. Soon I can associate a map from omega to L of Rn, Rn, some books you write just L of Rn in both spaces are there, where x going to f prime of x. And I want and this is a non linear space, you can make this to be a non linear space, and hence there is a continuity. I want continuity of this map that is the meaning, continuity of this map.

So, it is not just about the existence of f prime and mapping x going to f prime of x as a map from omega to a L of Rn to Rn I need theorem. Now, with this, we will go to the final part of this talk and what we call it the implicit function theorem and then I will stop it. So, that also requires some work. So, another 5 minutes or so, we will do that

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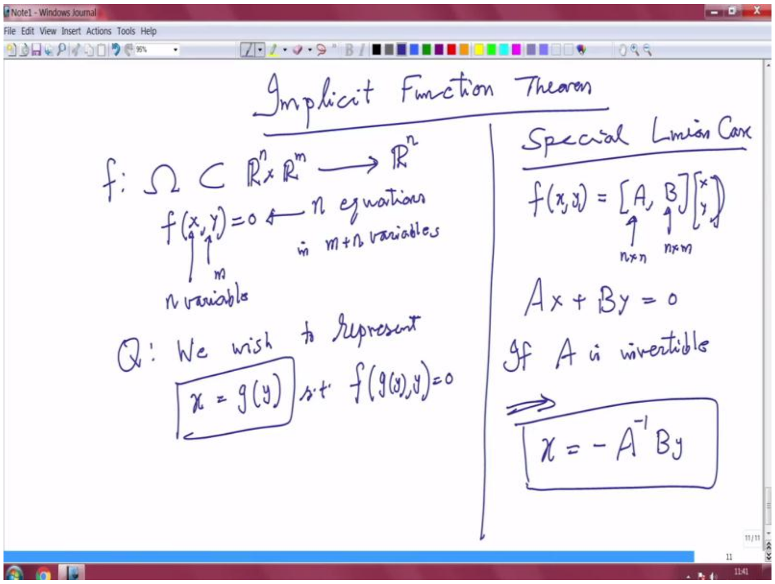

Implicit Function Theorem. So, what is the setup? So, I am providing a setup and then we will quickly do that one so, you have an f of x omega now, omega is a subset of Rn cross Rn. So, you have a map from Rn keep that tradition. So, what does this means, if you have an equation f of x y equal to 0 this basically m n equations, n equations in m plus n variables please understand that. So, you have n variable here, n variable, here you have m variable.

So, naturally you cannot solve it, you have only less number of equations than the unknowns. You have n plus n unknowns and you can you have only n equations. So, naturally you can only expect it to represent n variable in terms of the remaining m variables. So, it is only so, the so, the question is we wish to represent x in terms of y, so you want to solve this such that f of gy y is equal to 0.

So, that is what precisely you do it when you have an x square plus y square equal to 1 you want represent y in terms of x or x in terms of y. So, here you want to represent the n x variable in terms of the n m y variables. So, you looking for this kind of solvability. So, let me look into the special case linear case special linear case. What is the meaning of f of xy is linear? It exactly tells you that you can write it as linear equations like this A B, x y this is n by n this is an n by m matrix.

So, you get the n by m plus n matrix that is true f is from Rn to Rn plus Rn to Rn is linear, then the matrix representation is an n by m matrix. So, this equation the solvability of linear equation is Ax plus By equal to 0. So, naturally you can solve so, if A is invertible, invertible that implies you can solve your x is equal to minus A inverse of B of y. So, that is basically your linear form of the theorem. So, let me so, you have the linear form of a here. So, I want to do a linear version to continue, I want to write down something in a better some other way decomposed my matrix.

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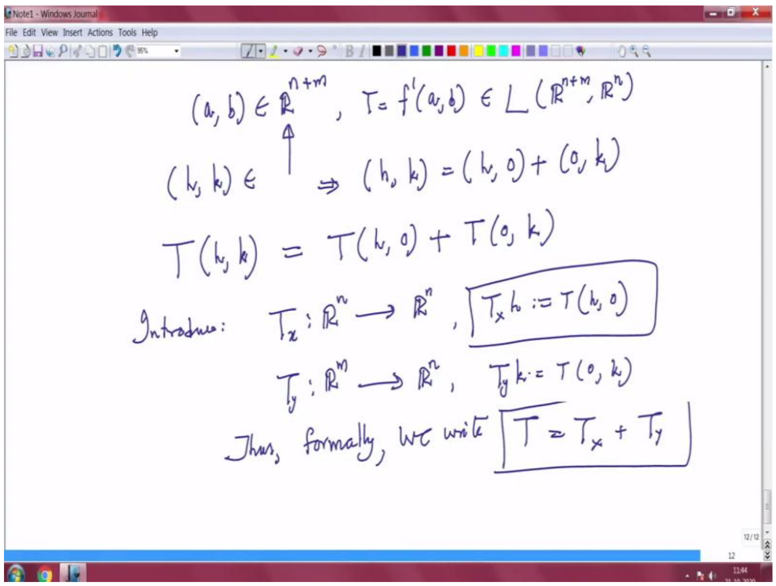

So, let us start with a b in Rn m cross n or n plus m, both are fine, you can write it. And then your f prime at ab is nothing but a linear transformation that we know R n plus m to Rn, it is the same, it is fine. So let us look at the point h k, h k belongs to Rn plus m which I can write it as h k is equal to x 0 this is the operations in R n plus m 0 k. So h is in Rn and k is empty.

So, let us look at the mainly in this I call it T for the convenience T means derivative. So, I can a T will act on h k and T is linear. So, I can separate it T at h 0 plus T at 0 k. I can write like that. So, this suggests to me introduce to linear operators, introduce to linear operators Tx from Rn to Rn this is a linear operator. So Tx will act on h, h is an Rn vector is nothing but this is by definition T of h 0 is the definition, nothing else and another operator Ty from Rm to Rn such that Ty of k is equal to by definition T of this is 0 k.

So, the previous thing does formally we write you have to be careful formally why I say formally we write T is equal to Tx plus Ty this is only a symbolic representation. So, in a be careful T is from Rn linear transformation from Rn plus m to n this is Rn to Rn this is Rn to Rn. So this representation meaning is what we have written here.

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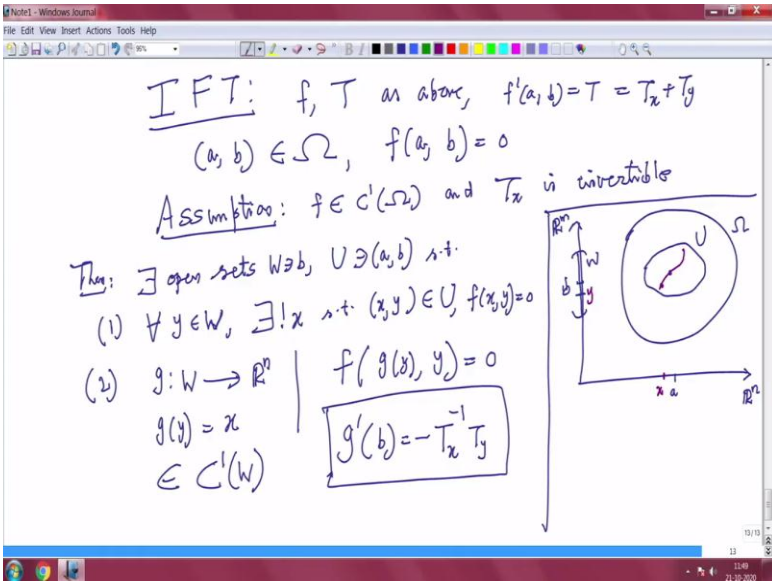

With this now let me state the Implicit Function Theorem. So, exactly so, we call f as above f, T f, T as above, ab you know that the f prime of a b equal to T and then T can be written as Tx plus Ty recall all that whatever just now stated here, where ab belongs to these are all assumptions ab belongs to omega. And therefore, for ab equal to 0, so that is what you look at it f of ab, so I want to fall in a neighborhood of b.

So, the first assumption is this is a with all that assumption, two assumptions as above; f is C1 assumption, f is C1 of omega and this part Tx part is invertible. And then the theorem tells you then let me just take there exists open sets, open sets W, which is a neighborhood of b and U, this is a neighborhood on ab, you have to be careful. It is a neighborhood of ab. Such, such that 1 for all y in W you see you can solve it there exists is unique a such that ab is in U.

You have to state like this, ab is in U, you need there exists a unique x such that ab is in U and f of ab equals to 0, f of xy, f of xy equals to 0. So you see, you can solve that one. And secondly, I can associate the map now g from W to Rn, what is my map? Where gy is equal to x. So, that is a unique solution. Again, I made a mistake, this will be xy we are doing a little bit in a hurry, because I want to finish it time is up.

So, you have g y, such that, so this g for which is all the x is y, so that implies f of gy, y is equal to 0. And you can also compute my g prime of b, g prime of b and this map is C1. So, you have g C1. C1 of W and g prime of b is nothing but minus Tx inverse of Ty a. Again g prime of this map a, so the corresponding component. So, let me finish by just giving a geometric direction, the geometric interpretation that is it, I cannot do much here.

So, you have a omega here, this is Rn where x belongs to this is my Rm where y belongs to and you are given a point a here, given a point b here. So, basically you are given a point ab. So, what this theorem tells you that there is a neighborhood here, there is a neighborhood here, there is a neighborhood here, this neighborhood is your U, this neighborhood is your W. So, you have a name and what it says that for every y here, so let me write in color every y here, there is an x here, such that xy here, that is what.

So, you have y varies along this thing, you will get x varies along you see, so, y varies in this neighborhood of W, x will vary here. But then xy will vary here. And the C1 ness of C1 and the map g is C1 tells you that there is a smooth curve passing through that. And that is a solution to this one, you see. So, that is exactly what you are getting it, every y in that neighborhood of b there is an x such that xy is in the neighborhood like and this varies very smoothly and giving you the solution curve there.

And as I said all these can be extended, but the you get familiarized we are gone through very quickly and probably you have to spend some amount of time to getting grasping and understanding this multivariable calculus is extremely important not just to study partial differential equation to study many other mathematical part the analysis of multivariable calculus studying that in more general setup is important.

So, we will stop this here in this, in the next couple of lectures that is in one hour. We will do another preliminary namely the Surface Measure in Integration there it is more delicate. I will be only able to give some intuitive picture about it. And it is your job is to learn that thoroughly. Thank you.