Partial Differential Equations - 1
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Lecture 36
One Dimensional Wave Equation

Welcome back. In this lecture we will continue the analysis of one D wave equation. Let me recall what we did last time.

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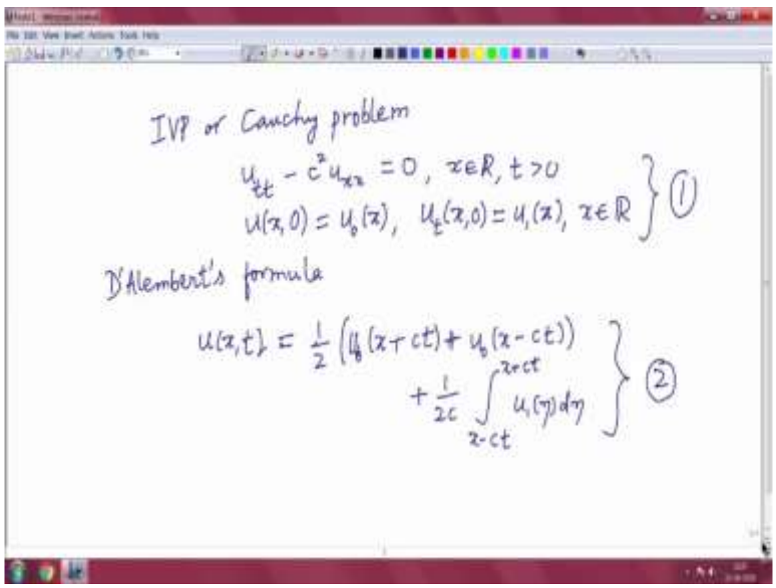

So we are discussing the Initial Value Problem or Cauchy problem for the one dimensional wave equation. Let me write it, utt minus c square uxx equal to 0, so it is homogeneous wave equation, so right hand side is 0, so this c is for x in the real line and t positive. And we are prescribing initial conditions at t equal to zero, so namely u, x0 equal to u0 x and ut, the first derivative of u with respect to t, at t equal to zero is equal to u1 x.

So here u0 and u1 are arbitrary given functions, u0 is a c2 function and u1 is a c1 function and the solution is given by the D'Alembert's formula, recall this, so we derive this in the previous class. So the solution is given by this D'Alembert's formula, so let me write it once again, equal to half u0 x plus ct plus u0 x minus ct plus 1 by 2c integral x minus ct to x plus ct, u1 Eta, d Eta. So this is D'Alembert's formula and so as just we derived this previous time.

So it is not necessary that, so this is again, x is in R, so we are providing the initial conditions at t equal to 0, so let me denote this by 1 and this D'Alembert's formula, so you just remember this D'Alembert's formula which is used repeatedly. So it is not necessary that we prescribe the initial conditions at t equal to 0, so we can prescribe them on any line t equal to t0, so let me write it, the another problem.

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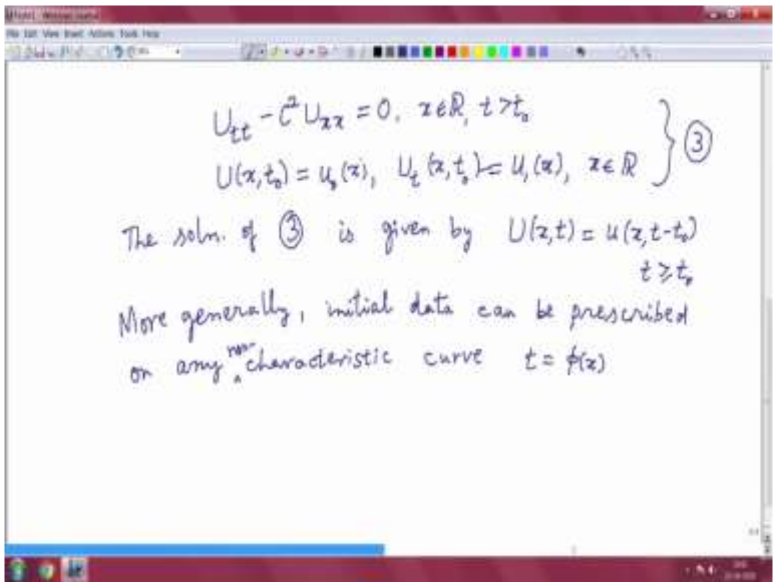

So let me write it, again, it is homogeneous wave equation, so let me use a different function here, equation is same, but now we are going to prescribe the initial conditions at some other t equal to t0. So this is again, x is in R, and now I am taking t, t0. So previously t0 was 0, but now I can take any t0 arbitrary, real number. And I prescribe the initial conditions at time, t equal to 0, so that is my initial time. So I use the same u0, x, t, u1x.

The wave equation has many nice invariant properties, so one of them is translation invariant, which is easy to verify. So if you change t to t minus t0, the wave equation does not change and exploiting this property we can write down the solution of 3, the solution of 3 is given by a very simple to verify u of x, t is equal to small u of x, now just you are translating, so t minus t0 and t is bigger than t0, where small u is solution of the problem 1, which is given by the D'Alembert's formula.

So this capital U is also given by the D'Alembert's formula, only thing is you have to change t to t minus t0, that is all. So this we are going to use a little later and more generally, so that is just a remark. So even for this solution of the problem 3 is neatly given by the D'Alembert's formula with t replaced by t minus t0, that is all. There is not much difference there.

More generally, so this is just a remark, initial data can be prescribed on any non-characteristic line curve, on any non-characteristic curve, t equal to phi x. But then there will be some conditions on phi in order to show that the solution exists and certainly solution is not given in any closed form, so this existence has to be proved by using some fixed point arguments. So the details we can find it in our recently published PDE book.

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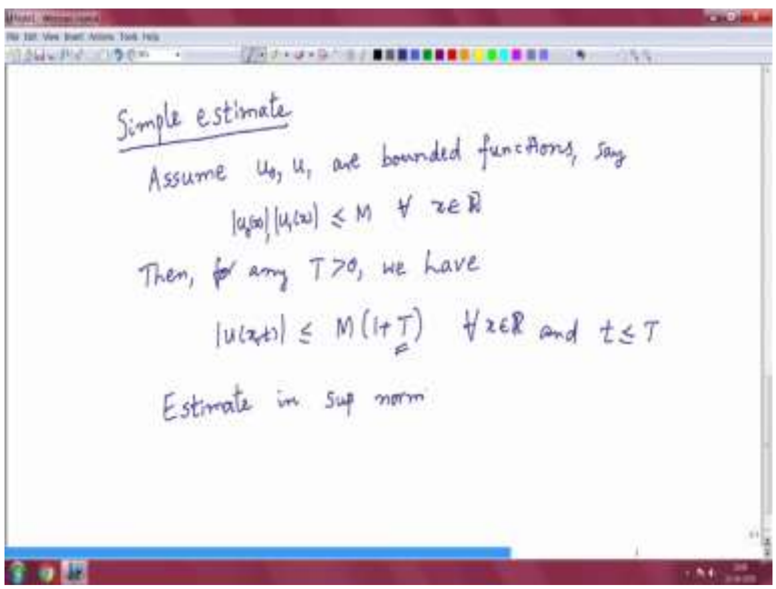

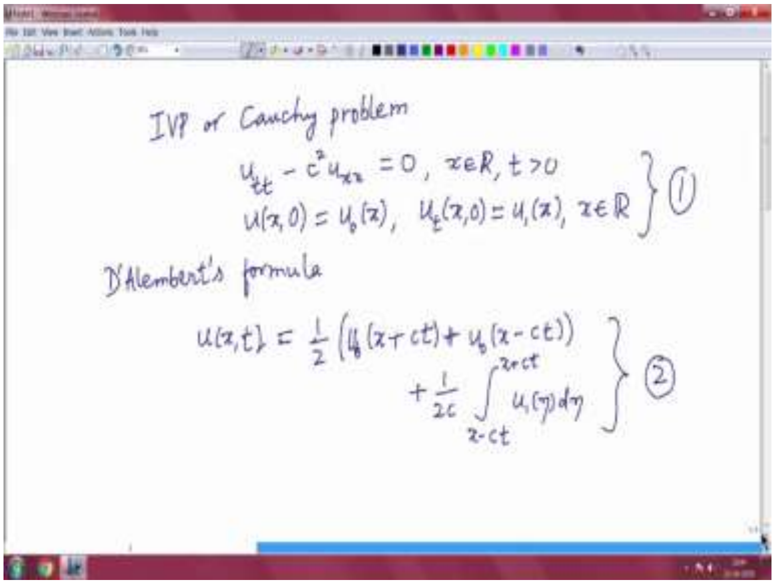

So we next use D'Alembert's formula to prove a simple estimate on the solution. So assume that the initial functions u0 and u1 assume u0 and u1 are bounded functions. Say, u1, u0x absolute value and u1x absolute value, both are less than equal to M for all x in R. Then you go back to the D'Alembert's formula.

So again, I just, yes, equation number 2, D'Alembert's formula, so now you take absolute value on both the sides, and by your assumption this u0, u0 is bounded by M, so you get 2M here, there is a half here so you get M. And again, in the integral side you take the absolute value and that is also bounded by M and then you integrate the constant, you get 2ct, again you get 2c, 2t cancels and what we get is, so this is a very simple estimate. I just write it.

Then for any t positive, so this is just direct consequence of the D'Alembert's formula. Then for any t positive we have mode of ux t, this I take less than or equal to M times 1 plus t for all x in R and t less than or equal to T. So there is a t here. So what this estimate says is that at any positive time t, u is also a bounded function of x, but it will not be a bounded function of t as t grows, so this right hand side also grows.

But this is useful, generally such estimates are useful in establishing uniqueness and continuous dependence on the initial data, so in fact this if you replace M by sup u0 and sup u1, you see that. So if you change u0 and u1 little bit, so the corresponding solution also changes very slightly and that is continuous dependence on the initial data. So in case c is not a constant then we do not have D'Alembert's formula.

In that case, deriving such estimates for solutions are very, very useful in establishing uniqueness and continuous dependence of solutions. A more physically relevant, so this is, you can say it is sup norm. So this estimate is in sup norm, because we are taking estimate in sup norm. We are taking supreme norm over x in R.

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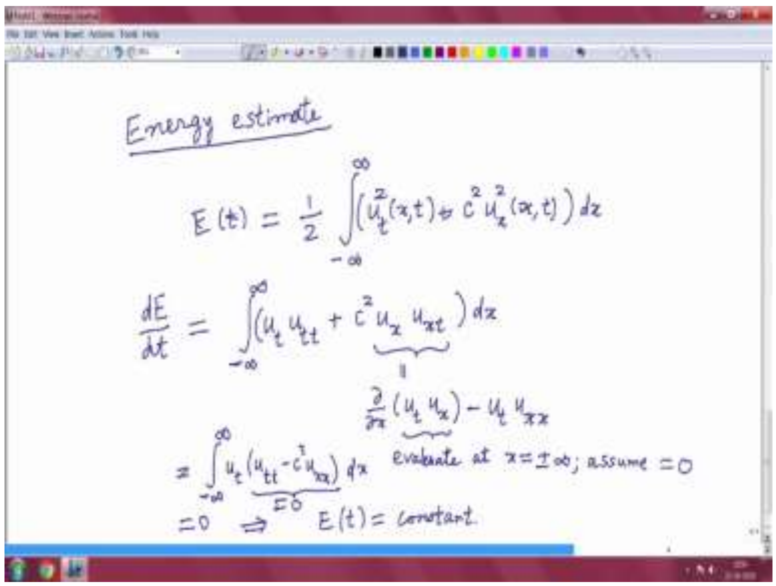

A more physically relevant norm is the energy norm, so let me briefly describe that and these are very useful in studying general hyperbolic equations, not only second order, but even higher orders and also systems. So, this energy, the sum of kinetic energy and potential energy, so this is total energy. So this is defined by at time t, so it is just half integral or minus infinity to infinity, u sup t, x,t square plus c square ux square xt and you integrate with respect to x.

So, the first term is kinetic energy and second term will be potential energy, so a sum of two energies. An integration by par, so let me just show you heuristically, so provided this integral is finite, if integral is infinite that does not make sense. So for the time being just assume that the integral is finite. Let me show you that this Et is constant, that means it does not depend on t.

For that, what we should do, we should consider this derivative with respect to t and show that that is 0. So again, formally, so assume that we can take the derivative inside the integral side, and you see that, so this is just nothing but minus infinity to infinity. The first term gives me ut, utt. So I differentiate the first term with respect to t, so that 2,2 goes away, so just to have utt, plus the second term, c square ux, uxt.

So, I am differentiating with respect to t, so that is what I get. These are just heuristic arguments; I will make remark at the end of this derivation. And now this one, you can write it as this is equal to, let me write it here, d by dx are ut, ux. So if I do that one term I get is this term and now there is another term by product role, so that I have to remove it. What is that term? That is precisely ut, uxx.

And remember we are integrating, so this is d by dx term, so I should just evaluate the limit of this at plus or minus infinity, evaluate at x equal to plus or minus infinity, and leave that out to dig the limit and assume they are 0, assume equal to 0. So they will not contribute anything to the integral, so what I get is simply minus infinity to infinity. So there is ut common here, there is ut here, there is ut here, so utt minus c square uxx, dx.

But this is, since u is solution of the wave equation, homogeneous wave equation, this is just, see the whole thing is 0, so that proves Et is a constant. So this is a conservative system. That can be expected because the equation is derived by using Newton's second law and most of the equation derived by Newton's second law are conservative equations. So the total energy is constant.

So in general, in the study of general hyperbolic equations, one considers such norms and tries to prove existence, uniqueness and other properties, because there are no explicit formulas for the solution.

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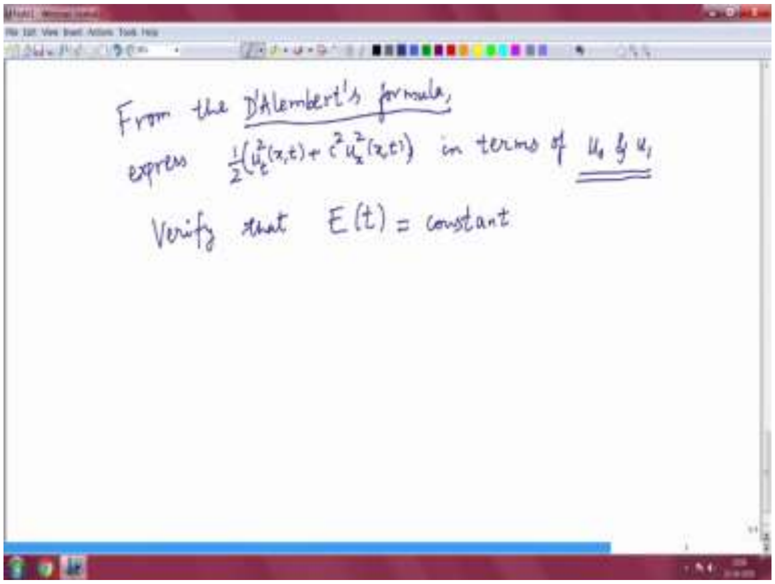

In the present case, so we have again the advantage of the D'Alembert's formula. So from the D'Alembert's formula, so this is an exercise for you, so it is a long calculation, so you have to do several, express this energy, namely this ut square x,t plus c square ux square x,t in terms of u0 and u1. So that is in terms of the initial energy, because at t equal to 0, u0 ux, for example, ux will be u0 prime and ut will be u1.

So you can express because we have the explicit formula joined by D'Alembert's formula and verify that Et is a constant. But this advantage of an explicit formula is missing when the coefficients are variables or higher order equations. So, one has to deal with the energy directly.

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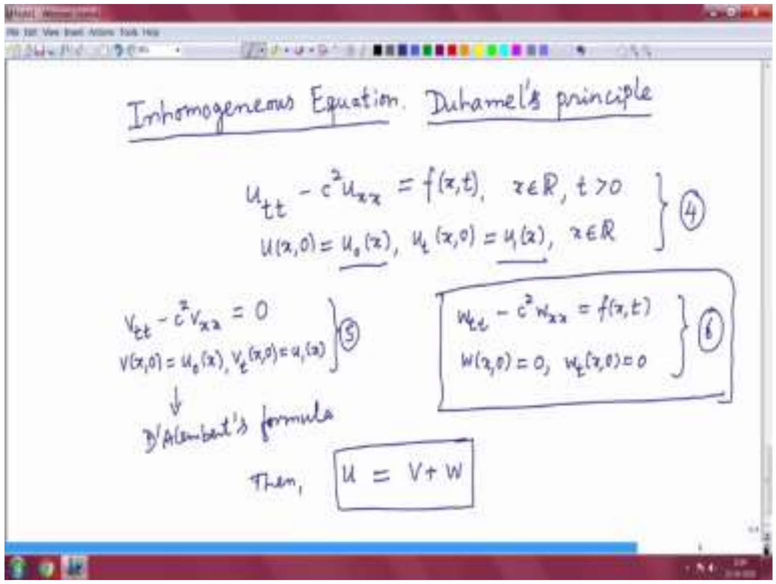

So with this remark, just to move on, so next discuss, so, so far we have discussed only the homogeneous wave equation, and now we will discuss inhomogeneous equation, and surprisingly even a formula for the solution of the inhomogeneous equation can be reduced to the solution of a homogeneous equation. And this is known as Duhamel's principle, an important tool, not only for the wave equation, but for many evolution equations.

We will see even for the heat equation this principle can be applied, very useful tool, so let me just describe that, Duhamel's principle. So in fact even for first order equation not, when you solve inhomogeneous equations you are using Duhamel's principle in some form, though you might not have noticed it, but it is hidden there.

So what is the problem? So this again wave equation, so instead of 0, now we have a forcing term, called so inhomogeneous term is called forcing term. So of course that will certainly affect the solution, so let us see how that. So again x is in R, and t positive and initial conditions, so that, let me write it, u1 x, so let me denote it by equation 4. So, obviously some continuity assumptions should be put on F.

So, let us first derive the formula and then we will see what conditions we should put on F, instead of stating in the beginning itself. So once you see the formula, we will know what condition to put on F, so that we will get again a c2 function. That is important. So as usual, this u0 is a c2 function and u1 is a c1 function. So that is always there, because even when F is 0, that we need to assume. So by linearity, so we consider two similar problems, so let me write it.

So this is homogeneous wave equation and I take the initial conditions as v,x0 equal u0,x and vt,0 equal to u1x. So let me call it 5. So let me not repeat where is the x and where is t, so that is understood now. And another problem, now I take the inhomogeneous equation, so wtt minus c square wxx, F xt and now, w,x0 is 0 and wt,x0 is 0, 6. For this problem 5, we already solved this and the v is given by the D'Alembert's formula, so there is no problem with that.

There is no problem. But we have not done this one. So we have by linearity, so that is an important observation, so then the solution u of problem 4 is sum of v and w. So linearity plays a crucial role here and you can easily verify that the solution u of problem 4 is given as sum of solution of problem 5 and solution of problem 6.

So problem 5, as I said, so it is already done there, so we have the solution given by the D'Alembert's formula, so what remains to do is this problem 6. So with this reduction it is sufficient to assume that the initial conditions are 0, so that is homogeneous initial conditions, only there is inhomogeneous term in the equation.

And this is solved by the Duhamel’s principle. And so to solve 6, we convert that into an initial value problem, so that is the idea of Duhamel’s principle.

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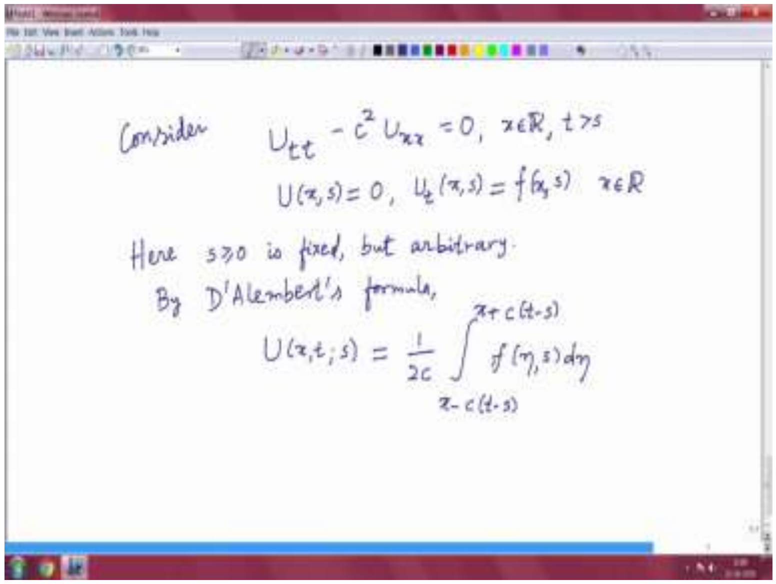

So consider now a homogeneous equation, c square uxx equal to 0, x in R, but I take t bigger than s, what is it, in a minute I will tell you. And now I prescribe the initial conditions at time t equal to s, not 0, but at s. So this I made remark in the beginning itself, so we can, x, this you provide it by x, x in R. So the inhomogeneous term, if you look at it, inhomogeneous term F appears as initial conditions for this problem.

So here, s is bigger than or equal to 0 is fixed but arbitrary. So as we change s, so the problem changes and this U also changes. So this U in principle is a function of xt and s arbitrary. So by D'Alembert's formula we have u of xt, just to stress the dependence on s, because it also depends on s, so we write this as s, that is to just indicate U also depends on s.

So if you again look at the D'Alembert's formula, the U is 0, so this will not contribute anything, so only the, first derivative is given and that is given by the integral of that initial condition. And now we have to replace t by t minus s, remember that. So, x minus c, t minus s, x plus c, t minus s, F for Eta s, d Eta. So we will complete the solution of problem 6 in our next class. So just remember this one and we will continue this in the next class. Thank you.