First Course on Partial Defferential Equations – 1

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Lecture 33
One dimensional heat equation-6

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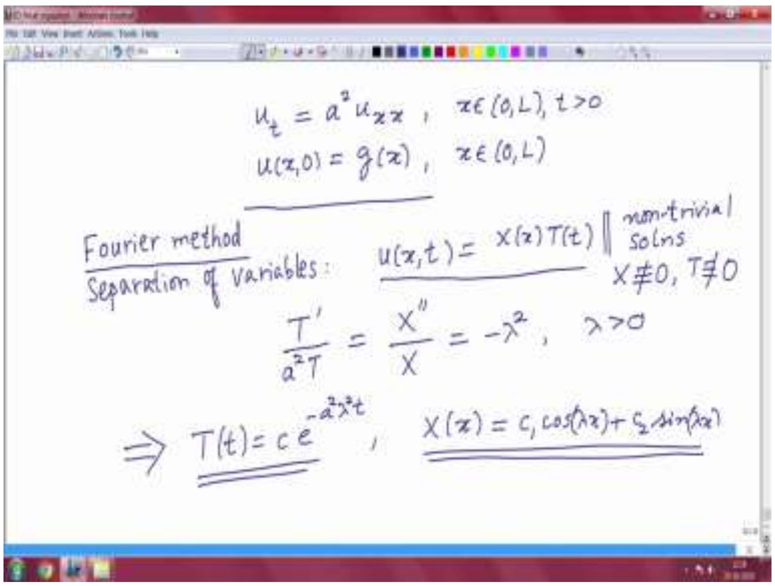

In this lecture we discuss the Fourier method to find the formula for the solution of the heat equation satisfying given boundary conditions. So again let me begin with the statement of the problem. So this is heat equation, so now we are working in a finite interval. So x is in, L is some positive number and t bigger than 0. So, initial condition is given, so before prescribing the boundary conditions at x equal to 0 and x equal to L.

So let us begin with the procedure for this Fourier method or separation of variables. So in this method the solution is sort in this form, namely X x, so this is what we did even while obtaining the solutions for the heat equation in the entire line, entire real line. This is what we started with, then you plug in this expression in the heat equation and you obtain T prime by a square T is equal to X prime by X and again as we discuss there this is the function of T only and this is a function of X only it must be constant.

And for physical reasoning we again take negative sign. Where lambda is positive and by the, so we are seeking non trivial solutions. So, that means these both the functions are not identically 0. But then this is just a 0 solution and it may not satisfy the initial condition. So whatever boundary condition we put at X equal to 0 and X equal to L they transform into boundary conditions for the function X.

So if you take positive sign again let me repeat that if you take positive sign that produces an exponentially growing T which is physically not acceptable. So we reject that the positive sign and take consider only negative sign. So in that case we immediately get the T of t some constant so let me write c1 c exponential minus a square lambda square t. And for x we have so these will be the common features for all the boundary condition we consider and we have to only see which boundary conditions give us non trivial solutions that is important for us.

So for all the things this will be the common things only things is we have to determine the values of those lambda for which we get non trivial solutions. So this is certainly non trivial solution for T all the time. If we take c not equal to 0 but here since x has to satisfy appropriate boundary conditions the choose of c1, c2 at least one of them non-zero is possible or not we have to see case by case.

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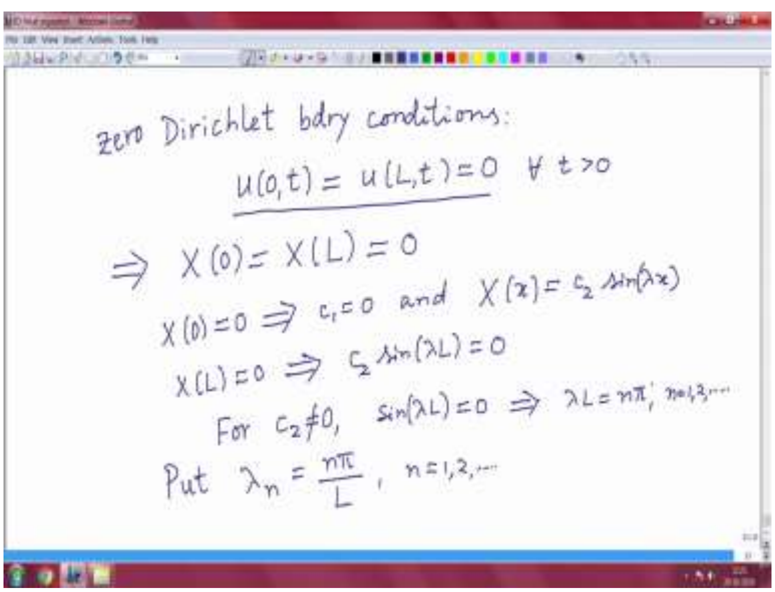

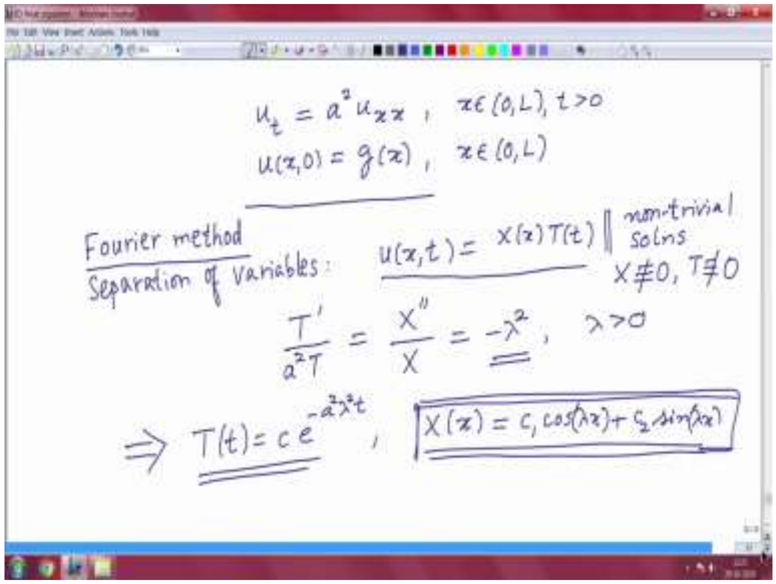

So first we begin with simple boundary condition Dirichlet zero Dirichlet boundary condition namely u of 0 t is equal to u of L t is 0 for all t positive. So let us begin with this boundary condition. And these two boundary conditions transform into the boundary condition for X so we get X 0 equal to X L equal to 0. So if you put so remember just let me write that X x is c1 x c1 cos lambda x and c2 sin lambda x so if I put x equal to 0 cos is 1 sin is 0 that implies c1 is 0.

So X 0 equal to 0 implies c1 equal to 0 and then X x becomes just c2 sin lambda x the other boundary condition namely X L equal to 0 implies c2 sin lambda L is 0. So if you take c2 equal to 0 then x is identically 0. We are not interested in the trivial solution. So for c2 non zero this sin lambda L should be 0 and that implies so lambda cannot be arbitrary. So lambda L just n pi L and if we cannot again take lambda equal to 0 so this is n equal 1, 2, et cetera.

So in that case we obtain, so there is desecrate set of lambda so call it lambda n so put lambda n is equal to n pi by L n equal to. So this is general feature of the method, so this in order to find the non-trivial solutions the lambdas will form only a desecrate set and in this particular example we have seen that desecrate set is given by n pi by L.

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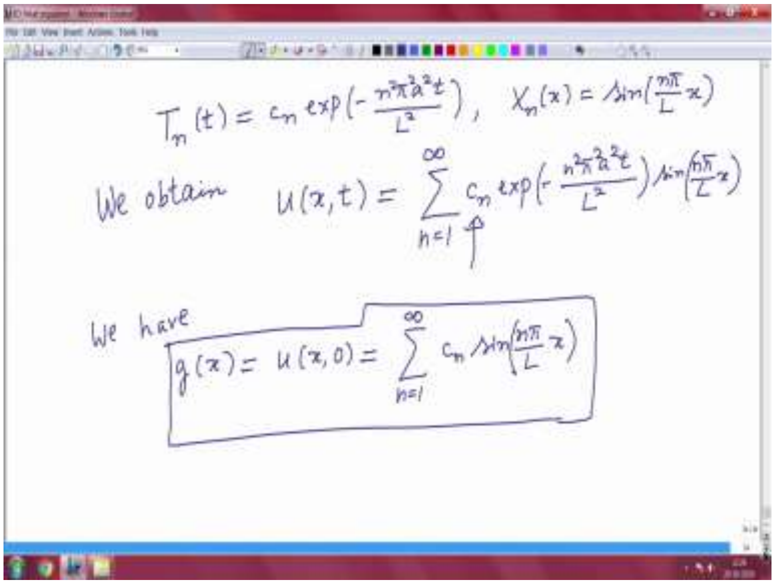

And the corresponding solutions so write it so Tn t some constant cn exponential minus n square pi square a square t by L square and Xn X so since we are multiplying t and x so that constant will not right here so we will just write simply sin n pi by L x. So again by linear superposition principle so we obtain a solution this is formally so just sum them u x t is equal to summation n equal to 0 to infinity n equal 1 to infinity n 0 cn exponential let mr write it again sin n pi by L x.

So we need to determine these constants and constant will be determine by using the prescribed initial condition. So we have g of x is equal to u x 0 so simply substituting t equal to 0 in the infinite series where cn. So this means the requirement on g is that it is expressed as a Fourier sin series which is convergent at all the points in that interval 0 n and this is provided by the Dirichlet theorem in the theory of Fourier series.

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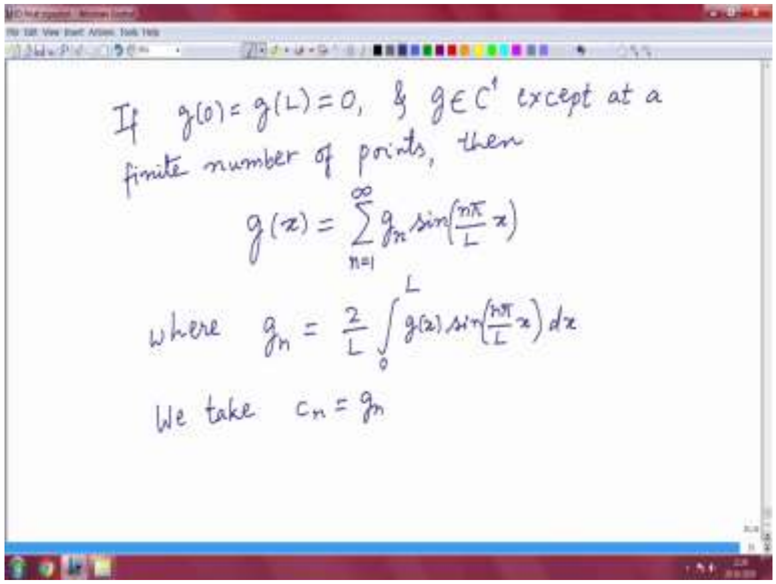

So we take minimum requirement on g namely so if g 0 is equal to g L equal to 0 if g vanish at the end points and g is c1 except at a finite number of points. Some more precisely you can take pice wise c1. So this Dirichlet theorem from the theory of Fourier series. So then g is expressed as Fourier sin series and the series is convergent to g x. so in general Fourier series you can always from that may not converse the function so there are some condition required.

And Dirichlet theorem provides these are sufficient conditions for the convergence of this infinite series sin series. Where gn are the Fourier coefficient of g where gn 2 by L 0 to L g x sin. So we take cn equal to gn and with this hypothesis on g then finally we obtain the solution.

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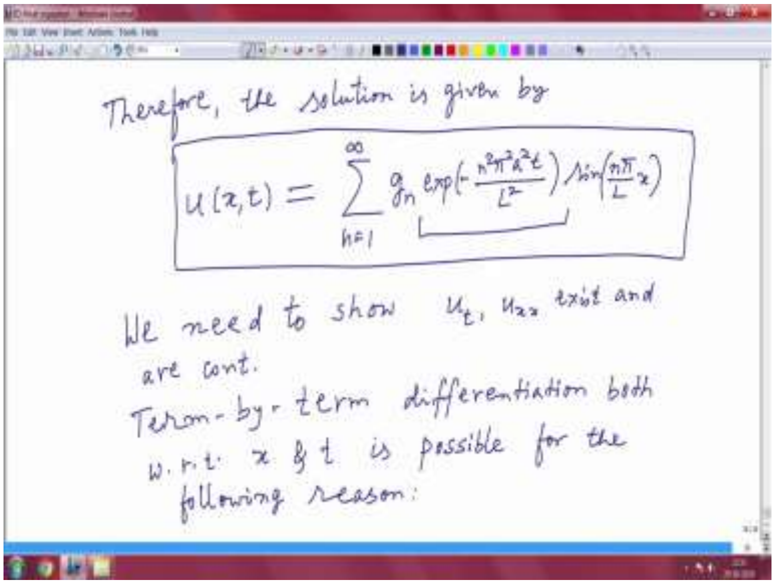

So therefore the solution is given by, I should not call it solution act, because we need to verify it is a solution. Solution is given by u of x t is summation n equal to 1 to infinity gn exponential n square pi square a square t by L square sin n pi by L x, where gns are the Fourier sin coefficients of the function g.

So with the stated hypothesis on g that is always possible, so we do get this infinite series involving sin function and exponential function. So what we still need to do is that u is, we need to show u is ut uxx exist and are continuous, so when u given by this infinite series the only way to check for the existence of these derivatives is again relying on the term by term differentiation and in this case this exponential term saves us.

So there is absolutely no difficulty in differentiating this infinite series term by term both with respect to t as well as with respect to x and the reason for that is, term by term differentiation both with respect to x and t is possible for the following reason. So let me write it separately.

So once we see in one particular case so if you just remember that so it smoothly goes through in other cases also. So look at these terms in the infinite series suppose I differentiate with respect to t I get an n square factor and if I differentiate with respect to x two times again I get an n square factor. So somehow that n square factor has to be tackled.

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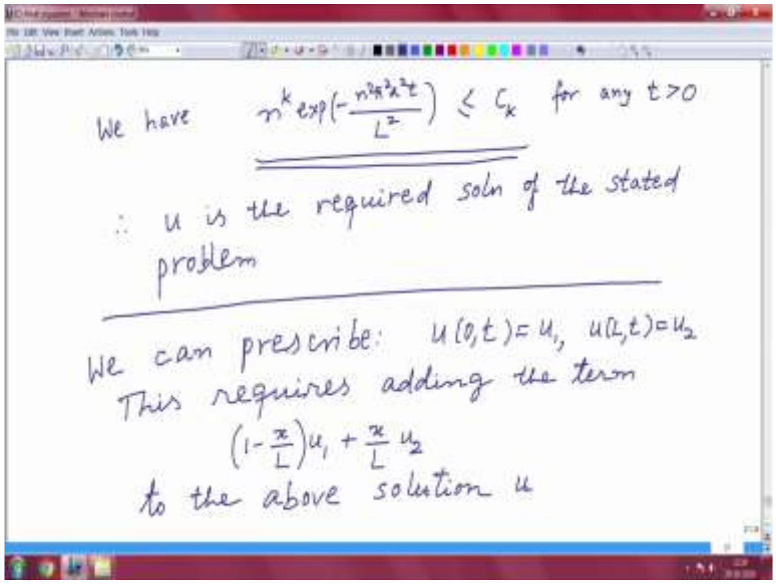

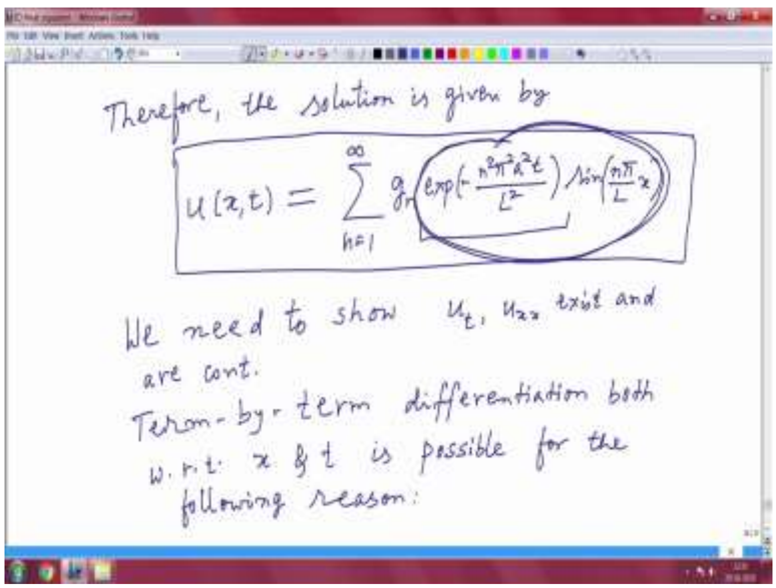

And in fact more generally we have n to the K exponential minus n square pi square a square t. So you consider this sequence as n varies is bounded for any t. So whatever the terms come there term by term differentiation I can always make some big factor of n in the denominator so that the series converges and then you are justifying term by term differentiation both with respect to x and t so you simply do without any problem because of this observation.

And once we do term by term differentiation so we already seen that this is solution of the heat equation. So if you do one differentiation with respect to t one time and differentiation of this sin term two times so each of this term satisfy heat equation so with this so there is absolutely no problem in doing term by term differentiation so therefore u is the require. So we already seen the uniqueness so this is the only solution of the problem required solution of the said problem stated problem. So this is a initial boundary value problem.

So we took the 0 directly boundary conditions here physically we say that the thin rod is insulated at both the ends. So this we can replace by some prescribed. So this is another so we can prescribe this is very simple with boundary conditions u 0 t equal to u1 u L t equal u2. So u1 and u2 are constants that means the end points of the rod are maintained at some fixed temperature. So this is not the difficult problem at all

what you do is we already have the solution with 0 boundary conditions for that you just add one term this requires again by linearity. Linearity is very important this requires adding the term. So what term you add? So we add x equal to 0 I want it to be u1 so you add 1 x by L u1 and L I want it to be u2 so this is just x by L u2.

So this function satisfies the boundary conditions and since it is only a function linear function of x so obviously its satisfy the heat equation. You just add this terms to the solution to the above solution u. And then you get the formula for the solution of this prescribed boundary conditions may be non-zero. So this is a simple observation.

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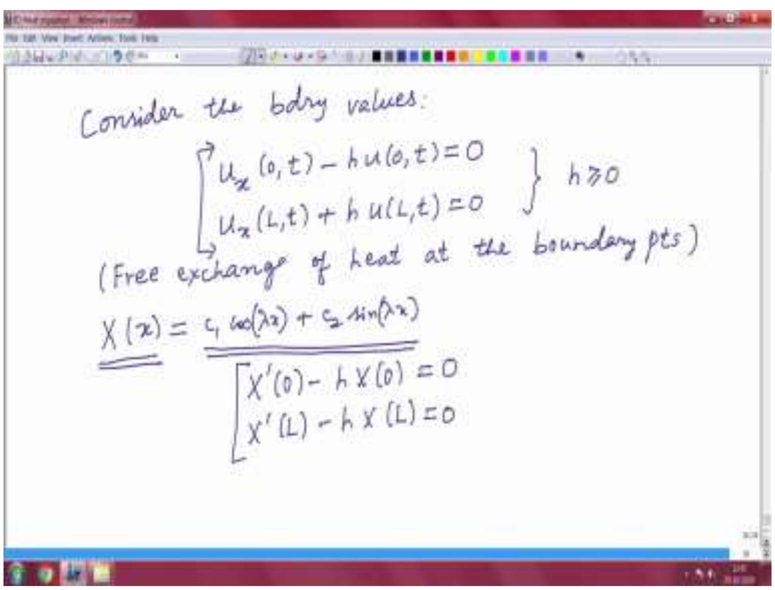

So let us take one more, but in this case we will not explicit formula for the lambda n's, so consider the boundary conditions. And this now requires the Sturm–Liouville theory from ordinary differential equations for the existence of solutions and the orthogonality conditions et cetera. So I will briefly discus this one and may be provide some notes later.

So these boundary conditions ux at 0 t minus h u 0 t equal to 0 and ux L t plus h u L t equal to 0. So here h is a constant, non-negative constant, physically this is refer to as free exchange of heat at the end points. So this u sub x that first derivative refers to the flow of heat across the boundary the points exchange of heat at the boundary points.

So now again recall this X x is c1 cosine lambda x c2 sin lambda x. So these boundary conditions transform to boundary condition for x at 0 and L. So that means we have X prime 0 minus h X 0 equal to 0 and X prime L minus h X L equal to 0. Now you write down these two conditions in terms of c1 and c2 that require the algebra.

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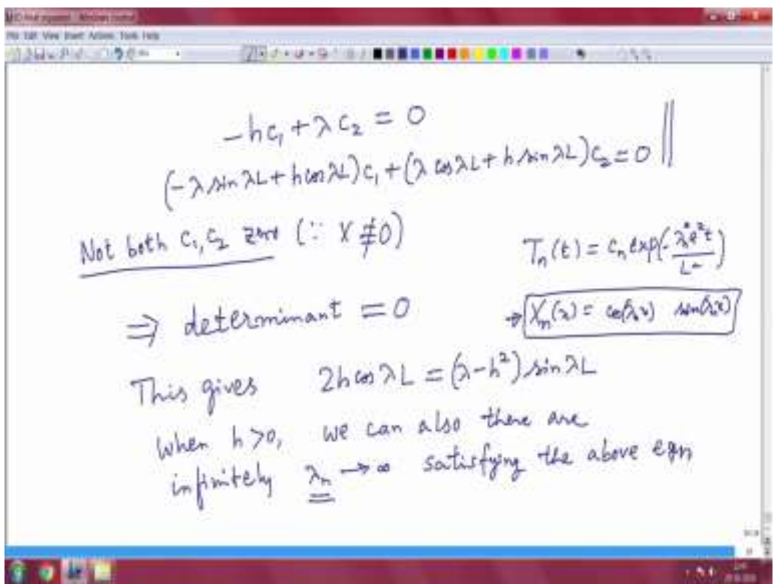

So let me write that so you have minus h c1 plus lambda c2 equal to 0 and another one big expression minus lambda sin lambda L plus h cos lambda L c1 plus lambda cos lambda L plus h sin lambda L c2 is 0. So we have got we have to determine c1 is in c2 and we have got two equations. What we want is not both c1 c2 zero. So this is requirement because we want X to be because X is not identically 0.

And now we have two systems of two equations for c1 and c2 linear equation and if you want, so these are homogenous equations but we want c1 c2 to be none zero solution. So that implies the determinant of the coefficient matrix must be 0, determinant of the coefficient matrix should be 0 in the system. And if you write down that so that this gives so you already seen some complication and these things are not explicit.

So 2h cos lambda L so you just compute the determinant and simplify it equate it to 0 and you get. So when h is 0 it is very simple so h is 0 this left hand side is not there though then again we get lambda sin lambda L equal to 0 and lambdas are easily given but h is positive we can only show there are infinitely many lambda n and they tend to infinity satisfying the above equation. So it is not possible to write down what exactly lambda n is but we can geometrically show.

There are infinitely many lambda n they are tending to infinity satisfying this equation and then again as usual now again this Xn and t n is known but lambda n is not explicitly known t n is again no problem. So you call it cn exponential minus lambda n a square t by L square may be pi is missing. But Xn is again, now it will involve both cosine lambda n x and sin lambda n x linear combination of both.

And now this time we cannot determine this lambda n explicitly, but we can still show that Xn's are, Xn's they form orthonormal family and any reasonable g can be written as Fourier series with respect to this Xn and finally we obtain a solution that requires little more computation and that will provide in the notes. How this Lambda n's can be obtain geometrically.

And though the formula is not explicit because of this lambda n's but nevertheless the procedure is the same and we do obtain again the solution in the form an infinite series. Again there is no absolutely no problem with regard to verification of term by term differentiation both with respect to both t and x variables and we do obtain the solution of the stated boundary value problem. So with that we come to an end of this discussion on heat equation. So whatever is missing those things will be provided in the notes. Thank you.