Partial Differential Equations - 1
Professor A. K. Nandakumaran
Department of Mathematics
Indian Institute of Science Bengaluru
Professor P. S. Datthi
Former Faculty, Tata Institute of Fundamental Research - Centre for Applicable Mathematics, Bengaluru
Lecture 39
One Dimensional Wave Equation

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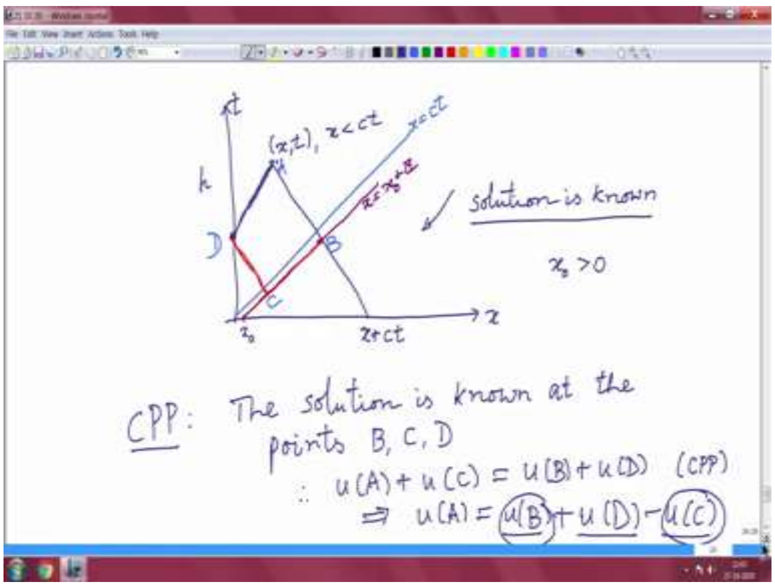

Welcome back. We will again now continue discussion on the Dirichlet problem which I started previous lecture, let me again just, so this is the Dirichlet problem we are trying to find the solution. So we have to find the value uB and uC, uD is provided by the boundary conditions.

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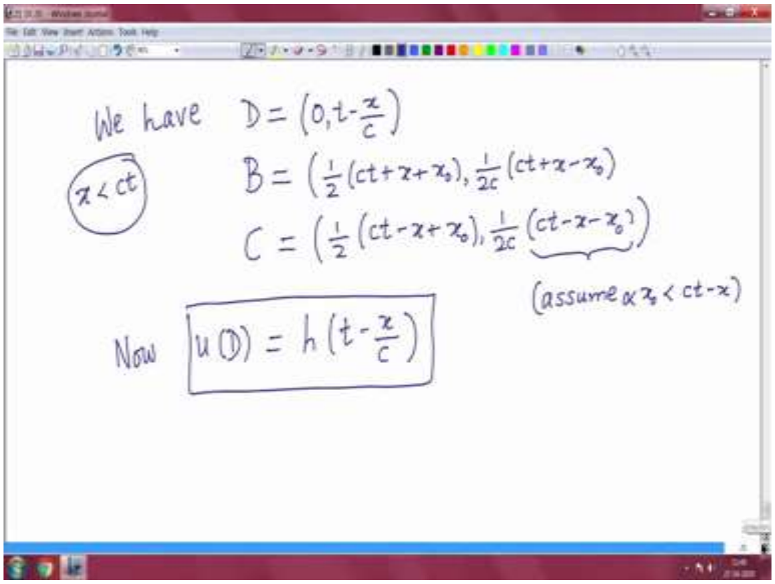

So we have, so let me write the coordinates of the point D, B, C. So D is nothing but, so it is on the boundary line, so that is x equal to 0 and its t variable is x by c. So here the characteristics are straight lines, so we have to just compute the intersection of two lines, so it is a little bit of work, but you do it. So that is no problem there, and, no this is not, let me just add in some notation here. That is D, D is, that is fine.

So then B, so this is the intersection of the characteristic passing through the point x0 and the characteristic passing through the point xt. So this is, we have to do some algebra here, I am skipping that algebra, plus x minus plus x0, 1 by 2c, ct plus x minus x0. So we have to remember this, we have to remember x is less than ct, we are in that region.

So this t minus x by c is positive, so it is in the positive t axis and the point C is half ct minus x plus x0, 2c, ct minus x minus x0. We want both this B and C in the region, in the first quadrant of the region, so the s coordinate and t coordinates obviously are positive. So for this reason, so this in order that this is positive, so we assume, so this is the small s, x0 is at our disposal, x0 is less than ct minus x.

And ct minus x is positive so I can always find an x0 satisfying that condition. So there is no problem finding the x0, and soon we will see that x0 does not appear in the final formula for the solution, so it is just like a catalyst. So now u of D is simply h of t minus x by c, D the point on the boundary line, so there is no problem at all.

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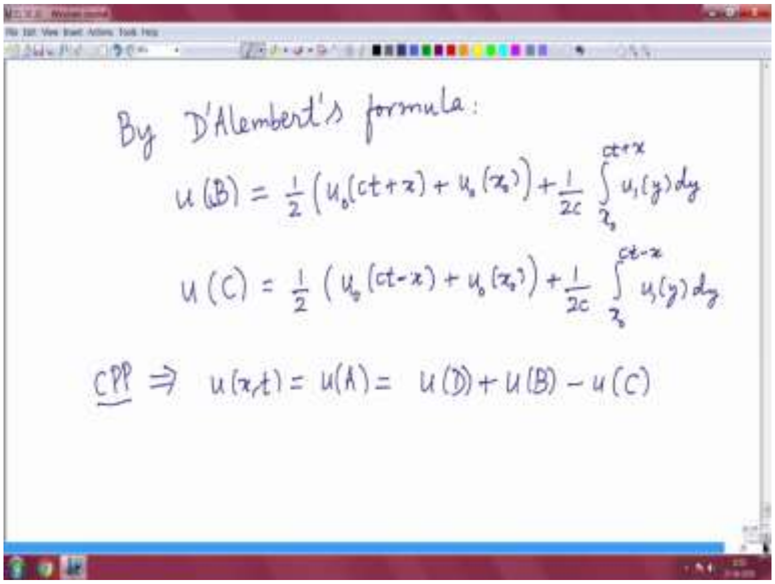

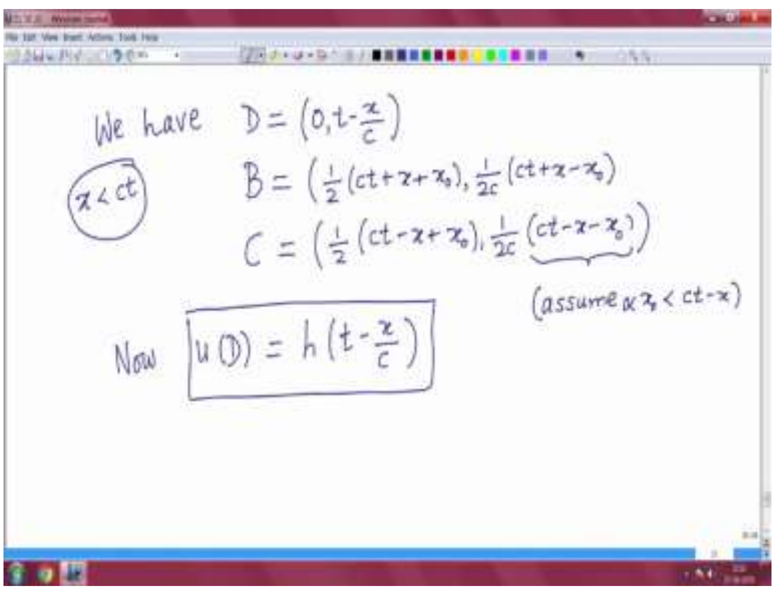

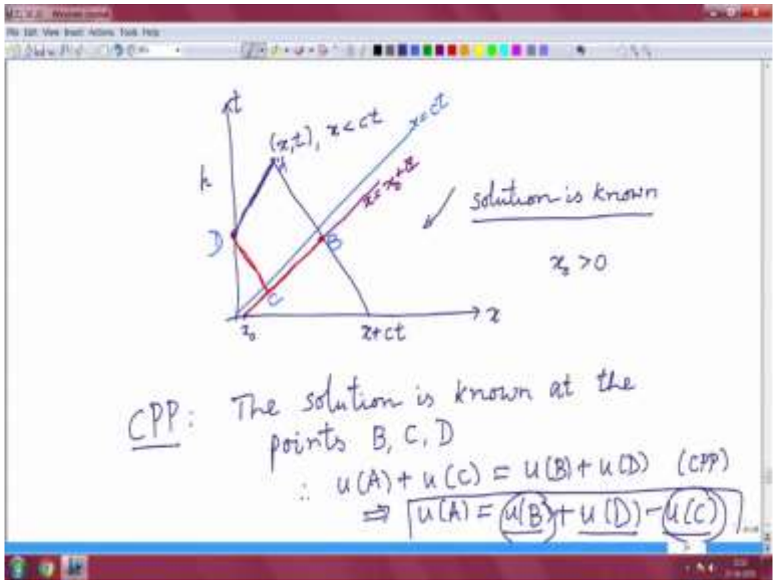

And now again I want you to do some simple computation, so by D'Alembert's formula, so you see how the D'Alembert's formula is repeatedly used. We have u of B, and u of C, u of B is half u0 ct plus x plus u0, x0. Just you find the domain of dependence of this point B and just apply the D'Alembert's formula, that is what I am doing, u1 y dy, some dummy variable of integration and similarly u of C is half u0 ct minus x plus u0, x0 dy.

So we have found the solution at all these three points, so at u of D it is provided by the boundary condition and at B and C it is D'Alembert's formula, that provides the solution. And now again you go back to CPP, so u of A is given by this, so just plug in there, that is all. So CPP, let me recall that, implies u of x,t, x,t we have denoted by the point A and this is just u of D, let me write that, u of B minus u of C. So we have the expressions for all these things, so let us put them together.

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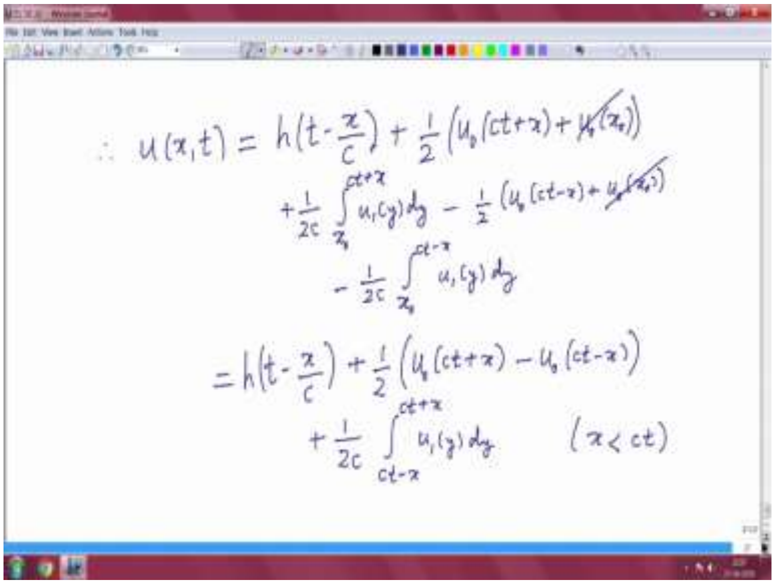

So therefore, u of x, t, u of D which is nothing but t minus x by C, plus u of B, so let us again recall that, u of B is here, half u0 ct plus x plus u0, x0 plus 1 by 2c, x0 to ct plus x, u1, y dy. So this term will be u of B minus half u0 ct minus x plus u0, x0 minus 1 by 2c x0 to ct minus x, u1 y dy. So let is simplify that, so that u0 half u0 x0 there is minus half 0 u0 x0, so x0 we can choose any arbitrary point, so it does not influence the solution at all.

So this h of t minus x by c plus half u0 ct plus x, now there is minus here, so u0 ct minus x. And now we can combine these tow integrals, so this is the integral from x0 to ct minus x, this is from x0 to ct plus x, so we can write this integral as x0 to ct minus x and then ct minus x to ct plus x and one term cancels with this, so we have just 1 by 2c ct minus x ct plus x, u1 y dy. So again, remember, let me stress that, so x is, this is in the region above the characteristic x equal to ct, so it is x less than ct. So let us combine both the formulas, so let us write in single formula.

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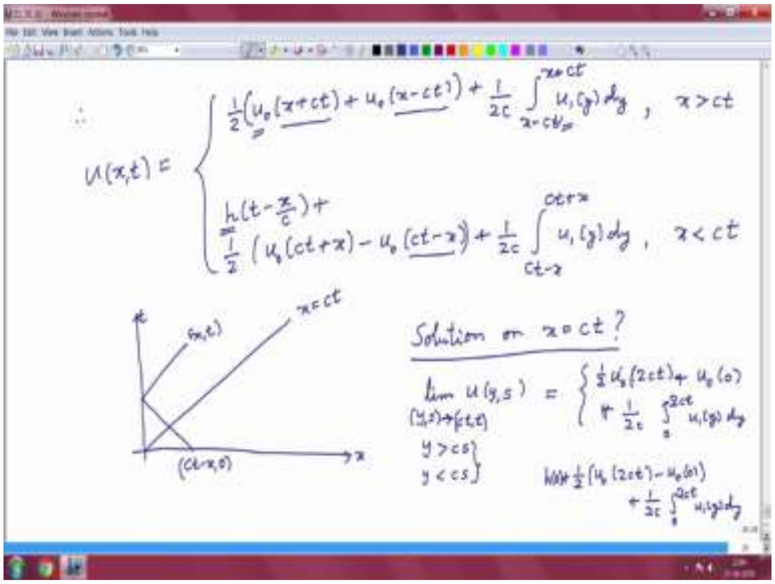

So therefore, we have u x, t depending on the position of the point, so in x bigger than ct is given by the D'Alembert's formula, so let me write that. This is for x bigger than ct. So when x is less than ct we obtain this one, so there is boundary thing coming, so let me write that separately, plus half u0 ct plus x minus ct minus x that is plus 1 by 2c.

So the expression for the solution is given separately for the regions x bigger than ct and x less than ct, so what about x equal to ct? So that part still missing and before that, so we understand what is this x minus ct and x plus ct, but what is the ct minus x? So let me just geometrically show where does that lie, when x is less than ct.

So again, just x equal to ct, so here is the point xt, so there is a characteristic meeting the boundary line x equal to 0 and then you draw the, so it is kind of reflected characteristics and this the point ct minus x0. And that is the reason it also appears with a negative sign here, so watch that one. So what about this solution on x equal to ct?

So we can simply take limit of the solution, so we have found the solution in this region and this region, so we simply take limit from either side. So obviously we want solution to be c2 and that imposes certain conditions on the boundary data and the initial conditions. So two expressions are here. So for example, let us compute the limit u x,t as x approaches ct or in fact you can even do better, you can take y, s, y, s approaches ct, t, t is any positive number.

The only condition is that. So we are taking limit when y is bigger than cs, and y is less than cs, so we have to take two limits now. So you get when y is bigger than cs we are in this region, so when you take approach y, s to ct, t so this goes away, x is ct. So what you get is half u0, 2ct plus u0, 0 plus 1 by 2c 0 to 2ct, u1 y dy. So if I use the first expression, that is what the limit I get and if I use the second expression, that is valid for y less than cs, that one.

So again, I get half u0 2ct, we have to do it carefully, minus u0, 0, plus 1 by 2c 0 to 2ct, u1 y dy. And there is of course boundary term, so that is coming h0. So I calculate this limit of the solution when the point approaches this line from above and from below, so I get two expressions. So in order that u is continuous so these should be equal and that gives us one condition.

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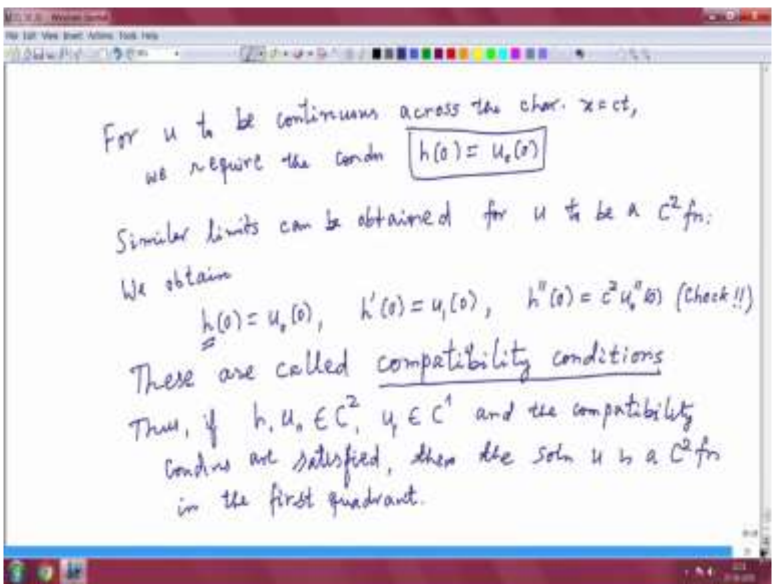

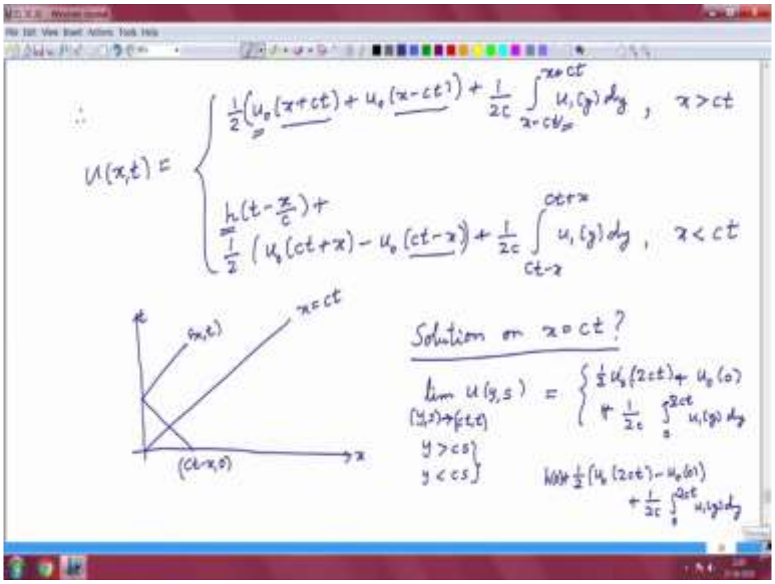

So for u to be continuous across the characteristics, x equal to ct, we require the condition h0 equal to u0, 0. So similarly, you can work out the conditions for the solution to be c1 and c2. So similarly, similar limits can be obtained for u to be a c2 function. So this is the minimum thing we require, c2 function. So let me write down, so it is a bit computation, so you can, so similarly you can differentiate this expression, differentiate this expression with respect either x or with respect to t and similarly uxx, uxt everything. So you will get two more conditions.

So we obtain, so this is bit work, so we have to do some computation, so let me write that first condition, h0 is equal to u0, 0. Then h prime 0, just check whether, u1 0 and h double prime 0 is c square u0, double prime 0. Check. So these are called compatibility conditions. So these are required for the smoothness of the solution, these are called compatibility conditions. So that are required to satisfy between the boundary data and the initial conditions.

So this h is boundary condition and u0, u1 are the, maybe there is c missing here, just check that, compatibility conditions. So thus if h u0 are c2 functions and u1 is a c1 function in their respective domains and the compatibility conditions are satisfied, then the solution u is a c2 in the first quadrant, that is important, and we have obtained a formula for that.

So in case any of the compat, here there are three conditions, any of the compatibility condition is not satisfied, then you will suffer that corresponding discontinuity either in the function itself or in the derivative across this characteristic. Again, just recall, so any discontinuity in the initial condition, we saw that propagates along the characteristics, and similarly if the compatibility condition is not satisfied, any of them.

Then the solution will suffer that kind of discontinuity across that single characteristic namely x equal to ct. So this completes the solution of the Dirichlet problem.

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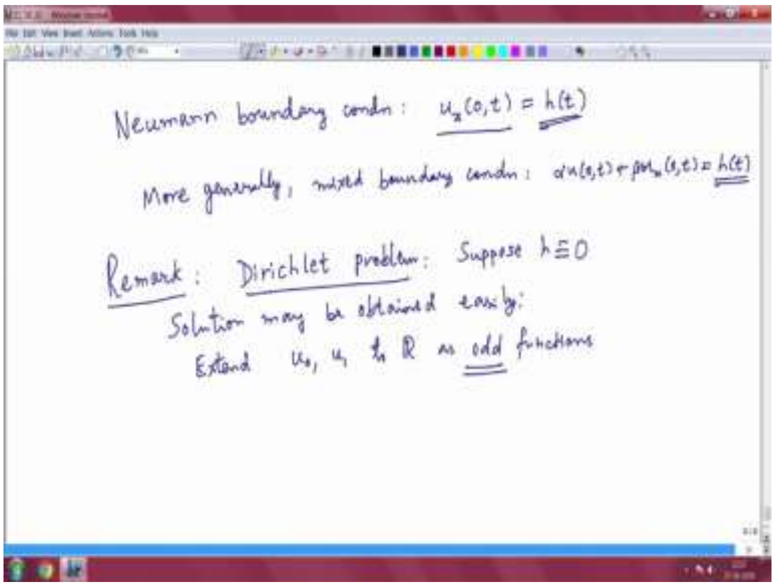

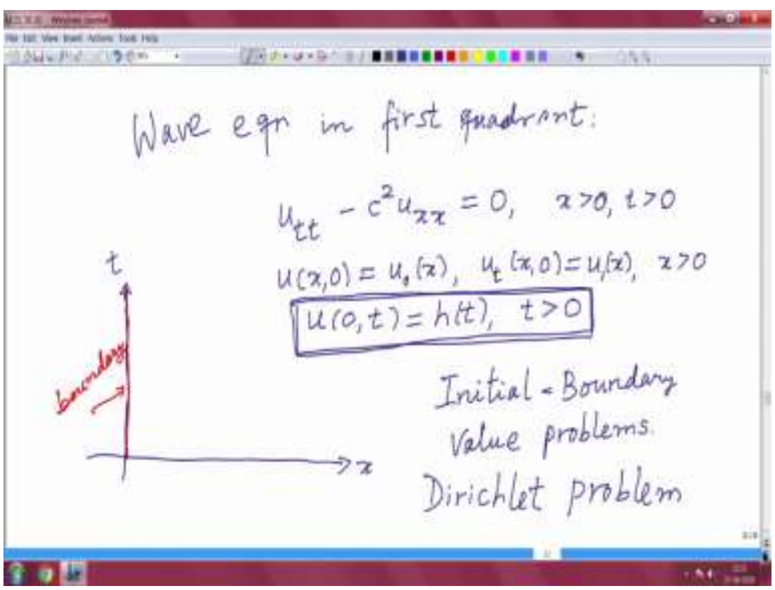

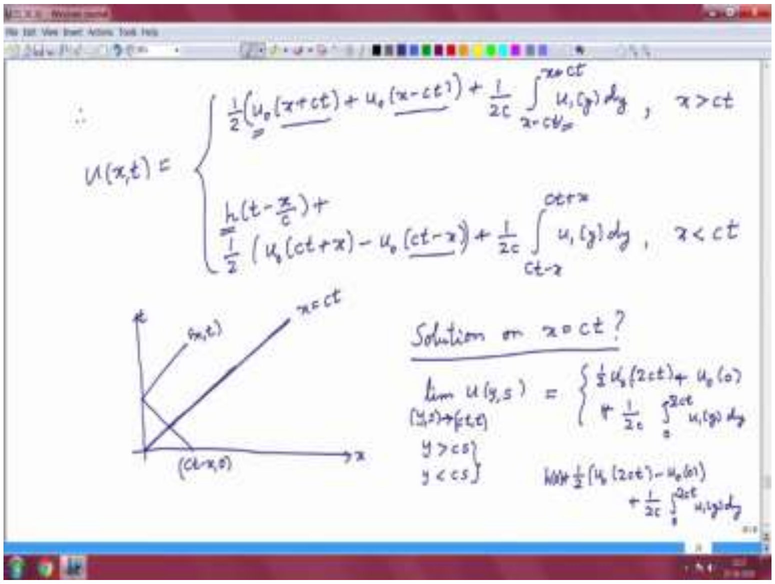

So let me just mention, so another kind of Neumann boundary condition or even mixed conditions. So Neumann condition, instead of u you provide the data for the first derivative with respect to x and that is in this case is the normal derivative of u, normal to that line is in the x direction, so we are providing the first derivative.

And this is more difficult than the Dirichlet one, we will see that. And more generally, we can also give mixed boundary condition, more generally, mixed boundary condition, alpha u, 0, t plus beta ux 0, t equal to ht. So these maybe different functions, I am just writing general things. So these are given functions.

So if alpha equal to 0 and beta is not equal to 0, we get Neumann boundary condition and beta equal to 0 and alpha not 0, we get Dirichlet boundary condition that we have already seen it and if both are non-0 then this is a mixed problem, we will see that. Just one remark, we will come to an end of this lecture, in the Dirichlet boundary value problem, Dirichlet problem, suppose the boundary data, suppose h is identically 0, so just let me show you that.

So I am taking this h identically 0, in that case there is a simple way of obtaining the formula by converting the problem into the entire line problem. In this case, so this is 0, solution formula, solution may be obtained easily, how we are doing that thing?

So extend u0, u1, so u0 and u1, they are defined only for x positive, so we extend to R as odd functions and use these extended functions as the initial value for the wave equation in the entire domain and then you apply the D'Alembert's formula and then you will get this, when h equal to 0 you precisely get just by solving the wave equation in the entire real line, h equal to 0.

So this you cannot do when h is not 0. So with that thing we will come to end of this class and in the next class we briefly study the Neumann boundary condition and see how the formula changes and that is a little more work, but we will do that so that you will get an idea how the formula changes. Thank you.