Partial Differential Equations - 1
Professor A K Nandakumaran
Department of Mathematics
Indian Institute of Science Bengaluru
Lecture 34
One Dimensional Wave Equation

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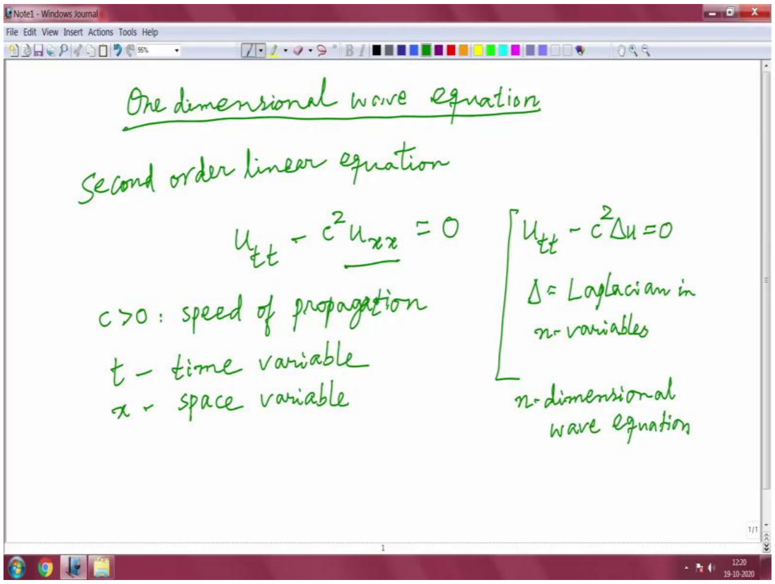

In this talk, we start the analysis of one dimensional wave equation. So, probably in the next five or six lectures, we will be studying many aspects of this one dimensional equations. So let me write down the equation first. So this is a second order linear equation given by utt minus c square uxx equal to 0.

So, this is called homogeneous wave equation and c is a positive constant known as speed of propagation, so we will see a little later why it presents speed of propagation. So, this t is the time variable, so let me just introduce this time variable, and x is the space variable. So, x is space variable. So, it is called one dimensional wave equation because x is a real variable. So, if you replace this uxx by the Laplacian u, so this just, so and Laplacian, Laplacian in n variables, so this, you study during the discussion on Laplacian, Poisson equation, Laplacian in n variables. And then this is termed as n dimensional wave equation.

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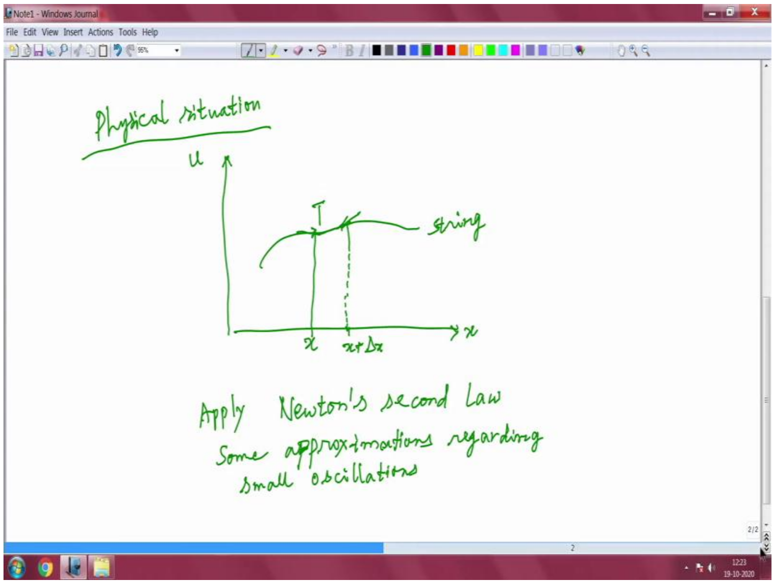

A physical situation where this equation arises, I just briefly discussed that, so a physical situation, so that generally represents vibrations in a string, physical situation. So this is the x axis and this is the u axis and so as typical string, so you are plucking it, so this is string. I am not going to give the details but just say how it is derived. So you take two close by points on the x axis, x plus delta x.

And you calculate the force acting at these two points, so that is the, so they act along the tangent to the string, call it t and then you apply Newton's second law. Apply Newton's second law, which is mass times acceleration, and that gives you del square u by del t square term and force gives you del square u by del x square term and that is how you derive the wave equation.

It is important that so that while applying this Newton's second law, some approximations are made, some approximations are made so that you can see in our book on PDE some approximations regarding small oscillations. So this is not valid for large oscillations. So if you take arbitrary oscillations then the equation becomes non-linear which is not difficult to analyze. So again, we go back to, so return to the wave equation again.

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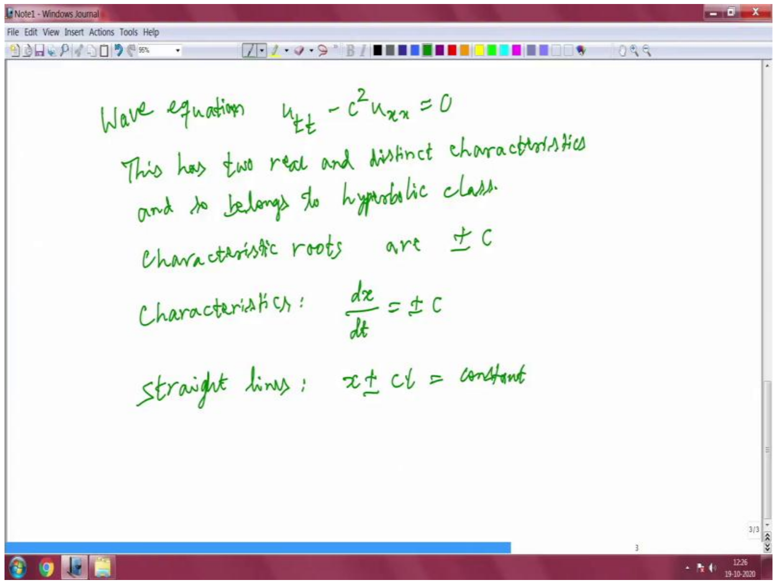

So we have already learned in the lectures on classification of second-order equations. So this has two real and distinct characteristics and so it belongs to hyperbolic class. So this is a typical example of a hyperbolic equation. So what are the characteristics? So you see the characteristic equation, maybe I will use a different notation here, so characteristic roots are plus or minus c. Hence the characteristics, so characteristics are obtained by solving the first order equations, characteristics are given by dx by dt is equal to plus or minus c. And these are the straight lines, so (characteristic) in this case straight lines, x plus or minus ct equal to constant.

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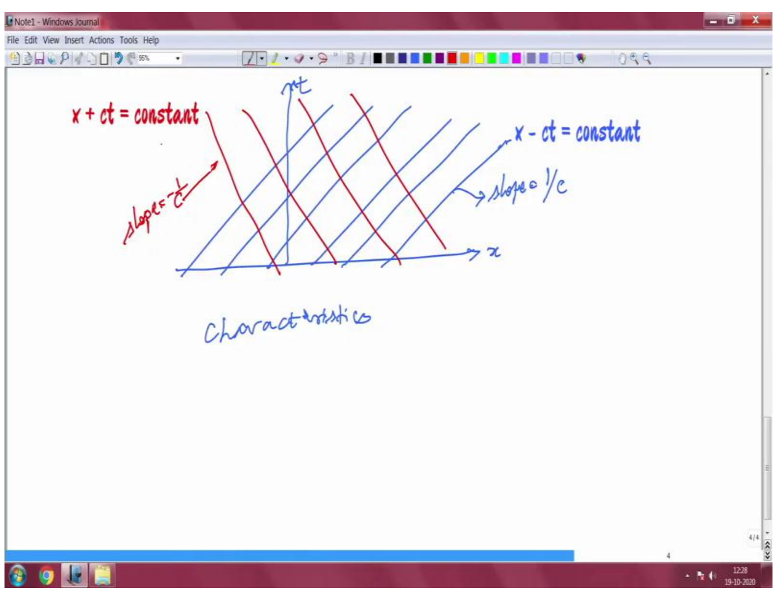

So, let me just draw some pictures, so typically you, so this is x axis and this is t axis. So the characteristics, so we have them. So, two families of straight lines, so these are x minus ct, maybe let me just, so this is x minus ct equal to constants. So this is one family of characteristics and another family, so let me use different color. So this is x plus ct equal to constant. So, the blue ones have slope. This is slope, so you are drawing, remember this, we are drawing x axis horizontal and t is vertical, so slope is 1 by c and the red one slope is minus 1 by c.

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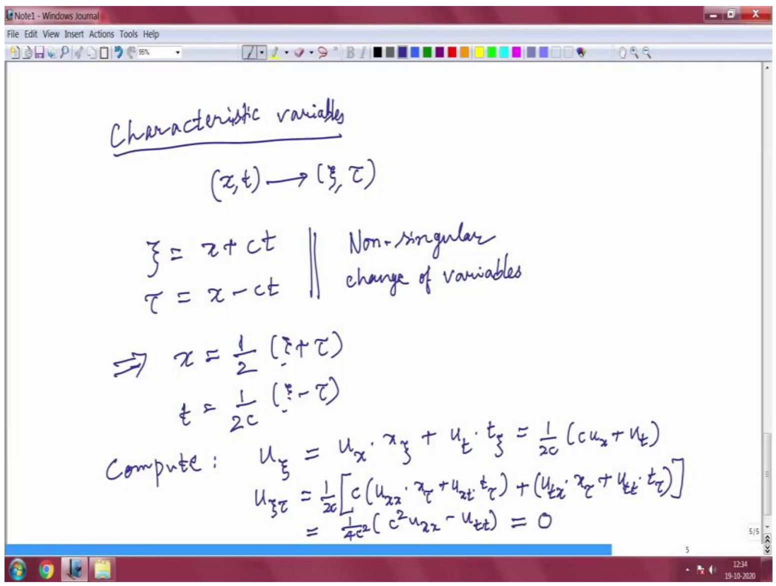

So this suggests us to use characteristic variables. So we are changing the cornice xt to xi tau, so let me write that. So what is xi, xi is x plus ct and tau is x minus ct. And this change of variables suggested by the characteristics explained in the previous sheet. The important thing to notice, this we already learned during the classification of second-order equations, the important thing is, this is non-singular change of variables. That is very important. So if you compute the Jacobian, you see that it is non-zero everywhere, change of variables. So, this helps us going from one set of variables to another set of coordinate system.

So, it is easy to see that, so this implies, so the inverse if you call it, so from xi tau we can go to xt, so the x is just, you do that, simple algebra, xi plus tau and t is 1 by 2c xi minus tau. So, now we will see that a simple implicit differentiation gives us. So, next compute, u xi, so this is by implicit differentiation so this is ux, x xi, let me write it, so you can work out t xi. And x xi you see is half and t xi is alpha 1 by 2c, so you get 1 by 2c, so simplification, c ux plus ut.

And now you do differentiation with respect tau, so u xi tau, so this is 1 by 2c is constant. There is a c constant there. And now I have to do this differentiation of ux with respect to tau. So for that thing again I use implicit differentiation, so x tau plus uxt t tau, so that is with respect to ux and similarly I do with ut and t tau. So if you now simplify and now x tau is again half, but t tau is minus 1 by 2c, so you see that you finally get, there is 1 by 2c here, 1 by 2c, so 1 by 4c c square, so some terms get canceled and you can easily verify that. So maybe c square here, 2c, 2c you get, and that is equal to 0.

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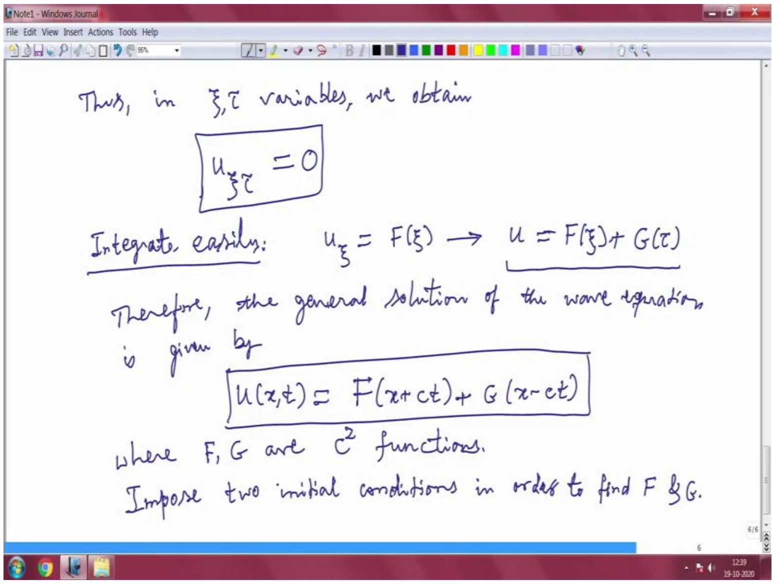

So with this change of variable we see that the wave equation reduces, thus in xi tau variables we obtain u xi tau equal to 0. So the wave equation reduces to this simpler form, so it is still second order, but this one we can integrate easily. So integrate easily. Wave equation we could not do, but now in these characteristics, characteristic variables we can easily integrate, for example, first you integrate with respect to tau, so you get u xi. So this is purely a function of xi. So when I differentiate this with respect to tau, it will be 0.

And one more integration, so if I integrate this, this is a function of only xi, so when I integrate again with respect to xi, so I will get another function of xi, but that constant of integration can be a function of tau. So that leads to one more integration, so u is equal to, again I write it F xi, but this is another different function because this is integration of that. But this one I have xi tau.

So you can easily verify that if F and G are two times differentiable, so any function of this form satisfies u xi tau equal to 0. So therefore, the general solution now, now we go back to the xt variables, original variables, general solution of the wave equation is given by uxt is equal to F x plus ct, xi is x plus ct and tau is x minus ct.

So, where F and G are arbitrary c2 functions. So c2 means two times continuously differentiable functions. So, where F, G are c2 functions. So, you can also easily verify if u is given by that and if you have here two c2 functions, then directly you can differentiate and verify that u indeed satisfies the wave equation. So there is no difficulty in doing that thing.

So, in order to determine this F and G, so there are two, these are two unknown functions, F and G, so you will need to impose two conditions in order to compute F and G. And that is done by imposing two initial conditions, so this is called initial value program, and that impose to initial conditions in order to find the F and G. And this will lead to D'Alembert’s formula which we will take up in the next class.