Partial Differential Equations - 1
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Lecture 38
One Dimensional Wave Equation

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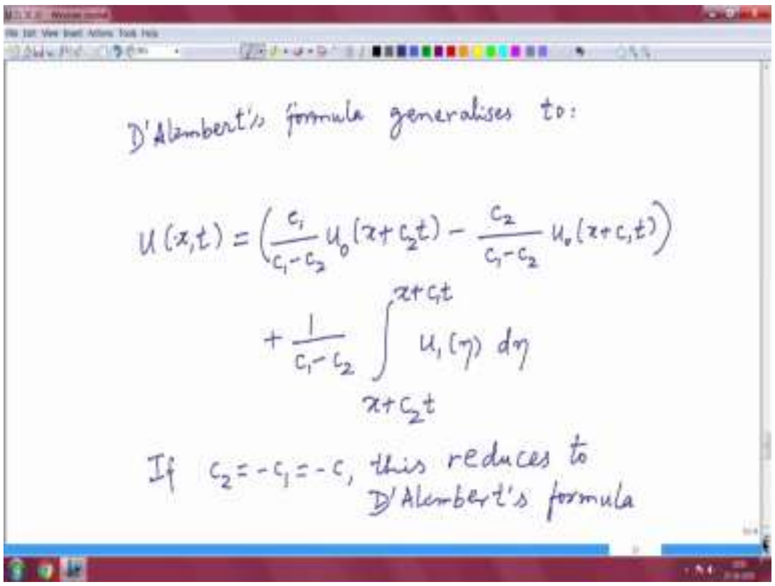

Welcome back. In today's lecture again we begin with general second-order equation with two different speeds. Last time we started that and we derived this formula for the solution of the second-order equation with two different speeds, c1 and c2. So it generalizes the D'Alembert's formula, known D'Alembert's formula.

Of course, when we take the c2 equal to minus c1 equal to minus c, this reduces to D'Alembert's formula. And now you asked the question, what happens if c1 equal to c2?

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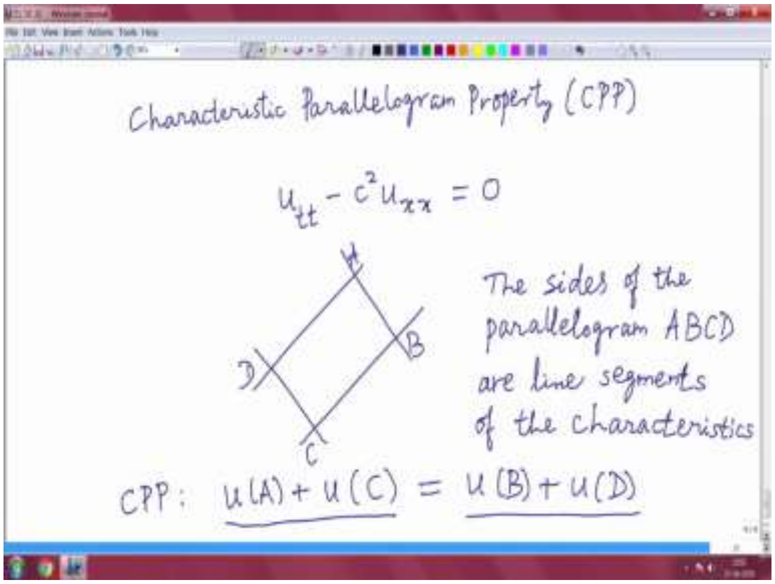

So before coming to that, so let me just mention one more important property of the wave equation, so this is called Characteristic Parallelogram Property, which we are going to use later. In fact, this property characterizes the wave equation as we shall see. Characteristic Parallelogram Property, for short I will use the CPP to denote this, so CPP.

So again, we consider the wave equation, so here the initial conditions are not explicitly used, but they are in the background, so this is the wave equation. Now you consider a parallelogram, not any parallelogram, but a specific one parallelogram, so here I denote this A, B, C, D. So the sides of the parallelogram, A, B, C, D are line segments of the characteristics. So that is why it is called characteristic parallelogram.

Then the important property, CPP states that U of A plus U of C equal to U of B plus U of D. So you consider any solution of the wave equation and then you evaluate the solution at the opposite vertices, A and C, and B and D, then you add them, so this is sum of the solutions at A and C and this is sum of the solutions at B and D.

And the characteristic parallelogram property states that UA plus UC is equal to UB plus UD. So if you know the values of the solution at three points, any three points of the characteristic parallelogram, then the value at the fourth point is determined by this property. So this is an important property which we are going to use later.

So conversely, if that is the characterization, conversely if U is any c2, so I am not necessarily I am not referring to solution of the wave equation, if conversely, if U is any c2 function satisfying CPP for all characteristic parallelograms then U satisfy the wave equation. So that is the characterization. So we are going to use it later.

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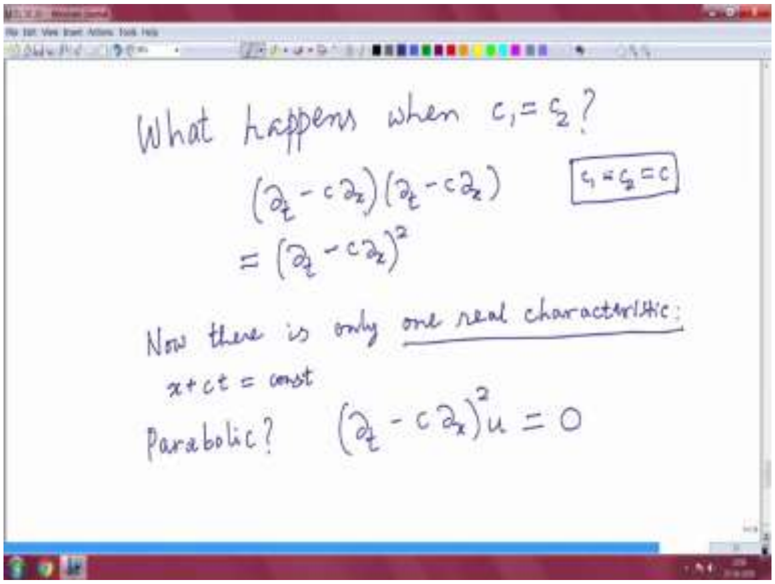

So again back to this second-order equation with only one real characteristic. So this equation, I will write, del t minus c, del x square u equal to 0, so homogeneous equation I am considering. So there is only one real characteristic, but this should not be called as parabolic.

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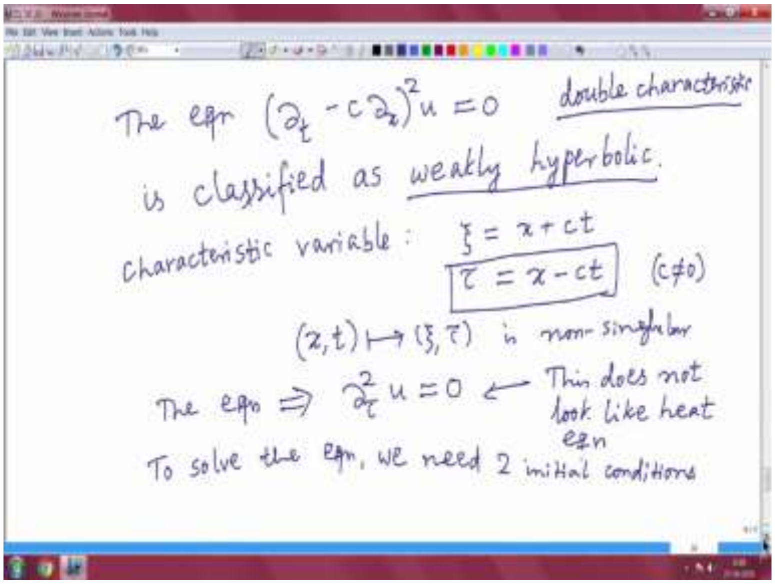

So the equation, del t minus c, del x square u equal to 0 is classified as weakly hyperbolic. So perhaps you are first time hearing this class of equations. I will give you two reasons why this equation should not be classified as parabolic. So the first reason is, so consider the characteristic variable, so there is only one characteristic, so xi is equal to x plus ct.

And now we choose another variable so that the change of variables is again non-singular, so Jacobian should not be a 0, so the easiest choice is x minus ct. But remember this time this is not a characteristic variable, so we are choosing a different variable, so that this xt change of variable to xi tau is non-singular.

So we are assuming of course, c not 0, if c is 0 then the equation reduces to OD, so it is not even a PD. So again, you do the computation, so you see that the equation implies, so some computation, you show that del tau square u is 0. So in these new variables, you show that del tau square u is equal to 0, and this does not resemble the heat equation.

So this is more like a OD, because it is only del tau square, so there is no del xi there, so it is more like an OD, so far from the heat equation, which is a typical example of parabolic equation. So this is one reason why this equation should not be called as parabolic. The another reason to solve this equation we need two initial conditions, whereas heat equation requires only one initial condition.

So for these two reasons, again I stress that this equation should not be classified as parabolic, so it is classified as weakly hyperbolic. Now let us quickly solve the initial value problem or Cauchy problem associated with this equation. So since there the c, c repeats here, so such we can call this double characteristic equation.

So if the multiplicity of this characteristic is more than 1, so generally it is called multiple characteristic, and in this case since there it only repeats twice, so it is a double characteristic equation. These are more difficult than, then in contrast, the equation so let me show you that.

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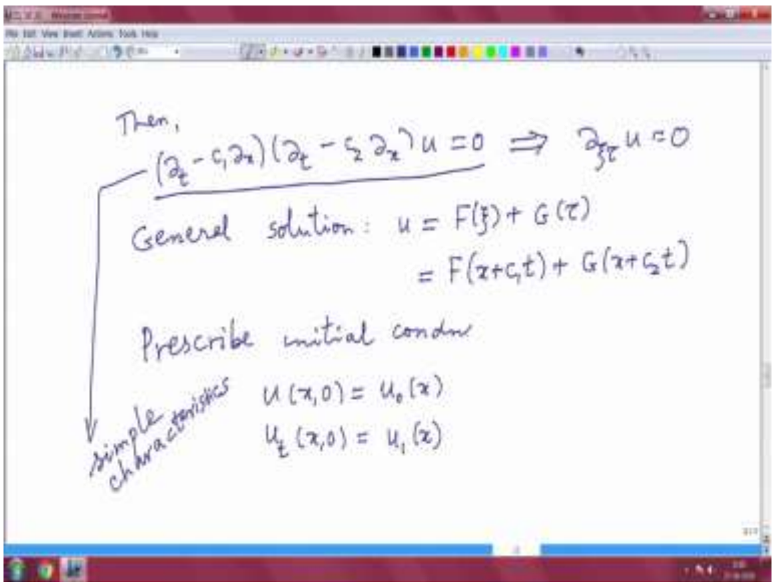

This equation is called equation with simple characteristics. So we are assuming c1 is different from c2, so this, there are two real and distinct characteristics here, so it is called simple characteristics.

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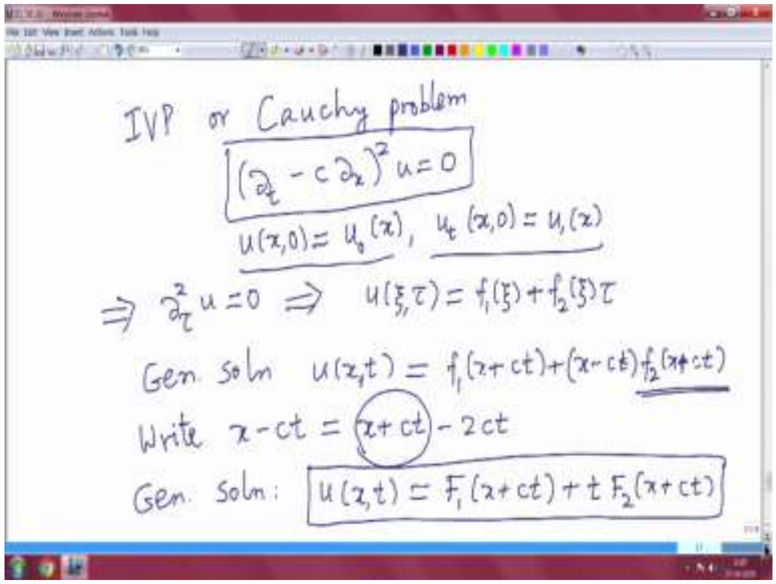

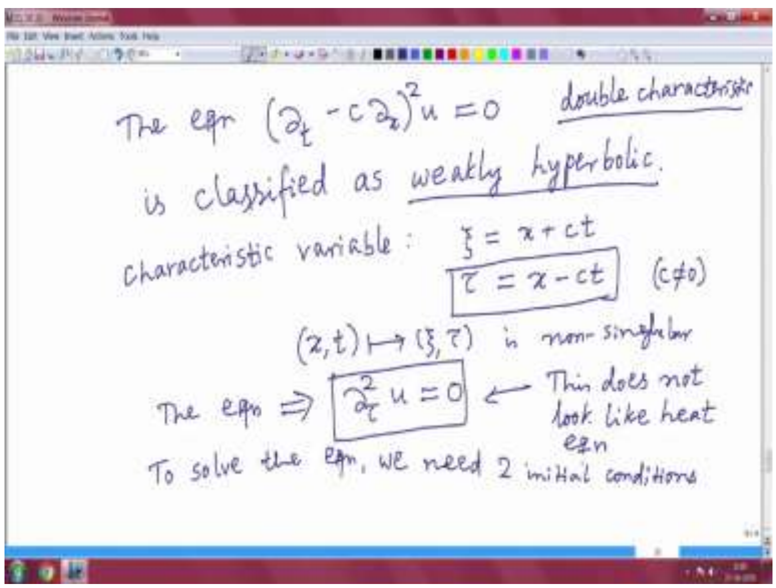

Now let us quickly consider the initial value problem or Cauchy problem of this double characteristic equation. So del t minus c, del x square equal to 0, as I said, it requires two initial conditions. In case of wave equation, we require u0 to be a c2 function and u1 to be a c1 function and let us see what difference we will observe here.

So for the time being assume u0 and u1 are such that we can obtain a c2 solution of the equation. So now since there is only one characteristic, we already done it here, let us go back, so we get this. With this change of variable in the new variable xi and tau, this equation reduces to del tau square u equal to 0. So that we can immediately integrate and obtain the solution.

So this implies del tau square, so let me not repeat again. So which again in term implies u of, remember, u is a function of xi and tau. So this is a linear function in tau, so we can write this as, so the constants can be functions of xi. So this is the general solution of this equation. So let us go back to the original variables, so general solution, u xt, so now again I write in terms of the original variables x and t, so this F1, x plus ct, so tau is x minus ct, F2 x plus ct.

So we can arrange the term little bit and write it in a neat form, so we write x minus ct as x plus ct minus 2ct, and you absorb this x plus ct term in this function, and so we have only a multiplication by t. So general solution we can write it as, one more time I will write it, so u xt, so let me use different notation, x plus ct plus t, F2 x plus ct.

So we can easily verify that the expression given by this for u satisfy the given equation, this is the equation. And now in order to determine F1, F2, again just recall what we did in the case of wave equation to arrive at D'Alembert's formula. So I leave it as an easy exercise, so you plug in the initial conditions in this general form and finally you obtain a solution.

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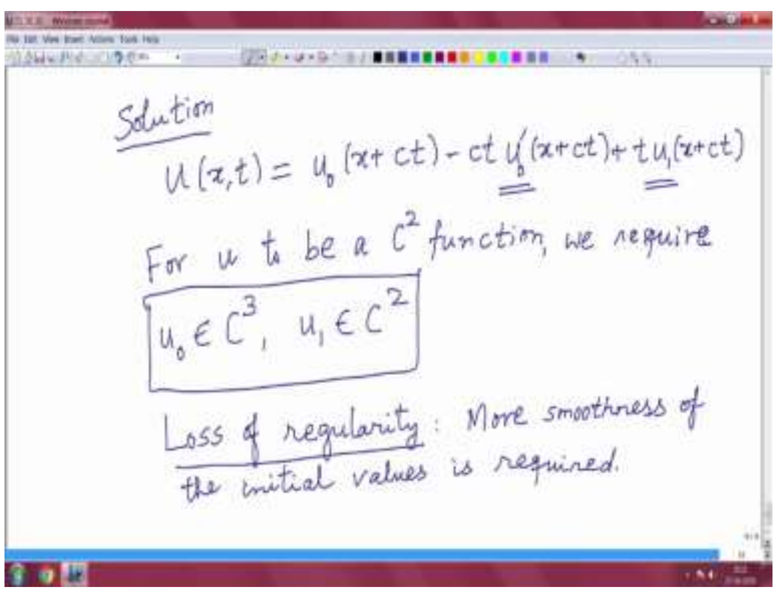

So I am skipping few steps, can easily point them and work out x plus ct minus ct, u0 prime, so this derivative of u0, plus t u1 x plus ct. So if you compare this form of the solution with the

D'Alembert's formula, you see a big change here. First of all, in the D'Alembert's formula no derivative of the initial conditions appeared, but here, the derivative of the initial condition appears, there is no integration of u1, but u1 appears as it is.

So now we require so far u to be a c2 function and that is what we need, at least two derivatives, u to be a c2 function of x and t. We require u0 to be a c3 function, because there is one derivative in the formula of the solution, so if you want u to be c2, so this has to be c3 and similarly u1 has to be c2. So we require more smoothness in the initial values to obtain a less smooth c, u0 is c3, but that gives u only a c2 function.

So this is referred to as loss of regularity and this is typical of weakly hyperbolic equations and systems, loss of regularity. So more smoothness of the initial values required.

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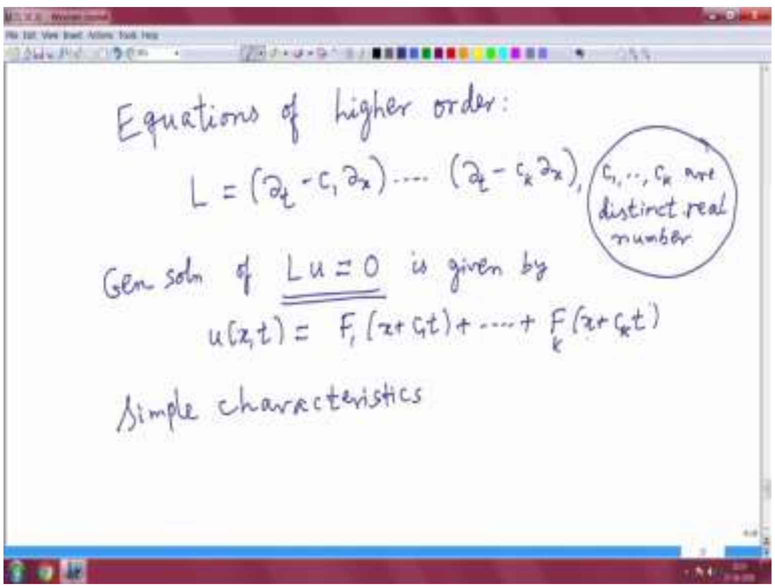

But now we can simply combine the two cases, so let me just mention as a general remark, so equations of higher order. So this is just follows from an induction argument. So first consider equation with simple characteristics, so the operator L, I write it, so this is c1 del x, so a k-th order differential operator, which can be factored into linear factors, so that is the form of the higher order equation, where c1, c2 all distinct real numbers.

So we did for it in the case of two values, but the same procedures are real. So what we can show is this general solution now, so this is just followed by an induction argument, so I will not do the details, you can do it yourself, general solution of Lu is equal to 0 is given by u of x,t is equal to F1 x plus c1,t plus F of x plus ckt plus where F1, F2, Fk are k times differential.

So here it is important there are distinct, otherwise you will not get this one. Of course, in order to determine this F1, F2, Fk, we have to prescribe k initial conditions on the solution, but it is more algebra, you have to solve k by k linear system. But in principle, it can be done. So that is with simple characteristics, so this is simple characteristics.

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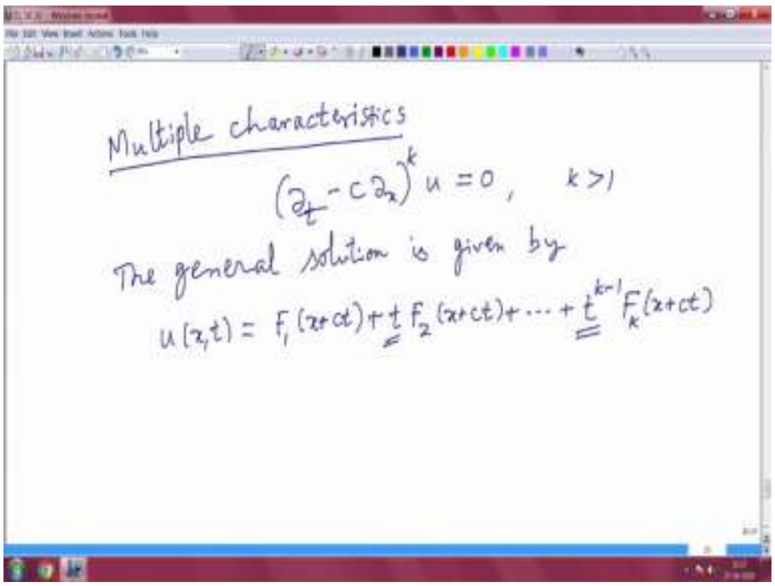

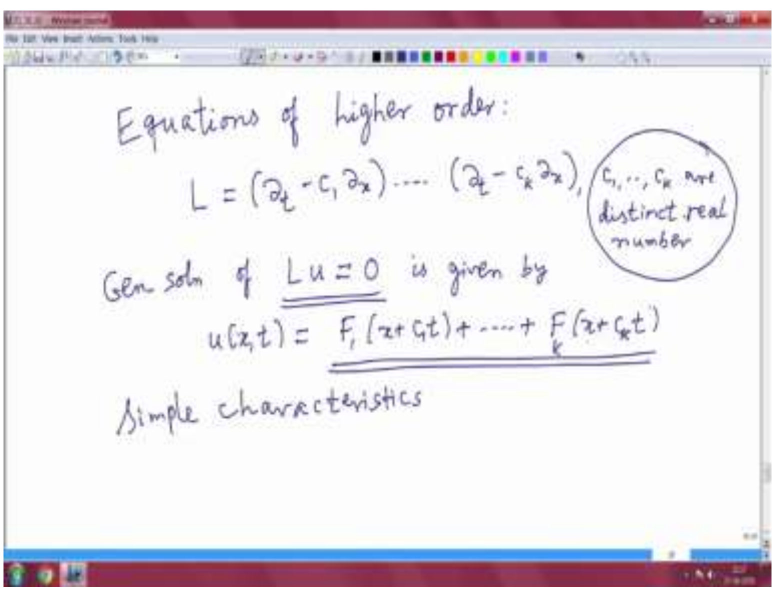

What about multiple characteristics? So it is similar thing. So this is the equation, del x k, u equal to 0. So k is bigger than 1, earlier we did for k equal to 2. So again, by an induction argument, this general solution is given by u of xt is equal to F1 of x plus ct plus t, this we saw for k equal to 2 and now you do an induction argument and you get this.

And you see this factor t, which was not there in D'Alembert's formula and even in this simple characteristic problem, c1 in simple characteristic problem, there is no t factor at all but when you go to multiple characteristics so you see this factor of t. So now we can combine these two simple characteristic equation and multiple characteristic equation into a single one.

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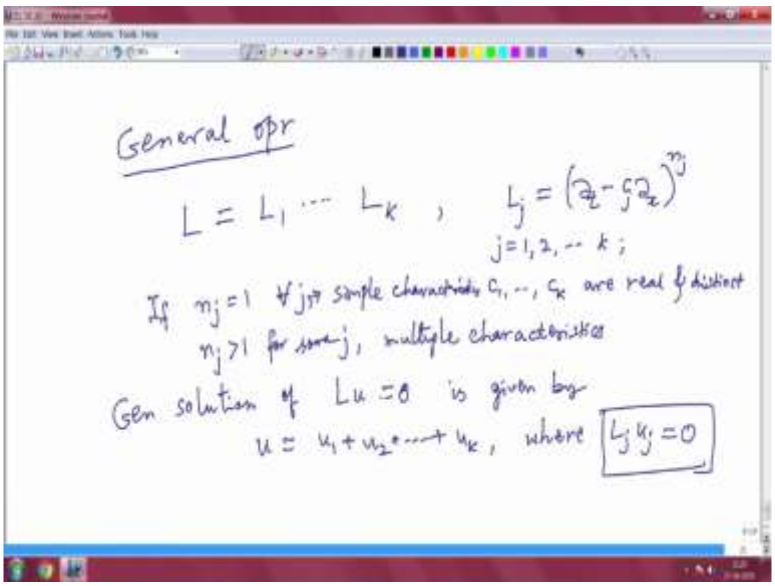

So, a more general equation, general operator. So L is equal to L1, Lk, so Lk is this operator Lj, let me with Lj, cj del x, I put nj. So j equal to 1, 2, k and c1, c2, ck are real and distinct. So if nj equal to 1 for all j, then we have simple characteristics, so that leads to simple characteristics. And if nj is bigger than 1 for some j, even for 1j, it is multiple characteristic problem.

So the general solution of Lu equal to 0 is given by u is equal to u1 plus u2 plus uk, where Lj, uj is 0. And this we have now learned how to solve that equation. So later on we will provide with some example just to list this general equations of higher order. There is more algebra in it, so that is I just want to make remark of that.

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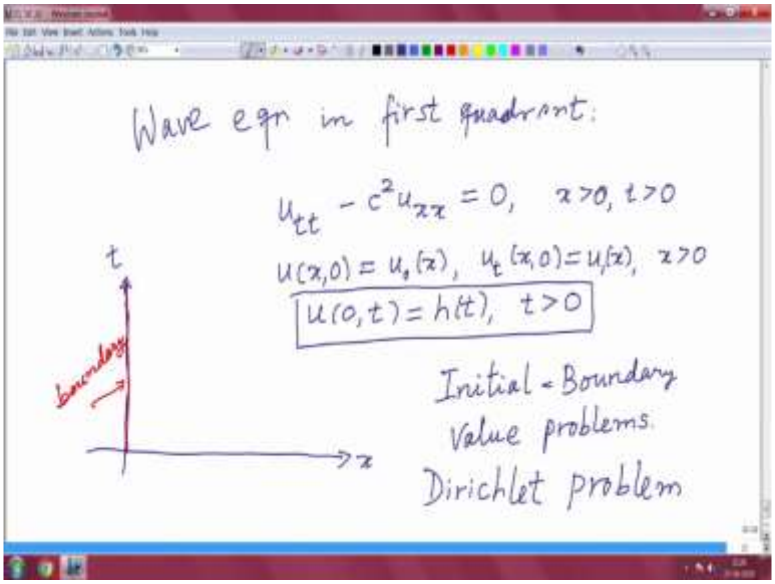

So with this thing now we will move on to again wave equation, but now wave equation in first quadrant. So, so far consider on the real line, but now we are considering only on the positive axis. So here is the problem, utt, again wave equation, let me again just consider the homogeneous one, so we can always use Duhamel's principle in order to get the formula for the solution of the inhomogeneous equation.

So now we consider this equation only in x positive, and as usual, t is always positive. So like the previous case, first we provide the initial conditions at time t equal to 0, as I remarked in the case of wave equation, you can provide on any t equal to t0 line, so there is absolutely no problem, by just translation the solution formula can be derived. But now only x is positive.

So let me draw this domain where we are interested, so this is the x axis and this is positive t axis. So we are interested in this solution, finding the solution in this first quadrant, x positive, t positive. And now, see earlier this was not there, but now there is a boundary here, so we distinguish the initial line and the boundary line, so let me use a different color here, a boundary. So what is the boundary, boundary is the t axis, so that is x equal to 0.

So we have to provide some conditions on the boundary, so for simplicity again let me just begin with, so x equal to 0 now, t, I put the h of t, and t positive. So such problems are referred to as initial boundary value problems. And in this case, since we are prescribing the value of the solution on the boundary, it is referred to as Dirichlet problem.

There are other kinds of boundary conditions one can prescribe, so I will mention as you go along, but let us now try to find the formula for this wave equation with this initial and boundary conditions. Let us try to do that. So since most of the work is already done in the case of wave equation on the entire real line, namely D'Alembert's formula and other things, so here it is much easier, as we shall see.

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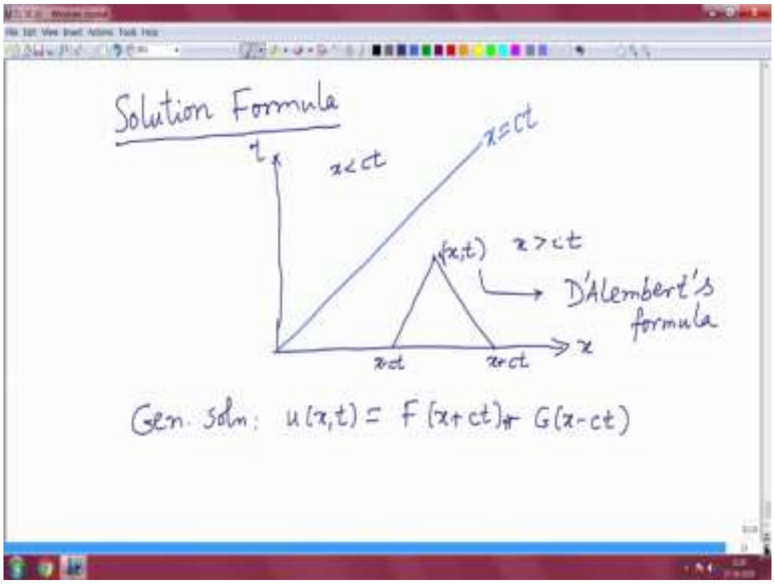

So solution formula. And let us see how the boundary condition influences the formula. So again, it is very easy to explain using diagrams. So this is the interface, there is only point here, the origin, and the origin which is the intersection of the initial line and the boundary line. Let us draw the characteristic here, so they play a role there, let me use different color here for the time being, let me use blue.

So that is the characteristic x equal to ct passing through the origin. While we are doing that one, let me just see that. So if I take any point x, t here, so below this characteristic, that means x is bigger than ct here and x is less than ct here. So the geometry changes, that is what is important, and if you look at the, so again, general solution of the wave equation, we already know that.

So, general solution in whatever domain is of this form, F of x plus ct, there is no change here, we can use just characteristic variable and obtain this. So F and G are two c2 functions. So again, if you look at the domain of dependence property, so this value of the solution at this point is determined by the initial values in this interval, that is what is the domain of dependence.

And since x is bigger than ct, this x minus ct lies on the positive axis, thus the value of the solution at such a point is determined only by the initial values, because this will not see the boundary, so the solution simply obtained by D'Alembert's formula.

So geometrically also we can see that, so the value of the solution at this point below the characteristic is not influenced by the boundary value. But situation is quite different, so let draw again one more picture, so a different picture.

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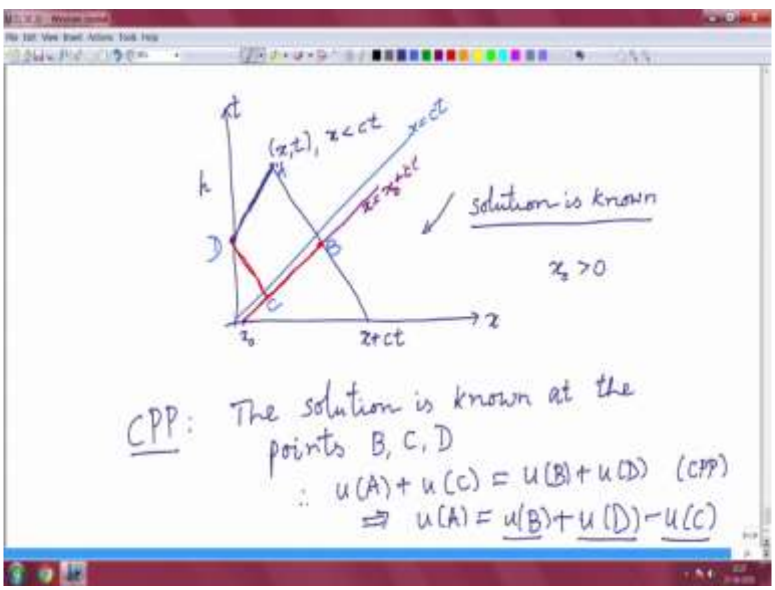

So if I take xt here, so now remember, x is less than ct. So again if I draw the characteristics through this xt, so one characteristic intersects, see the earlier case it was not doing that and one characteristic there is no problem, so this is x plus ct is there. But this characteristic does not meet the initial axis in the positive x axis and here we do not have any data, but we have data here, wherever it hits, we pick up the value h there.

So this is okay, but how to incorporate this value to find the value of the solution at this xt. So here we use, this is one way, so you can again use the general solution and try to determine G, but let me use the characteristic parallelogram property to find the solution of u at this point. So we have construct a characteristic parallelogram, so there are already two sides, we see that.

So this is also a characteristic and that is also a characteristic, and two points, so one to find the solution there and the data is given here, so we also know the solution at this point. So we need another two points where solution is known. But in this region we already know the solution, the solution is known here and that is given by the D'Alembert's formula.

So if we can choose two appropriate points in this domain, namely x we get then ct, and which form a characteristic parallelogram, then we are able to find the solution u at the point xt. And for that, so let me just use different color again, so you pick as point x0 here, x0 small, so we will see how small it is, x0 is positive, small.

And now you draw a characteristic, so this is the characteristic, so this, you can write this x equal to x0 plus ct. So this blue one is x equal to ct, that is passing through the origin and now this is passing through the point x0. So that is a characteristic, so there is now third side and so we know the value here because that is in the region where solution is known.

So we need one more point, so just you draw a line, a characteristic parallel to this. So this is one characteristic and this, so it does not look like a parallelogram, but we can just, so this is A, this is B, this is C, this is D. The solution is known at the points B, C, D, so therefore uA plus uC equal to uB plus uD. This is CPP that implies u of A is equal to u of B, B u of D minus u of C.

So we have to find out what is u of B and what is u of D, u of D is just very easy, so that is the boundary data and u of C we have to find. So this we will continue in the next class.