Partial Differential Equations - 1
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Lecture 37
One Dimensional Wave Equation

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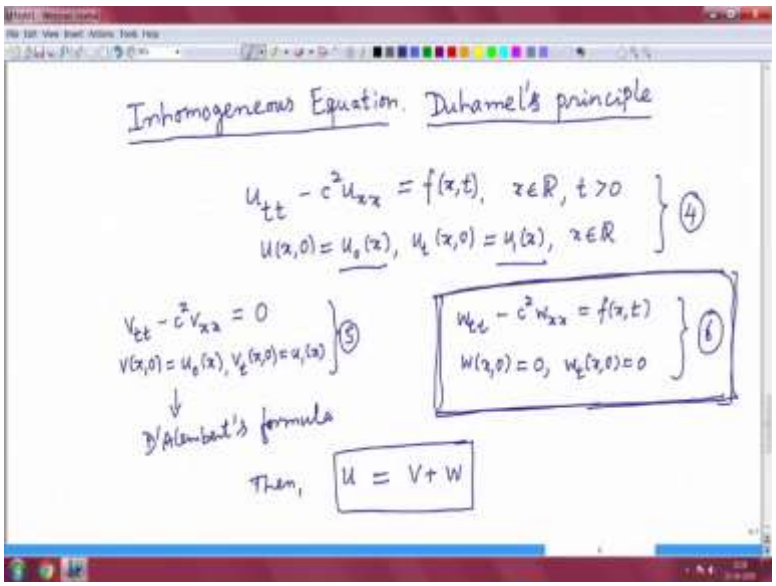

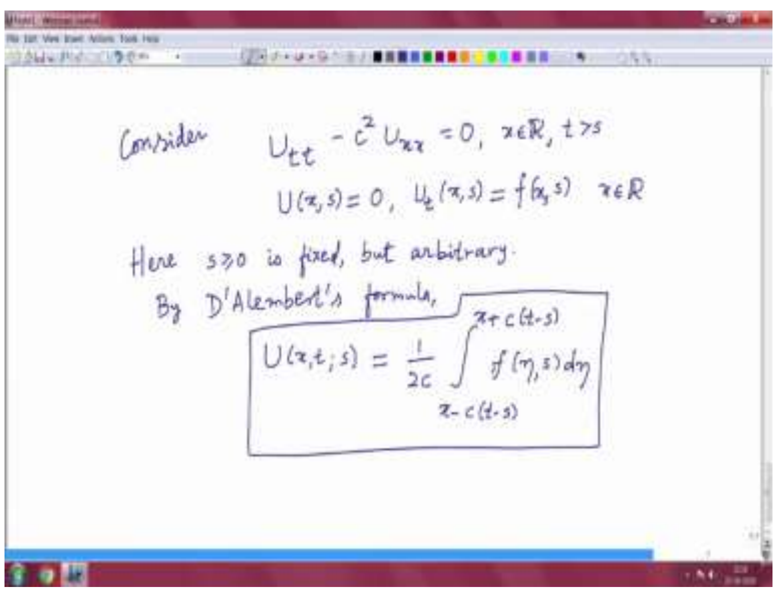

So welcome back, and now we will complete, we write down the formula for the solution of this problem 6. So let me write once again, that formula. So that w is given in terms of this U, so just you remember this U, U is the solution of this homogeneous equation with these initial conditions.

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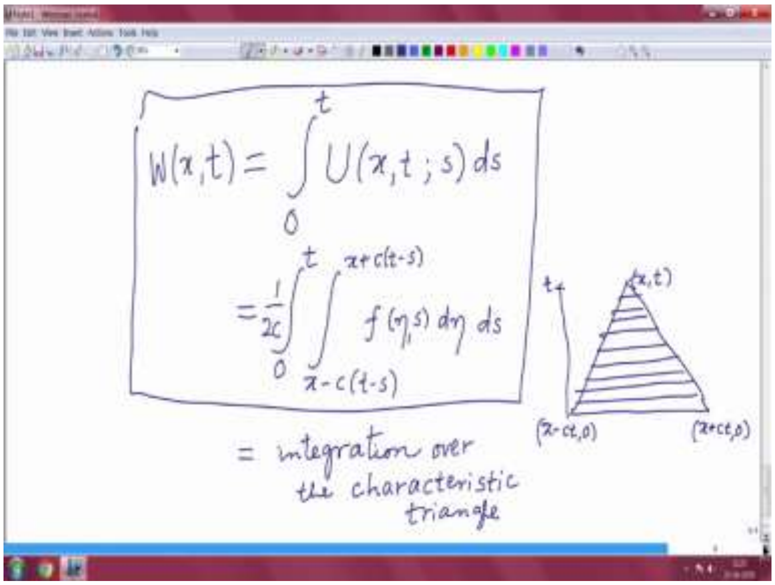

So W is given by integral 0 to t, U of xt, remember there is an s, and now you integrate with respect to s. So using the formula for the U, we can write this as 0 to t, and U is 1 by 2c, that is a constant, that comes out. And let me write what the formula is, so just have a look there. This is x minus ct minus s, x plus c, t minus s, f of Eta s, d Eta.

So this is U xt s, and now we are integrating with respect to s. So it is double integral as you can easily verify, so this is nothing but integration, double integration over the characteristic triangle. So let me write that what that is, I will just show you that. So what is this characteristic triangle? So a point xt is given, so t is positive, and we want to evaluate that solution at that point.

So what you do is you just take these characteristics, so this is the x axis that stay equal to 0 line, so this is x plus ct 0 and this is x minus ct 0. So these two sides of the triangle are characteristics and this integrally nothing but, so first you are integrating along the horizontal lines. That is integration with respect to Eta and then you are integrating from 0 to t.

So that is this integral, so integral over this characteristic time, 0 to t. So it is not immediately clear why this w is solution of problem 6. So remember, problem 6 is, so it is inhomogeneous wave equation with 0 initial data. So now we will quickly verify that the formula given by this double integral is indeed solution of the problem 6.

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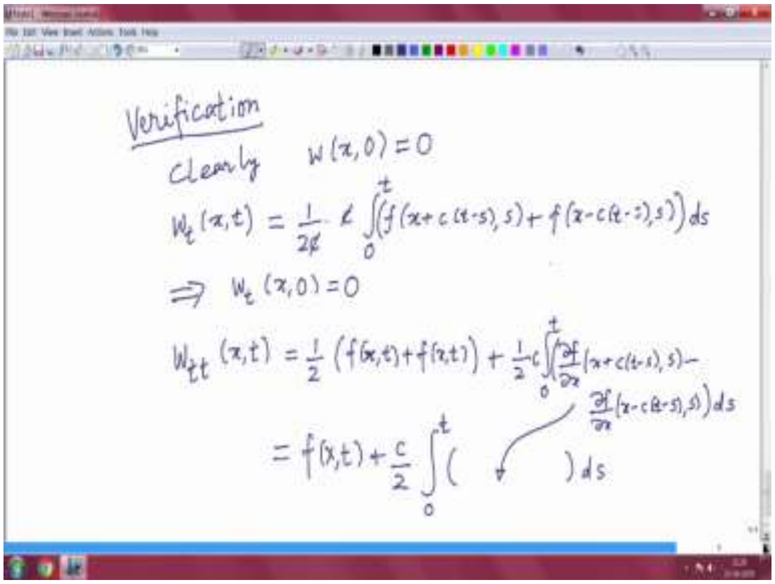

So verification, it is a quick verification. So just remember, look at the formula for the w, t is inside the integrant as well as it appears in the limits of the integral. So we have carefully do the differentiation with respect to t, as well as with respect to x. So again, just remember that, this formula. So if I put t equal to 0, so this integral vanishes, so certainly that is 0.

So clearly, w x0 is 0. So this is one of the initial conditions. So we want w tx 0 is also 0, so for that we have to compute wt, so differentiation with respect to, so as I say, t appears both in the integrant and in the limit, so we have to use appropriate formula from calculus. So this is differentiation and that the integral sign, so let me write that.

So this is 1 by 2c, that is constant. And then the c comes from the differentiation of the integrant, so this is f of x plus ct minus s plus, you do carefully, you see these terms come there, s the whole thing, ds. So after first differentiation, the integration with respect to Eta variable disappears. So this again immediately gives us, so this c, c goes away, so you have half there. That does not matter, x0 is 0, because this integral is just 0, 0, so it just becomes 0 again.

So let us compute the next one, wtt, so this half is there, so c has gone there. So if I differentiate with respect to limit this t, so I simply get f of, you just put s equal to t here, so you get x,t and another one there. And another term, now I differentiate the integrant, so that is 0 to t, because t is also sitting there, so del f by del x, the arguments are different.

So you have to pay attention to that. So x plus c, t minus s, s and when I differentiate with respect to t, there is c coming, that c is there. And if I differentiate the second one, there is minus c. So that makes it minus del f by del x, x minus c, t minus s, s ds and there is a c coming. And this just, let me write it, f of xt plus c by 2, that integral. So let me not repeat that because there is no change there, so this whole term comes here.

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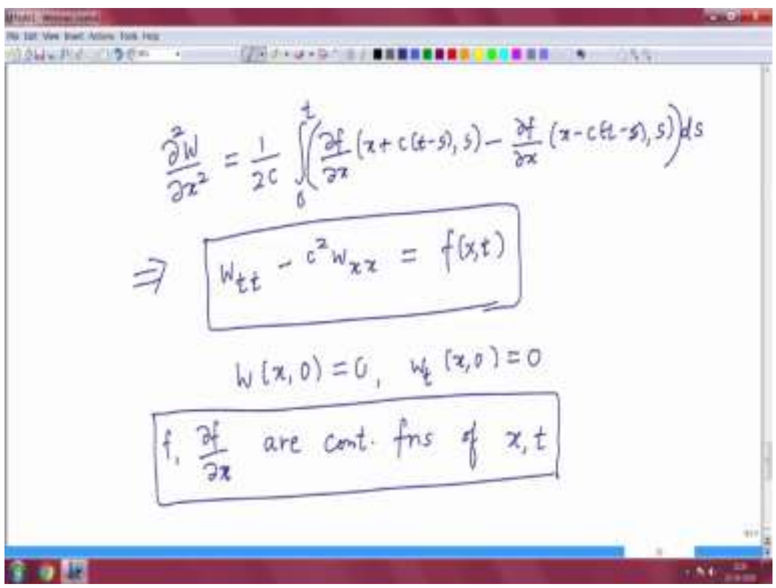

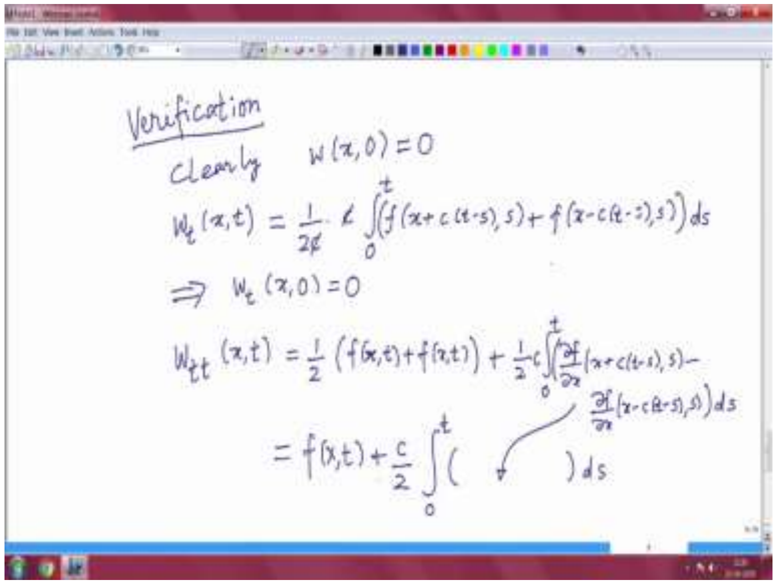

So similarly, del f, I write directly, because x is only in the integrant, so there is not much difficulty here, so just we get 1 by 2c integral 0 to t, same thing, del f by del x, x plus c, t minus s, s minus del f by del x, x minus c, t minus s, s ds. So using this computation, so you immediately, it is not f, it is w del square term.

So let me show you that, so wtt is f, xt plus c by 2, this integral and wxx, the second derivative with respect to x. There is no ft term here, so immediately we see that wtt minus c square, wxx, so this square term will produce c by 2 and there is a c by 2 term here also, there is a c by 2 term here also, so that gets canceled and what remains is only f of x,t. So indeed w given by this Duhamel’s formula from the Duhamel’s principle, this namely, this formula satisfies equation 6 and with the initial conditions.

So let me just write again, once again, w x0 equal to 0, wt x0 equal to 0. So if you look at the verification, so what we require on the inhomogeneous term, so we require, so these are conditions on f, and it is first derivative with respect to x are continuous functions of x and t. This is hypothesis on f, so with this hypothesis, so we can write down the solution of the inhomogeneous problem 3, so this is finally we would like to do that.

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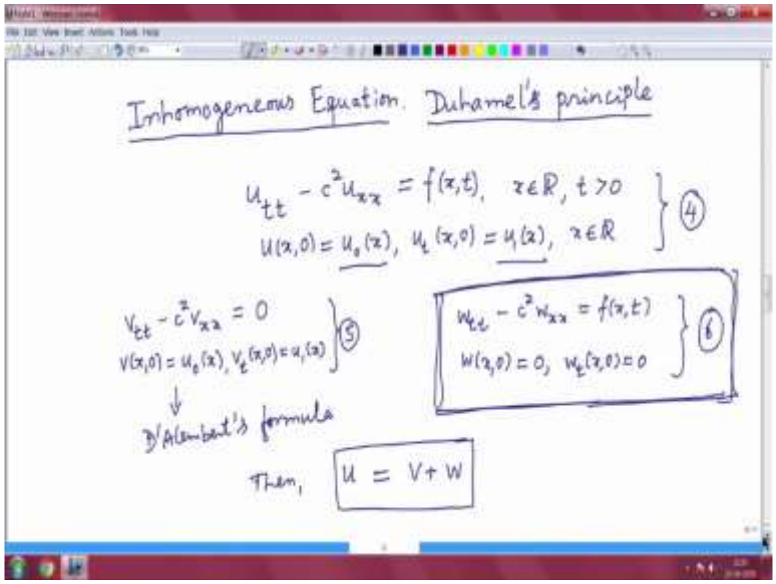

So this is problem 4. So one part comes from D'Alembert's formula and another part comes from the Duhamel's principle. So now we can write down the complete solution for the equation 4.

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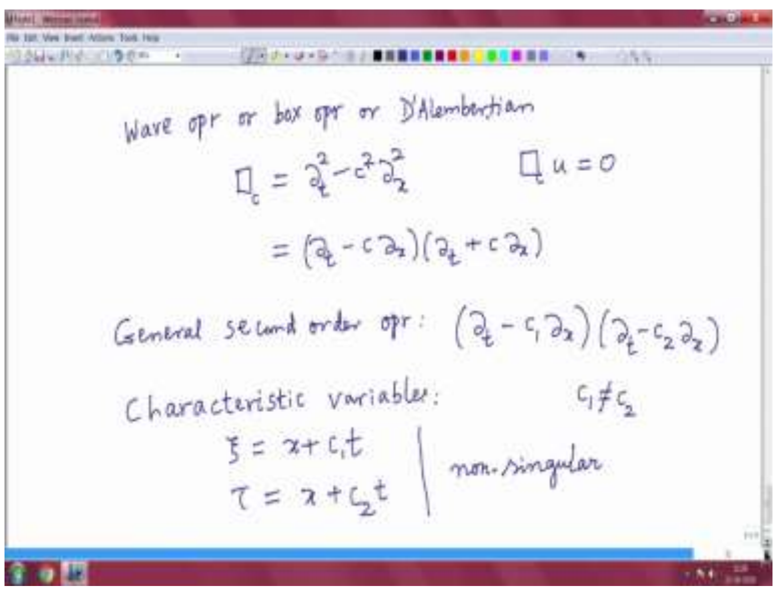

So now we quickly take a small digression and describe general second order equations. So this, we will use some notations, so these are again standard notations we will be using given in later. So this is just like Laplace operator, so we have here wave operator or it is also called box operator because it is denoted by box or D'Alembertian.

So all these are used in literature, different authors use different terminology, D'Alembertian. So what is this? So this is denoted by just like Laplacian, so it is just box. So this is the operator, partial differential operator, c square del x square. So wave equation was just nothing but, so we can rewrite, so this is the wave equation, homogeneous wave equation.

So the advent is, since c is a constant, so this we can split into linear factors. So if c is not a constant, we cannot write this, you can easily verify this. You take any c2 function and operate this on that c2 function, you see that this is same as this. So this suggests us to consider a general second-order equation operator, second-order operator with linear factors.

So namely, so instead of plus c minus c, now I consider c1 del x, so where we take c1 not equal to c2. In this setup, when you consider this general second-order operator, so there is no constraint on the signs of c1 and c2, c1, c2 can be of the same sign, both positive or both negative, or they can be of different signs. So 1 maybe positive and 1 maybe negative, and in fact one can be 0, as long as c1 is different from c2.

The analysis is same for this operator also, so here the characteristic roots are again c1 and c2, so you see the characteristic variables, because this we have done for the wave operator, so similar computation, variables. So this is also hyperbolic. There are two real and distinct characteristics, so they are given by x plus c1t and x plus c1 equal to constant, x plus c2 equal to constant.

So we again introduce the characteristic variables c1t, tau is equal to x plus c2t. So since we are assuming c1 not equal to c2, so this is again non-singular change of variables, so that is important. So whenever you want to make change of variables, first you have to verify that it is non-singular, so you can simply compute the Jacobian and verify yourself that it is non-0 because c1 is different from c2.

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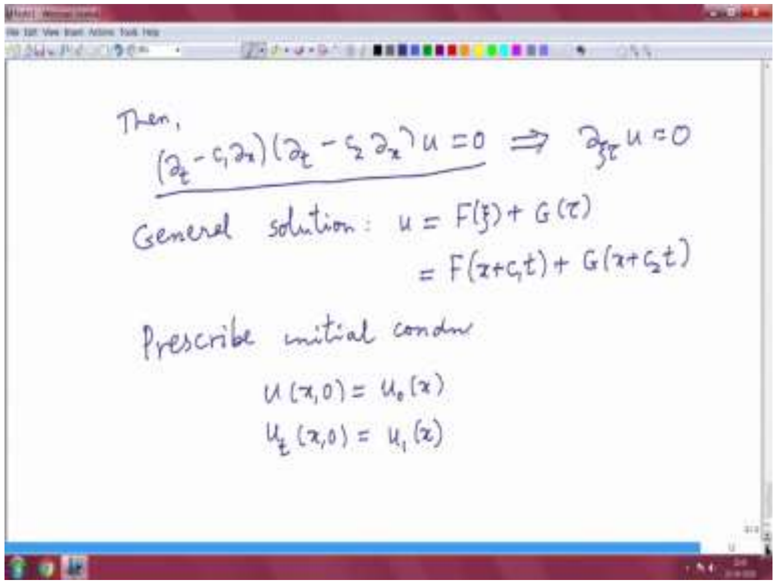

So similar computation, what we did earlier, so then del t minus c1 del x, del t minus c2 del x, u equal to 0 implies del xi tau, u equal to 0. So this equation in the characteristic variables reduces to this second-order equation. So again we can easily integrate that, so general solution we can write down, general solution. So u is equal to, so first let me write in the xi tau variables, so this is the general solution of this equation. So once we go, then we will go back to the x, t variables, where F and G are arbitrary c2 functions.

So, you can easily verify that u is solution of this equation, and any solution of this equation is given by this. So again, if I prescribe initial conditions, u x, 0 equal u0, x. And now you plug in these initial conditions in this general formula, and determine F and G. So I am omitting some computations, but they are easy ones, just like we did for the D'Alembert's formula.

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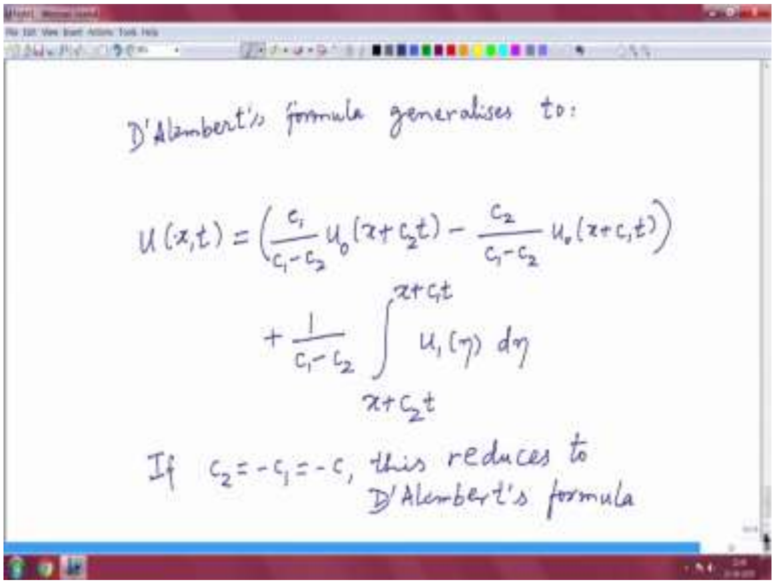

D'Alembert's formula generalizes to, so let me write down the solution, u xt is equal to c1 by c1 minus c2, it is very similar to D'Alembert's formula, u0 x plus c2t minus c2 by c1 minus c2, u0 x plus c1t. So this is the first term and then the integral term is given by 1 by c1 minus c2, x plus c2t, x plus c1t integrate from x plus c2t to x plus c1t, u1 Eta d Eta.

So similar to D'Alembert's formula there is one just algebraic expression and one integral term. Of course, if c2, this is special case, if it takes c2 equal to minus c1 equal to minus c, this reduces to D'Alembert's formula.

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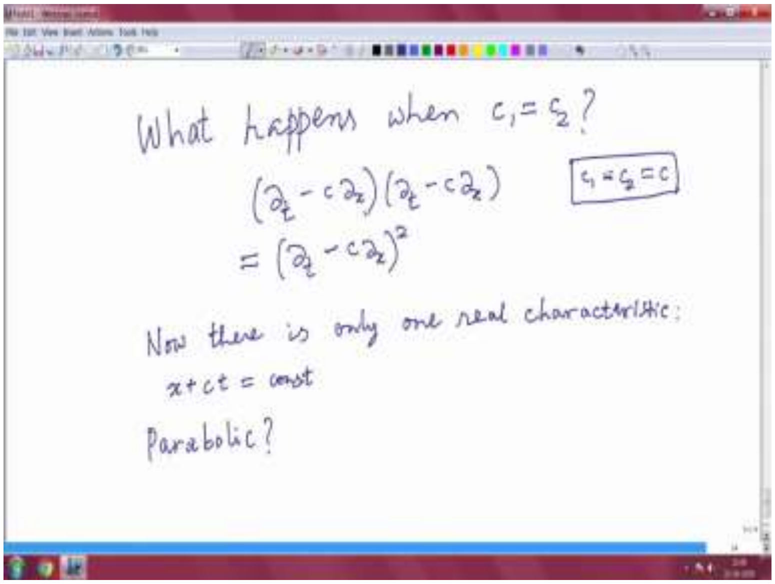

Now an interesting case happens here, so in this general setup when you take a second-order operator with 2 speeds, so next question is what happens when c1 equal to c2? So what the operator we get, the equation, so this c1 equal to c2 equal to c, so that is c del x, this is the operator. So c1 equal to c2 equal to c, I am putting it, c.

So this we rate it as c delta x, so this is also a second-order operator. But now there is only one real characteristic that corresponding characteristic family is given by x plus ct equal to constant. These are the characteristic lines. Should we call it a parabolic, this is parabolic or not, that is a question, because in the classification, if there is one real root of the characteristic equation, that is usually called parabolic.

In this case, should we call it parabolic or not? That is the question. So we will continue the discussion on this double characteristic problem in the next class. So thank you.