First Course on Partial Defferential Equations – 1
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Lecture 32
One dimensional heat equation-5

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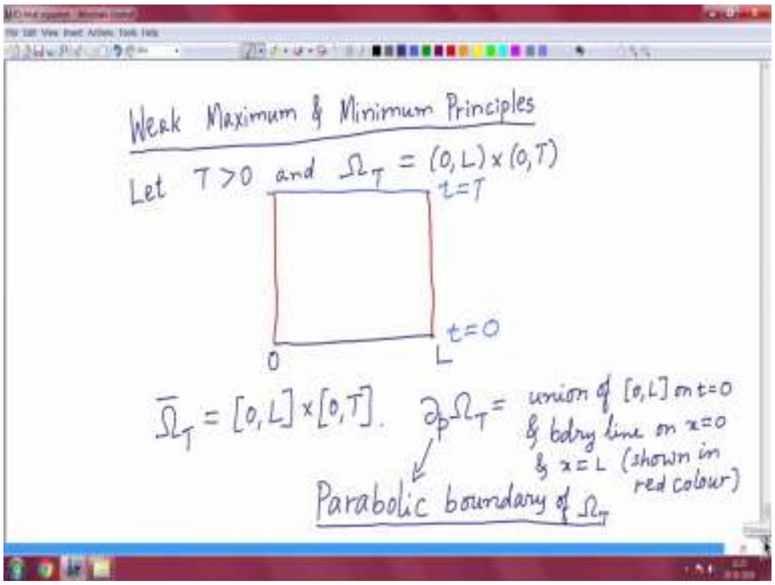

Welcome back, in this lecture we discuss again with regard to one dimensional heat equation, two important properties of the solutions, namely weak maximum and minimum principles. Such properties are also enjoyed by the solution of the Laplace equation. So in fact the solution of the heat equation say this property of the solution of the Laplace equation. So first let us fix some notations, so let T be positive and denote by omega T this open set 0 L cross 0 T.

So, 0 L can be replaced by any open interval on the real line for simplicity I just taken it to be zero L. So let me draw the diagram, so this is 0 L. So let me use different color here. And this is level let me just t equal to T. Let me remove this t equal to T and this is t equal to 0. So the boundary of this open domain consists of 4 lines. So here I put two red one on black and blue one so these three lines, so what is omega T closer so that is simply 0 L 0 T.

Now you consider the union of this two red lines and this black line denote that by del p omega T. So this is union of so let me just write of the interval 0 L on t equal to 0. And boundary lines on x equal to 0 and x equal to L shown in the red color. So this has a name this is called so we are omitting one part of the boundary of omega T namely this the line t equal to T. So this is refer to as parabolic boundary of omega t.

So its boundary minus just one line segment on t equal to T with these notations now we state the theorem. So these are weak maximum and minimum principles.

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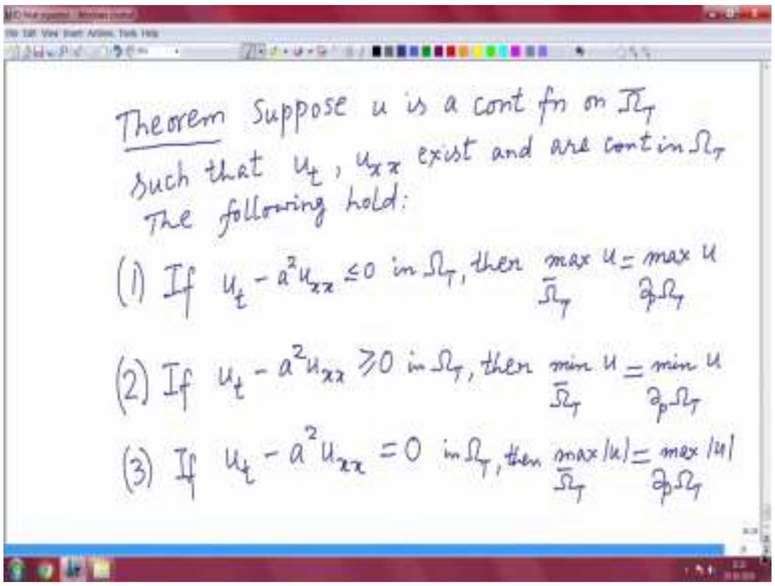

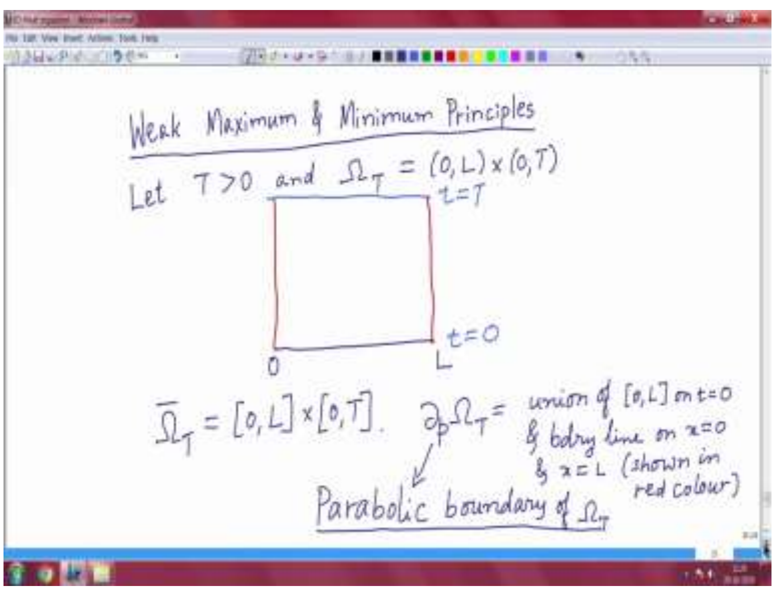

So theorem suppose u is a continuous function on omega T bar. So including the boundary so since this is a compact set then you will be bounded on omega T bar such that its just like the previous theorem. So the first derivative with respect t exists and the second derivative with respect to x exists and they exist so this is the hypothesis and are continuous in omega T. Then the following hold or there are three statements conclusions.

The first one if ut u sub t minus a square u sub xx is less than or equal to in omega T then maximum of u in omega T closer as I said it is a we are assuming it is a continuous function so it is bounded and this is less than or equal to maximum of u on the parabolic boundary. That is suppose that was our notation del p omega t. Of course this does not true value that you may attain its maximum even somewhere in omega T.

But it will not be more than the maximum that occurs on the parabolic boundary. If now we change the sign here ut minus a square uxx is bigger than or equal to 0 in omega T. Then minimum u in omega T bar is bigger than or equal to omega and finally equality if ut minus a square uxx equal to 0 in omega T. So u satisfies the heat equation in the domain omega T then so we combine these two so we get maximum of mod u in omega T bar is no sorry.

I am sorry the statement so this is equality. So before prove just uses the ideas of the calculus where it is not very difficult so I will provide a prove of this.

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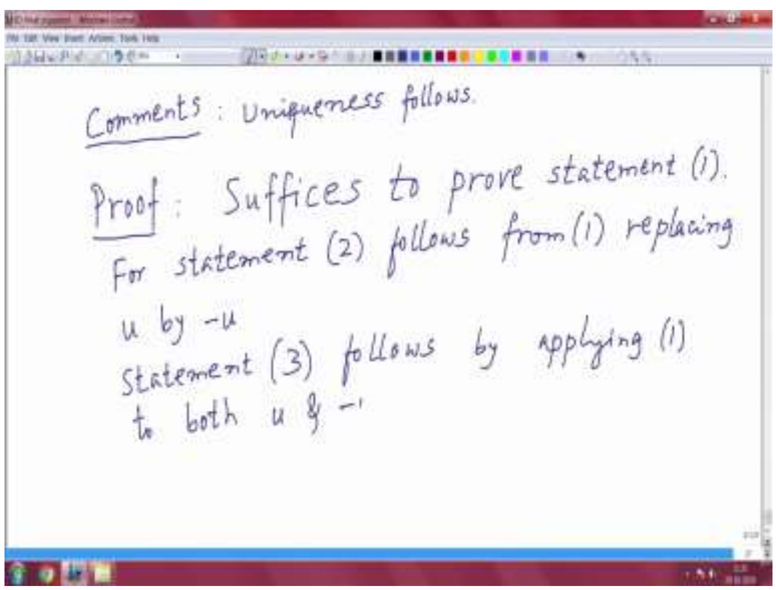

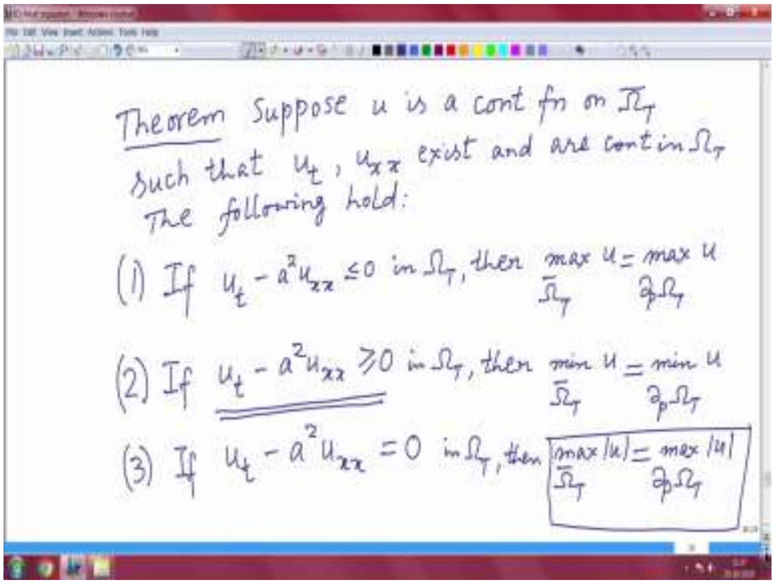

But before proving the theorem so comments. The theorem we obtain uniqueness of so let me not write 1. That may be cause confusion so let me first comment is uniqueness follows, uniqueness of the initial value problem for the heat equation follows and the another advantage of such uniqueness results is that if we try to find a formula of are this solution by some method as a Fourier series, Fourier transform then we need not worry about uniqueness.

Uniqueness is separately proved so that is why such uniqueness results they play an important role. So this just one comment so another comment I just of so regarding the proof. So first some observations so suppose is to prove statement one. So this is the first observation prove statement one why is that? for statement 2 follows from 1 replacing u by minus u just see here. if I changed u to minus u in the second statement that then we get this inequality.

So if we are already proved this one then we can apply. So this then you get maximum of minus u and maximum of minus u and that becomes minimum minus of minimum so 2 follows. And 3 follows again statement 3 follows by applying one or 1 to both u and minus u. So when there is equality here, so it is both less than or equal to 0 and greater than or equal to 0. So I can apply, again statement 1 to both u and minus u and then that results in this statement.

So it suffices to prove just statement what that is the observation. And just remember in the statement we are not using any formula for u, u is any continuous function satisfying these properties. So we are not referring to any Fourier Poisson formula or any Fourier series anything.

So that is why such results are important in the analysis of PDE. We always look for such results. So as general as possible.

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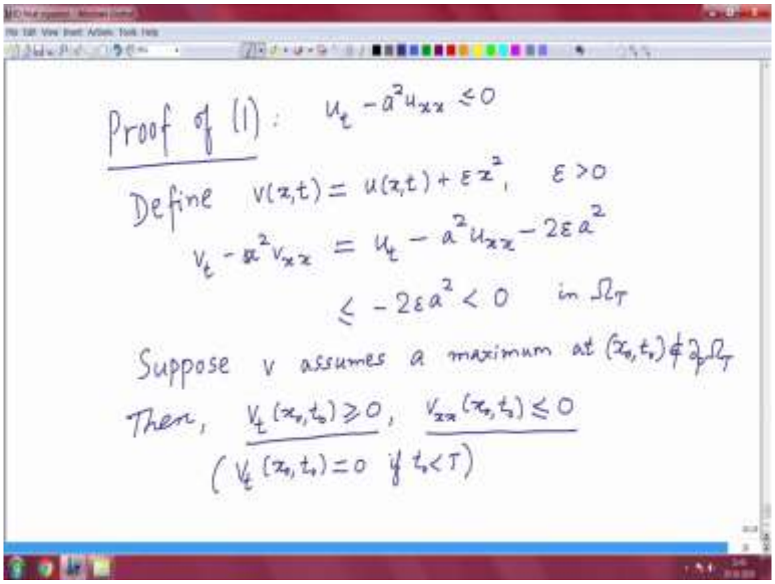

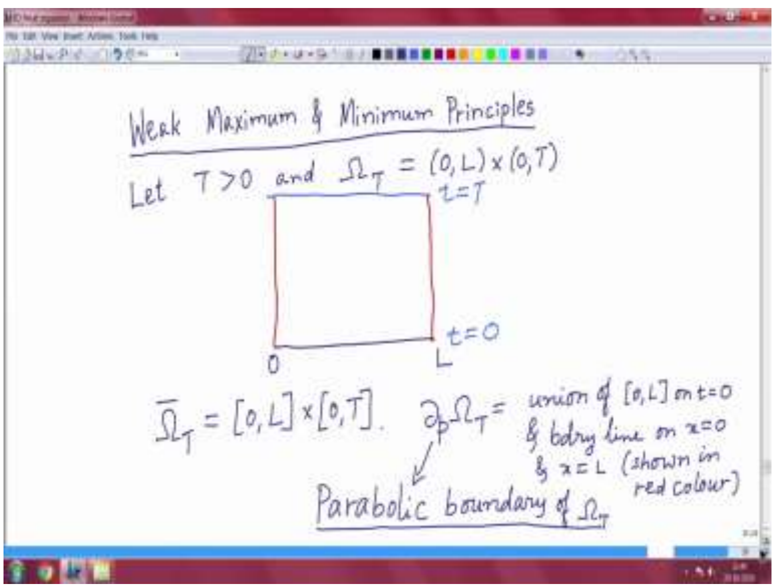

So prove of what? So define, so now we considering an auxiliary function. So define so this means what we are assuming ut minus a square uxx is less than or equal to 0 in omega T. So define v of x t is u of x t plus epsilon x square so epsilon is a small positive number. So then you compute v t minus a square vxx this is same as ut minus a square uxx with respect to x another term coming here namely two epsilon and there is a square there.

So you get minus 2 epsilon a square and by assumption this is less than or equal to 0. So this is less than or equal to 2 epsilon a square which is strictly negative in omega T remember that. So now assume v assumes suppose v assumes a maximum at x0 t0 that does not belong to the parabolic boundary. So we want to obtain a contradiction but we intend to show is the maximum of v can occur only on the parabolic boundary.

So we are assuming this x0 t0 is somewhere here where the function v assumes a maximum the t0 can be on the blue line. So I am not ruling out that but I am assuming it is not on the parabolic boundary. So this two red lines and this one black line so t0 could be on the blue line just remember this diagram. Then from the calculus we see that vt x0 t0 is bigger than or equal to 0. And vxx at x0 t0 is less than or equal to 0.

A comment about this though this is first derivative I am not putting equal to 0. It is equal to 0 so let me just make that point separately vt x0 t0 is 0 if t0 is less than T but T if this point lies on the blue line that means t0 equal to t then we can only claim this is bigger than or equal to 0. That is this is enough for us. So at that point x0 t0 where we are assuming we takes a maximum which is not a point on parabolic boundary then this first derivative with respect to t is non negative. And the second derivative with respect to t is non positive. But then we get a contradiction.

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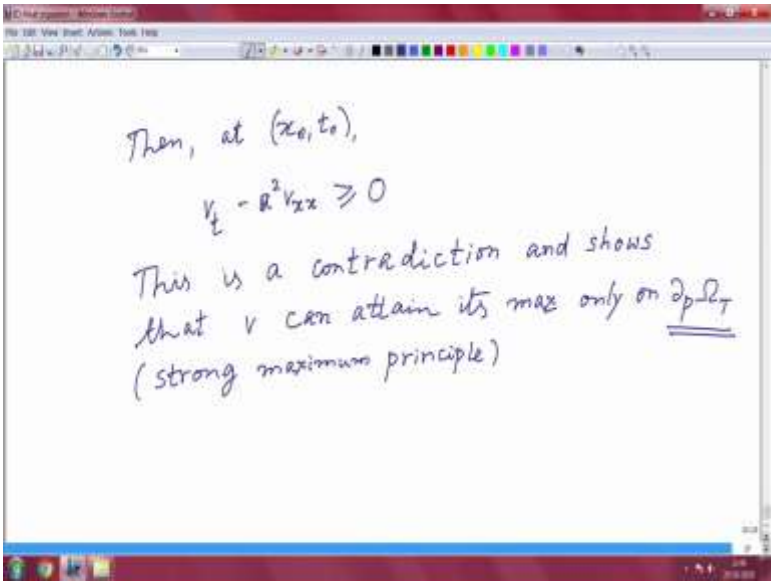

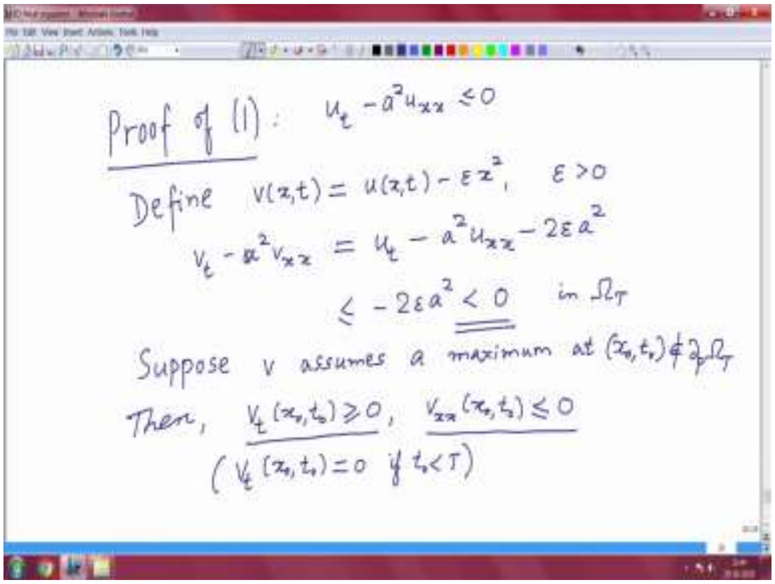

So at then at x0 t0 vt minus a square vxx so this is non-negative this is non positive but there is a negative sign here so the quantity is non negative. But we have just shown by hypothesis that it is strictly negative. So this is a contradiction and shows that we can attain its maximum certainly it does because v is a continuous function and we are in a compact set maximum only on so what we have proved for v is refer to as strong maximum principle.

So check claim we cannot make for u that is why we call it weak maximum principle so these are again important results in the analysis of heat equation which provide uniqueness and comparison results some of them we will see. So as I against t we are stated weak maximum principle but what we have proved for v is refer to as to as strong maximum principle.

So we can attain its maximum only on the parabolic boundary. So if we can make such a statement for u also then its refer to as strong maximum principle. And from this we will obtain the result for u. How is that? that is just one step.

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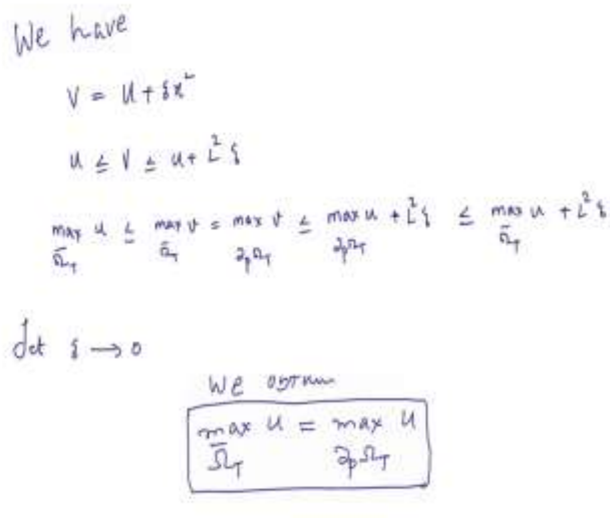

So we have so since epsilon is arbitrary so let epsilon tends to 0. So then we see that the first and the last term of this chain of inequality are the same as epsilon tends to 0 this term goes away. So I have maximum of v over the parabolic boundary. And here also maximum of v over the parabolic boundary that implies equality should be there everywhere. So in particular so we obtain maximum of u omega T bar same as. So as you see it just uses some simple calculus results to prove this result which is important in many many aspects.

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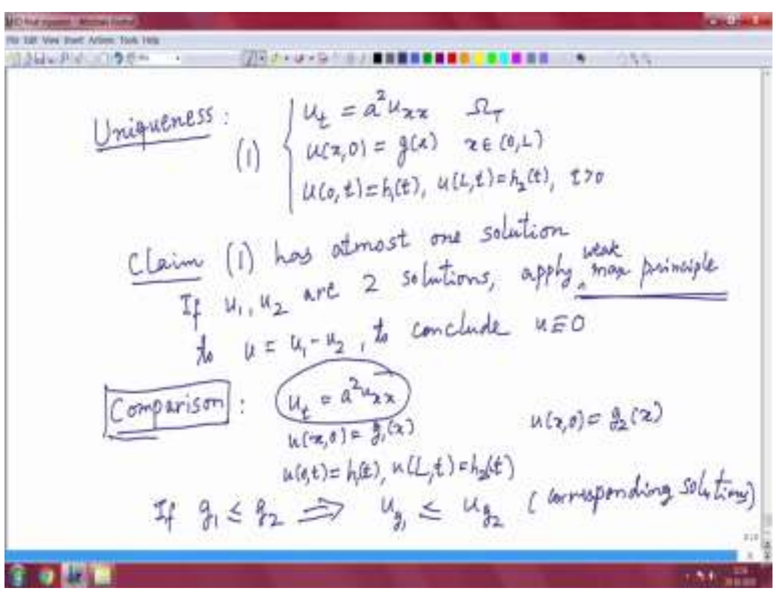

So let us see some consequences first T is uniqueness as usual. So again you consider the heat equation so u x 0 is g x and since these are boundary values so we just provide 0 t equal to some h1 t u L t h2 t so t positive x in the interval 0 L and this is in omega T. So claim is 1 has at most one solution. So we are not claiming that 1 has a solution but if it has one at most one solution then that is unique.

So again if there are two solutions you take the difference if u1 and u 2 are solutions apply weak maximum principle to u equal to again you take the difference to conclude u is identically 0 that gives you uniqueness. Comparison what is comparison? So now you consider problems with different initial values and possibly different for simplicity let me take same thing but you can take different things just for writing things and you also take another one another initial data.

So comparison means so if g1 is less than g2 then you get corresponding solutions. So let me call it u g1 u g2, corresponding solutions. So we also take different boundary data and if some inequality holds between them then the solutions, corresponding solutions will also maintain that inequality, so these are called comparison results. So they all follow from this either weak maximum principle or weak minimum principle.

We can also consider the inequalities in the equation. That is also possible because inequalities are alone in the statement of the theorem. So there are many more interesting results, so because of lack of time we will provide some them in the notes, but now we will proceed to find solutions of this heat equation in a bounded interval using Fourier series. So we consider again different kinds of boundary conditions namely Dirichlet, Neuman and mixed one and see how we can apply the Fourier series method in order to find the solution. Thank you.