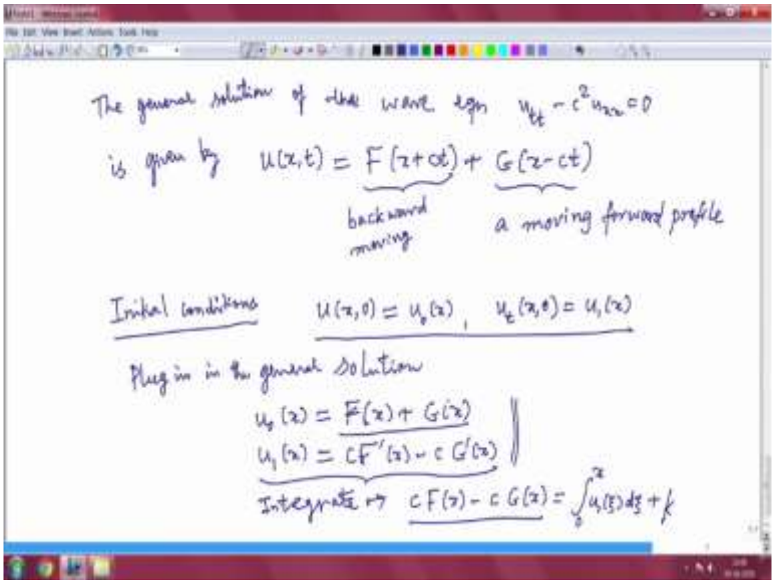
Partial Differential Equations - 1
Professor A. K. Nandakumaran
Department of Mathematics
Indian Institute of Science Bengaluru
Professor P. S. Datthi
Former Faculty, Tata Institute of Fundamental Research - Centre for Applicable Mathematics, Bengaluru
Lecture 35
One Dimensional Wave Equation

(Refer Slide Time: 0:35)



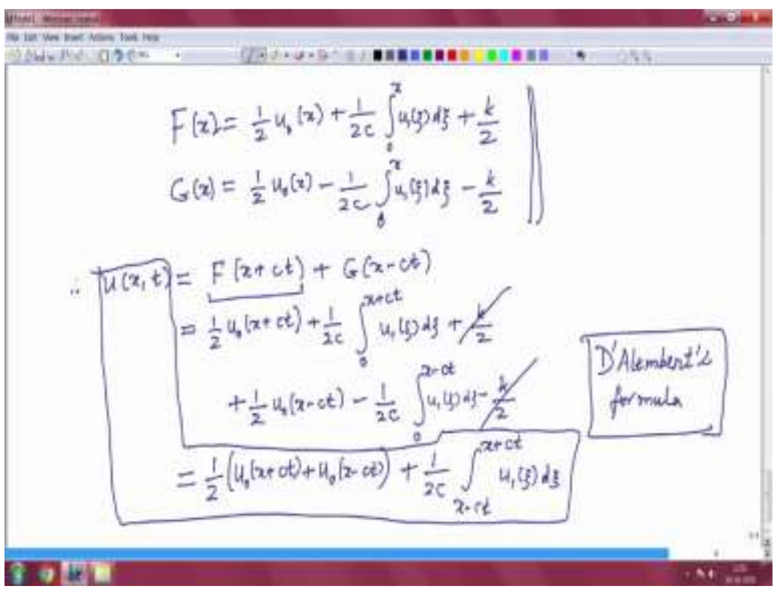
So, in the previous lecture we saw the general solution of the wave equation, utt minus c square uxx equal to 0 is given by u xt equal to F of x plus ct plus G of x minus ct. So, this actually represents a wave moving forward direction, maybe later I will show you some pictures, so you can yourself take some simple function and try to draw these functions for different values of c and observe this phenomenon moving forward and this is moving properly, so backward.

So, in general, the solution of the wave equation is a combination of forward moving profile and backward moving profile and now we will determine F and G given two initial conditions. So, let me just, initial conditions, so u x0, u0 x and ut x0 equal to u1 x. So, here we have taken t equal to 0 at the initial time, so the first condition represents the initial profile of the string for example, and the second condition represents the initial velocity in the string.

So, now we will see how this initial conditions help us to determine F and G in the general solution. So, if you plug in these initial conditions in the general solution, we see that, so plug in, in the general solution. So, we get u0 x, I put t equal to 0 here, you see that, so F of x plus G of x and now you take time derivative of the general solution and again put t equal to 0, so left hand side you get u1 of x and when I differentiate with respect to time there, so I get cF prime x minus cG prime x, because there is minus c there so that is why I am getting minus c.

So, these are linear equations but only thing is now F and G are not numbers, they are functions, but we can easily determine them, so you integrate the second equation. So, you integrate it. You get cF x minus cG x is equal to integral 0 to x u1, let me give the variable xi d xi, plus some constant where k is a constant of integration. So, we will see that that is irrelevant, we will see that so. So, now we have one equation with Fx plus Gx and another one essentially Fx minus Gx, so we can easily solve them.

(Refer Slide Time: 6:38)

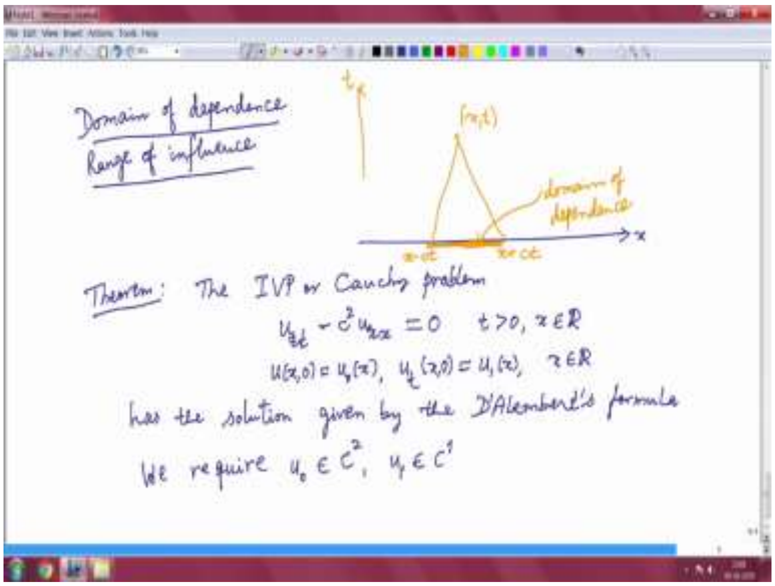


So, I will write the formula here. So, we get F of x is equal to half u0 x plus 1 by 2c 0 to x, u1 xi d xi, do not confuse that xi with characteristic variable, this is just a dummy variable for integration, k by 2 and G of x half u 0x minus 1 by 2c, 0 to x, same thing, minus k by 2. So, these expressions for F and G we will be using many times in the future also, so just remember how they are derived.

Now, we go back to the general formula and now find out the solution for the wave equation. So, therefore, u of xt, so we have that, F of x plus ct plus G of x minus ct, so now I have expressions for F of x and G of x, so just we have to replace x by x plus ct and x by x minus ct. So, I get half u0 x plus ct plus 1 by 2c, 0 to integral, 0 to x plus ct, u1 xi d xi, t is a constant so that will not change, so just give it. This is for F of x plus ct. And similarly for G of x minus ct, so I have half u0 x minus ct minus 1 by 2c integral 0 to x minus ct, u1 xi d xi minus k by 2.

So, you immediately see that constant of integration has no role to play, so that will just vanish and so we will simplify this half u0 x plus ct plus u0 x minus ct, combining these two terms, and combining the integral, so we have 1 by 2c, so 0 to x minus ct we will write it as minus x minus ct to 0 and combine it, so we get x minus ct to x plus ct u1 xi d xi, so we write that. So, this is called D'Alembert's formula. So, remember it, because we will be using it throughout the discussion on one dimensional wave equation.

(Refer Slide Time: 11:35)



So, let us analyze this formula a little bit, so now we discuss two concepts; one is, domain of dependence and another concept, range of influence and there is also domain of determination, that will come next, a little later, range of influence. So, what are these? So, again, let me draw the picture, x and this is t. So, we want to determine the solution, so let me write that as a theorem maybe, let me write it.

The Initial Value Problem, IVP are also called Cauchy problem, utt, so let me write it, uxx. So, this is in the region t positive and x is in R, and we are given the initial profile, u1 x, again x is in R. So, the equation and the condition, the two initial conditions in this case and this is referred to as Cauchy problem, has the solution given by the D'Alembert's formula. So, let me not repeat it. So, a word about the concept of solution again, so this we have discussed earlier also.

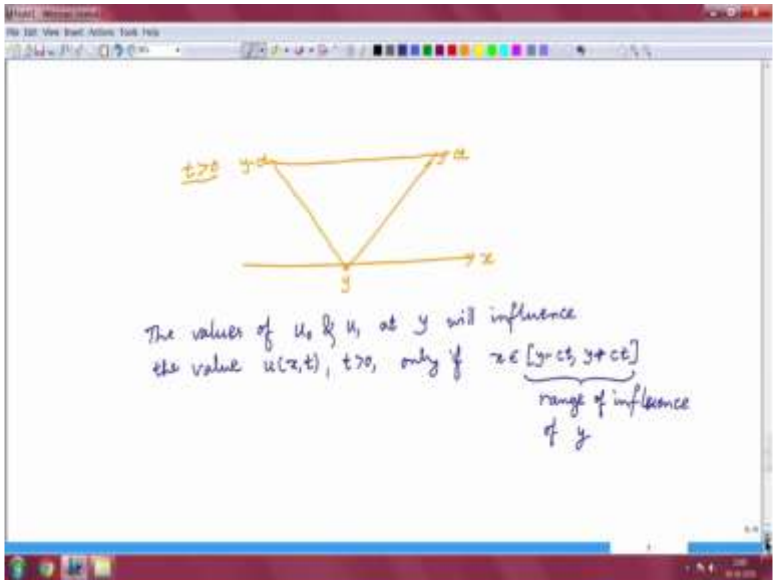
So, here we are seeking c2 solution to the wave equation satisfying the given initial conditions. So, for that we do require, if you look at the D'Alembert's formula, we require u0, a c2 function and since in the D'Alembert's formula u1 gets integrated, so for that we only need c1. So, then, if u0 is a c2 function and u1 is a c1 function, then the wave equation has the solution given by D'Alembert's formula, hence it is unique.

So, here we do not have to prove the uniqueness separately because we are deriving a formula for the solution. But in many other cases that uniqueness has to be proved separately that I will make some remarks. So, if you look at the D'Alembert's formula, so let me just remove this thing and just put a t here, so we are trying to find the solution at a point x, t, so if you look at the D'Alembert's formula, it depends on the initial conditions.

So, this is, if that is x, t, this will be x plus ct and this is x minus ct. So, D'Alembert's formula tells us that the value of the solution at the point x, t, t is positive, depends only on the values of the initial values only in this interval. That is important. Nothing else. So, u0 x plus ct, u0 x minus ct and u1 is integrated over this interval.

So, this is referred to as, this interval, domain of dependence. So, in other words, the value of the solution at the point x, t depends on the values of the initial data only in this interval, so that is how the finite speed of propagation comes into picture. So, what is range of influence, so that is another…

(Refer Slide Time: 19:36)

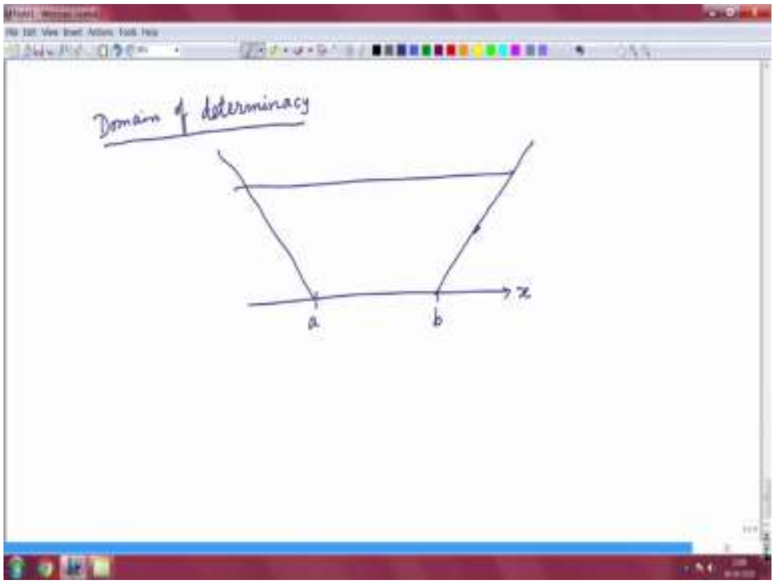


So, we take a point y on the initial line t equal to 0, that is x axis are t equal to 0 and now we would like to see the value of the initial data at this point, how it influences the solution for a positive t. So, you take a positive t, fix it. So, this, again you look at the D'Alembert's formula, you see that, so this go there, this goes here. So, this is y minus ct, y plus ct.

So, let me go back to the color. The values of the initial u0 and u1 at y, that is on the initial line, will influence the value u xt, so t is positive only if x belongs to y minus ct and y plus ct. So, as you change t, of course that interval gets expanded. So, this is referred to as range of influence. So, these are two important concepts and these are also used in devising numerical schemes for solving the wave equation, when you want to solve numerically range of place of all.

So, these are two important concepts. And now if you combine these two, so that is what we next get, so instead of taking one single point, you take an interval.

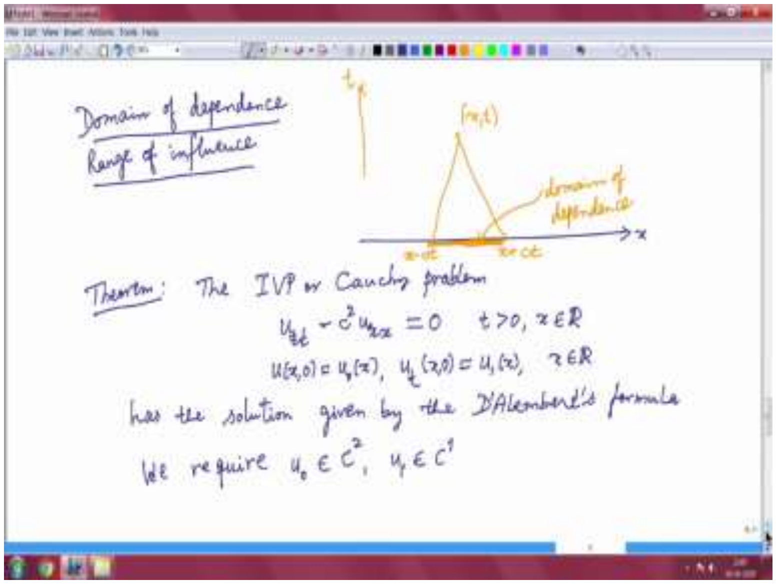
(Refer Slide Time: 23:07)



So, this another concept, so instead of taking one point on the, so domain of determinacy. And again on the initial line t equal to 0, now you take an interval A, B, so now you determine the range of influence of all these points in this interval, and what you get is this. So, at any time this will become the domain of determinacy.

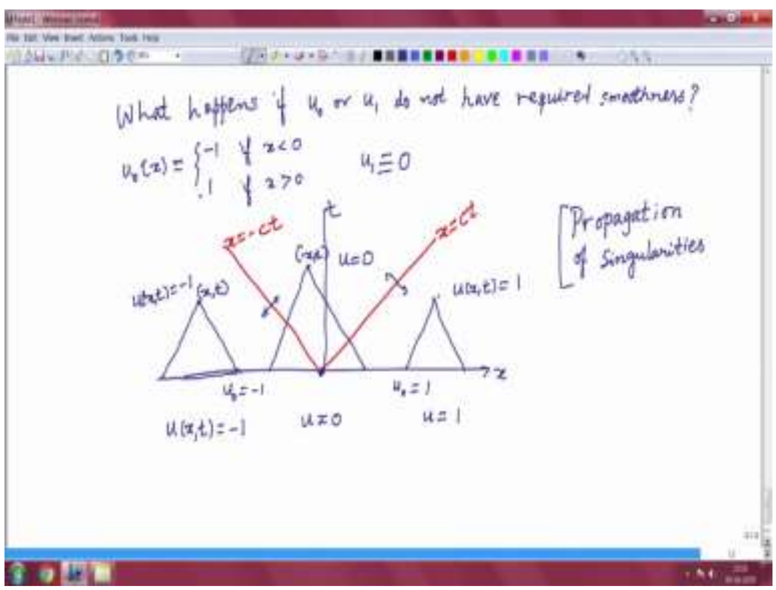
So, this is domain determinacy. So, you take the union of the range of influence of all the points in this interval, so that is the concept of domain of determination. So, before we go further, just let me make one comment. So, again, let me stress that.

(Refer Slide Time: 24:45)



So, in order to get a c2 solution of the wave equation we need to assume that u0 is a c2 function and u1 is a c1 function. What happens if I relax this condition? So, let us take some examples and see what we get. Of course, we do not expect a c2 solution, but we will see what happens.

(Refer Slide Time: 25:42)



What happens, so that is the question. What happens if u0 or u1 or both do not have required smoothness? So, this is an important concept in modern theory, so I just want to make a remark, we are not going smoothness. So, let me take simple example, u0 x has a jump discontinuity, minus 1 if x is less than 0 plus 1.

So, you see that u0 has simple jump discontinuity at x equal to 0 and I am u1 for simplicity, you can work out many problems. So, let me explain the solution graphically, so that is much easier, so this x and t. So, I have here u0 equal to minus 1 and u0 equal to plus 1. So, we expect some difficulty at this origin. So, let us draw the characteristics to that. So, this is x equal to ct, characteristic, so let me draw this another one, x equal to minus ct.

They are straight lines, though they do not look straight lines here, but okay. I am writing that. So, if you take any point here x, t, what is the relation between x and t in this case, so it lies below the line x equal to minus ct, so x is less than minus ct. And if you look at the D'Alembert’s formula, is domain of dependence, so that is why it is important, very important.

So, its domain of dependence lies entirely in the negative x axis, that is very important. If you take any point below this line, its domain of dependence lies in the negative x axis where u0 is minus 1 and u1 is identically 0, so you get u of x, t, look at the D'Alembert’s formula, you will simply get minus 1. So, this region you get u of x, t, so let me write here.

For all the points below this characteristic x equal to minus ct. And similarly if you take any point below the characteristic x plus ct and again its domain of dependence lies entirely on the positive x axis, so you get again u of x, t is equal to plus 1, because u0 is 1 here, so just plug in the D'Alembert’s formula and you get that.

But if you take any point between these two characteristics, you take any x, t, here so its domain of dependence now, one endpoint lies in the negative x axis and another one lies in the positive x axis. Again, if you plug in the D'Alembert’s formula, you see that u is minus 1, plus 1, because u1 is anyhow 0, so you get u is 0 here. So, u is minus 1 here, in this region u is 0, and u is 1 again.

So, what we see here is that in three different regions, so this is separated by two characteristics, x equal to minus ct and x equal to ct, so u is c2, in fact it is as smooth as u0, they are c infinity here, so it is also c infinity, no problem there, it is constant functions. And you see it has jumped discontinuity across this characteristic and it has jumped discontinuity.

So, the jump discontinuity in the initial value at one point now propagates along these characteristics emanating from that point, not any other characteristic, only those two characteristics emanating from that origin, origin is the point of singularity of the initial conditions. So, this is termed as propagation of singularities and this is an important topic and nowadays used in many inverse problems, but it is very difficult topic.

So, those who are interested can look into literature to have a glimpse of what this propagation of singularities for general hyperbolic equation, so this is the simplest case, we have D'Alembert’s solution, so we can see where these solutions has discontinuities. So, it is important to note that the solution has discontinuity only along the characteristics emanating from that one point where the initial condition has jumped discontinuity.

So, I will stop here and next time I will make a little more elaboration on this propagation of singularities and then we will move on to other topics in one dimensional wave equation. Thank you.