First Course on Partial Defferential Equations – 1

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Lecture 31

One dimensional heat equation-4

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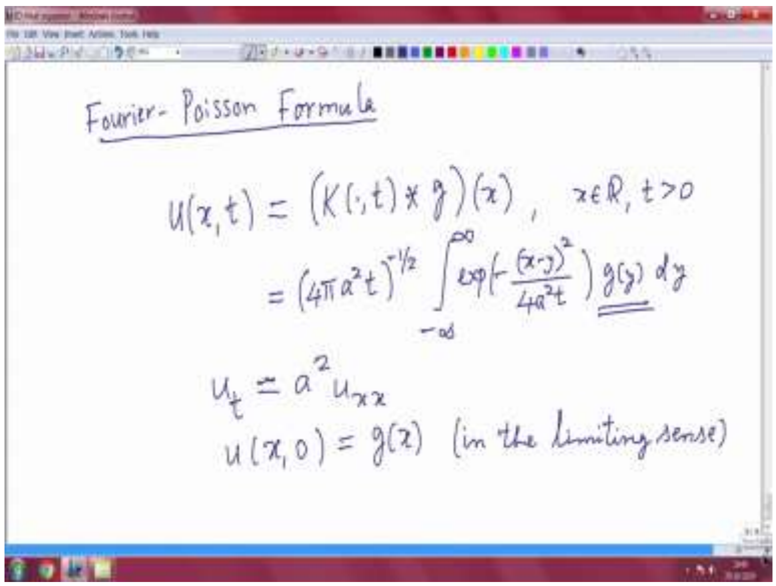

Again in this lecture we will continue the discussion of the solution of the one dimensional heat equation. In the previous class we derived the Fourier Poisson formula. Let me recall that u x t is equal to K dot t star g of x where K is the fundamental solution of the heat operator and this is the convolution of operator. So explicitly return, so this is for x in R and t positive.

So this one is pi a square t to the minus half integral minus infinity to infinity exponential minus x minus y square divided 4 a square t g y dy. So we showed in the previous lecture that u satisfy the one-dimension heat equation, namely ut is equal to a square uxx and satisfies the initial condition in the limiting sense.

So we also saw smoothing effect of this fundamental solution that makes. So even though the function g is merely continuous and bounded or even bounded or and miserable for t positive u becomes a c infinity function and that is because of this heat fundamental solution of the heat operator it is also called heat Kernel Gaussian, so use these terminologies repeatedly. So one immediate consequence of this Fourier Poisson formula in that.

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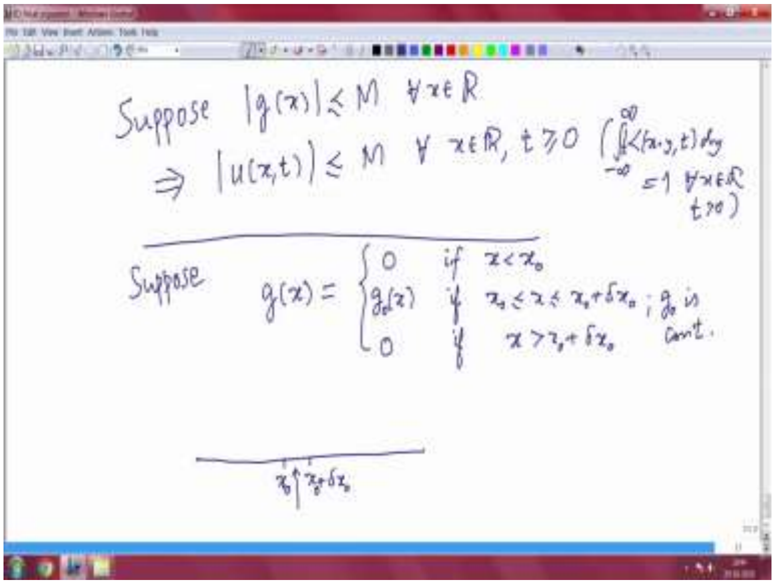

Suppose g is a bounded function and bounded by M for all x in R. So if you look at the Poisson formula that immediately implies u of x t is also less than or equal to M for all x in R and t bigger than or equal to 0. This is because the reason for this is as we saw yesterday this integral of K x minus y t dy is 1 for all x in R and t positive.

So g is and that is physically meaningful. So if you for example thing rod if you heat in a bounded fashion the temperature that is you want in time also remains bounded. So, another testing consequence so for quick reference suppose we just heat the rod in a small interval. That is the next one so locally what happens so that also we can see from the Poisson formula suppose g of x, so the initial temperature is 0 if x is less than x naught and some continuous function let me call it g0 x if x0 and again 0 if bigger than x naught plus delta x naught.

So here is the picture, so you just want to see how the temperature evolves in time so if we just heat the rod in very a very small segment of the rod. So everywhere it is 0, so only heating is done in that interval, so g0 is continuous. So if you look at the Poisson Fourier Poisson formula.

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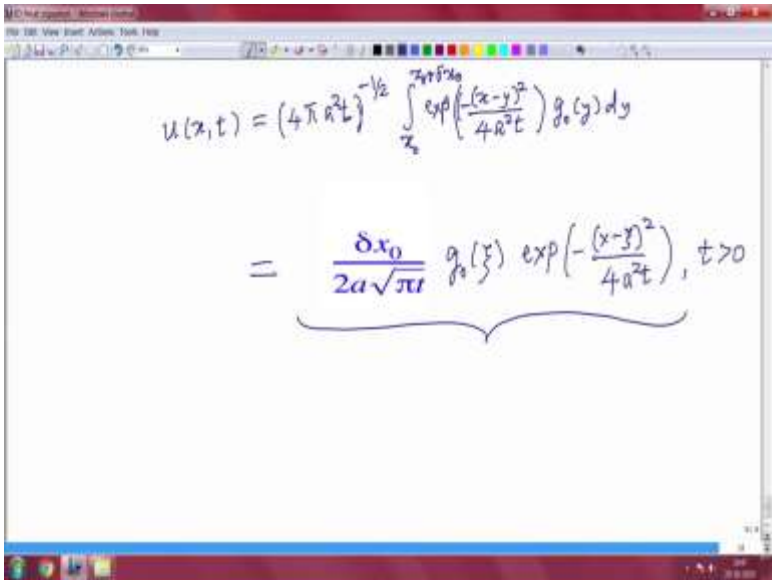

You see that the solution is given by u x t is 4 pi a square t minus half integral, now only non-zero values of g r there in the small interval. So this is the Fourier Poisson formula reduces to integral over the small interval by the mean value theorem of integral this can be written as g0 xi exponential x minus xi square by 4 a square t so t positive. So delta x0 is very small so even xi you can take to be approximately equal to x0 itself.

So we know so this is what happens when you heat the rod in a very small interval so we not need one bother about the integral. So approximately this will be the temperature for a positive times. So this is helpful positive times. So this is helpful in quickly reading the temperature.

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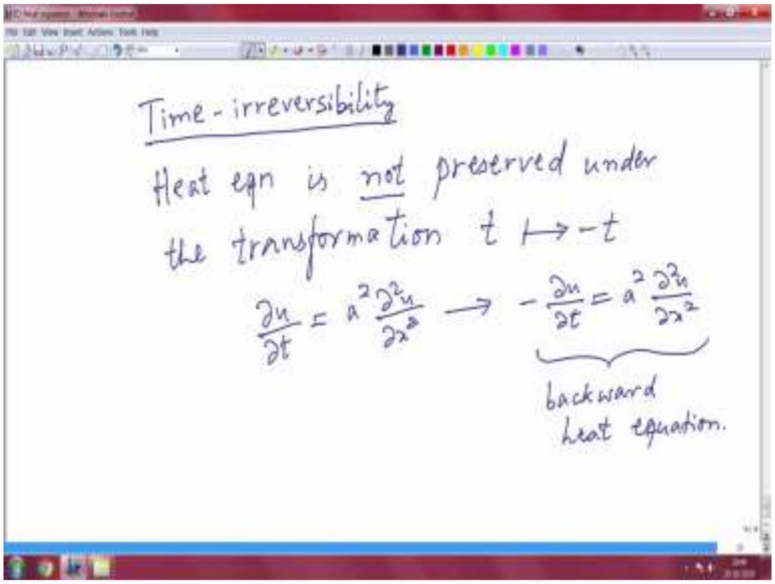

So another aspect I would like to mention about heat equation is the time irreversibility so unlike the wave equation where the wave equation is not change if we change the time to from t to t minus t so heat equation is not preserved under the transformation t going to minus t. So physically this means that suppose we are given temperature in a rod thin rod at the present time we cannot predict what was is temperature sometime back.

That is even from our daily experience we can say that so this heat equation so if we change this heat equation changes to, so del u by del t equal to a square del square u by del x square under this transformation this goes to minus del u by del t. And sometime this is refer to as backward heat equation.

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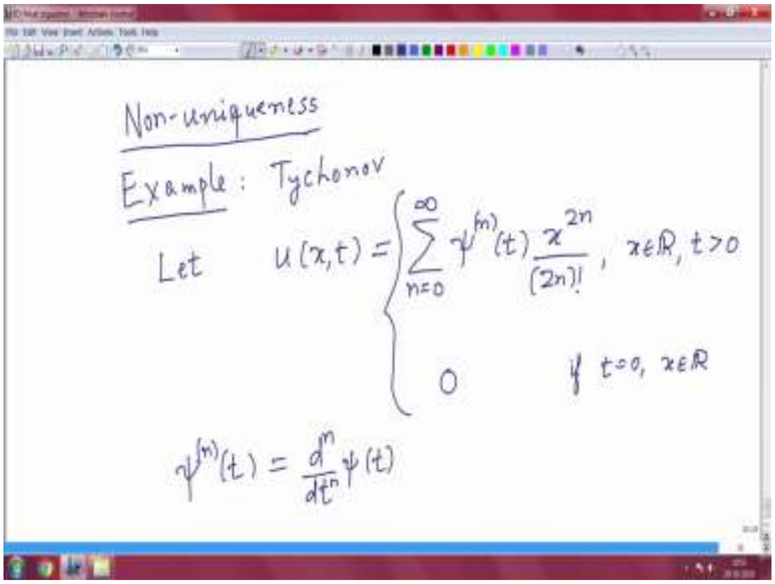

Now we will move on to some non-uniqueness questions. So I was telling in the beginning so we are dealing with characteristic initial value problem so there could be non-uniqueness. If you look at the Poisson formula it is very difficult to prove that is the unique solution because even while deriving that Fourier Poisson formula we made several assumptions but the end formula was neat. So we could prove that u given by the Fourier Poisson formula satisfy the heat equation with the prescribed initial condition.

And again producing non-uniqueness is not easy in this case, so is generate to just produce two solutions and say there is non-uniqueness. So this in many instances even proving non-uniqueness requires lots of analysis. And this is one example where we need to do some work. So this example standard example it has become standard example is due to Tychonov, Russian mathematician and example is the following.

So let, we define a function u x t, so this is not given by the Poisson formula remember. So this now an infinite series, a power series, let me write n equal to 0 to infinity, psi K psi n t x to the 2n by 2n factorial. So event to come up with this example, you can see so much work would have gone into that. So this is this let me, x in R and t positive and 0 if t 0 and x is in R. So what is psi n? Just so psi n first of all definition that is a derivative psi n t is nothing but the nth derivative of the function psi evaluated at t, here is the function psi.

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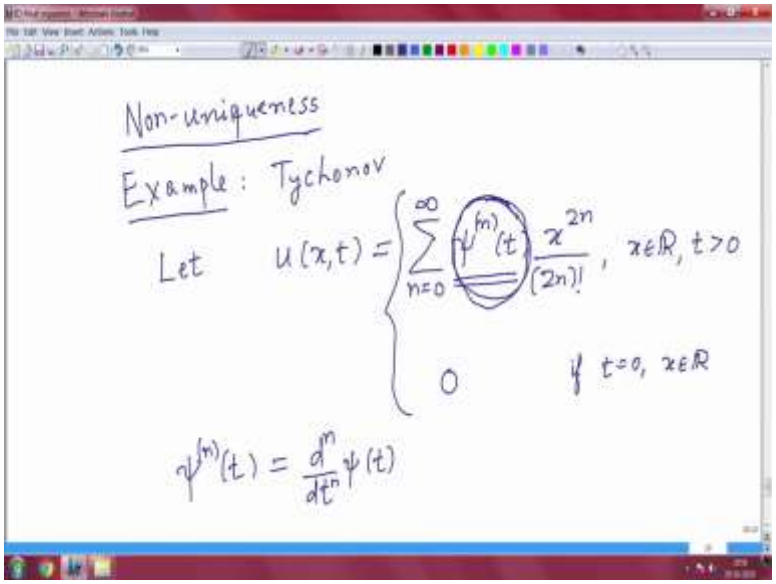

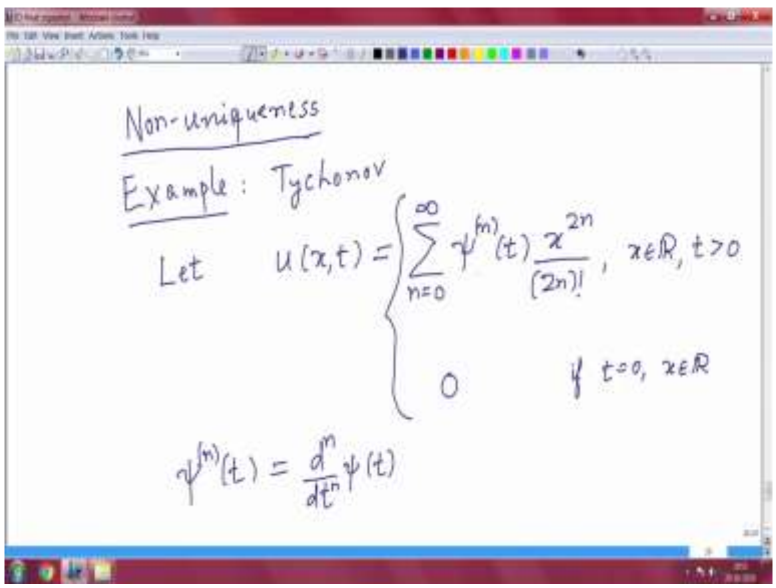

So it, so where psi z so it is function of a complex variable, so we can see the difficulty even in producing this example z naught 0 and 0 if z is 0 where z belongs to c. The set of complex numbers and what is t, t is the real part of z. So if you look at the definition of u. So but for this factor it almost look like e to the x square if you just this is the two n factor is here but it is not.

So something like e to the x square but this makes it more complicated that psi and t. So what Tychnov showed was then u satisfies the heat equation u t equal to uxx. So a equal to 1 here a equal to 1. And u is certainly not identically 0 and since and it is 0 by very definition and u of x 0 is 0. That is how it is define. So u is a solution of the heat equation with a equal 1 with 0 initial data. But identically 0 function is also a solution.

Thus so it is there is non-uniqueness in the problem. So we have two solutions here one is not identically 0 and another one is identically 0 both satisfying the same initial condition namely 0 so there is non-uniqueness. So proving that u satisfies the heat equation require some work and this will be provided in the notes.

So we have to essentially estimate psi and t and show that this power series converges and we can do term by term differentiation both respect to t and x and then verify that u satisfy the heat equation. And that requires some computations they are little hard because we have to go to function of complex variable and this is estimated using the Cauchy formula so the details will be provided in the notes.

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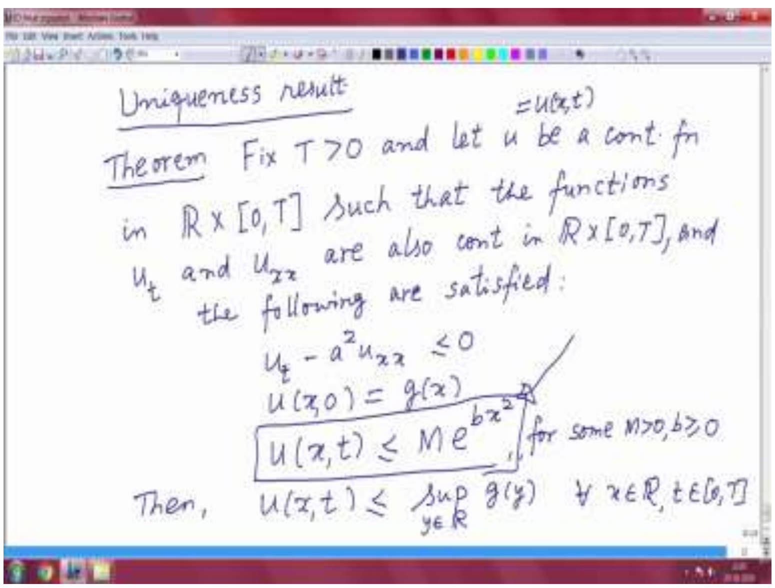

Not that there are uniqueness results so this examples certainly tells us to look for some conditions on the solution. So that we have uniqueness in the problem and one such result I just mention here. So uniqueness result as I commented in the beginning so in order to get uniqueness result we may have to put some additional conditions and this is one such result. So let me state it as a theorem again prove of this theorem is little lengthy details will be provided in the notes.

But your information and knowledge this is important to be noted fix T positive and let u be a continuous function in R cross 0 t. So function of two variable x and t so let me make stress it u of x t x is in R and t varies in this interval such that the functions u sub t. So that is so is differentiable with respect to t the first derivate and the second derivative with respect to x are also continuous in same thing.

Sorry if the following are satisfied let me write it in different way. We will continue here itself and the following are satisfy. So this is all part of the hypothesis ut minus a square uxx is less than or equal to 0. So let me not write again it is valid in this domain R cross 0 t u x 0 is g x and here the additional condition comes. So in particular we can take the heat equation so here with inequality also it is true.

So u x t is less than or equal to M times b x square for some M positive and b non-negative. So this is the additional condition require on the function u then with all this hypothesis then u x t is less than or equal to supremum of g of y y is in R so for all x in R and t. So in the week of Tychonov example, so if we cannot relax such a condition that is the important thing. So if you assume that this function has at most exponential growth with a quadratic term then goes and this immediately implies uniqueness. Why?

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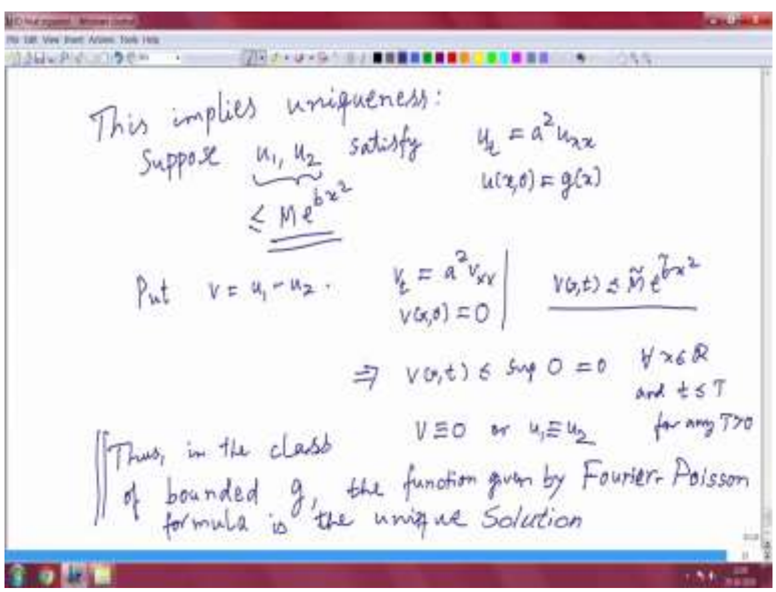

So let me just show this implies uniqueness this theorem. Suppose u1, u2 satisfy the same equation ut equal to a square uxx and u x 0 is g x same initial condition so we have to show that u and un equal to u2. So you are assuming also that they have, they are less than or equal to M e to the b x square, so that is additional thing because we are going to use the theorem.

So put v is equal to u1 minus u2 and by linearity that is important so vt v satisfies a square vxx and v x 0 is 0. And because of that assumption, so v also satisfies such a growth condition, may be with different constants that does not matter. There could be b1 and b2, so we can choose smaller or bigger as the case may be.

So then we apply the theorem to this equation and that implies for any t positive, so v x t is less than or equal to supremum of 0, that is 0 for all x in R and in a t t is less than or equal to T for any t positive so we can take arbitrary large and that implies b is identically 0 or u1 is identically equal to u2. So once we have that theorem uniqueness of the problem easily follows.

So thus in the class of bounded initial bounded g let g the function given by Fourier Poisson formula is the only unique solution of the Cauchy problem but its though to begin with who did not know the function given by Fourier Poisson formula with unique solution we certainly knew that is satisfy the Cauchy problem but uniqueness are not there with the help of this theorem now we also get uniqueness of that Fourier Poisson formula. So there is also just let me mention one more result.

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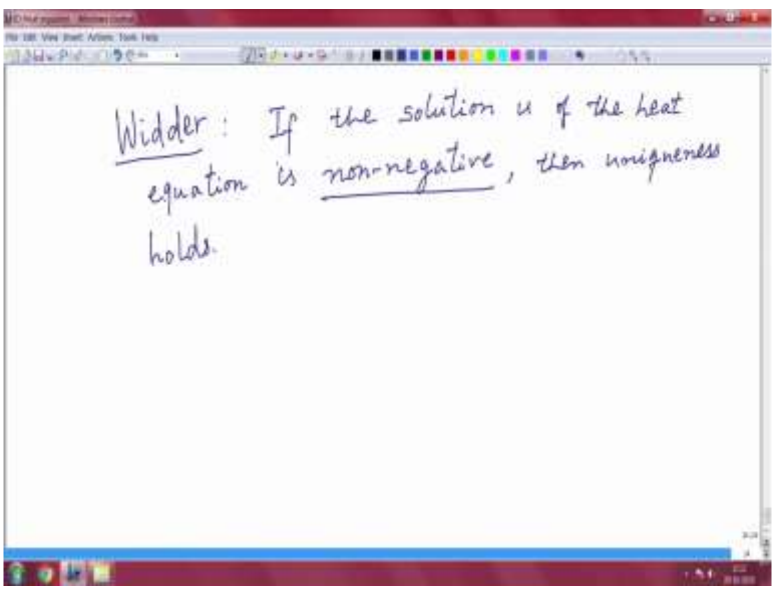

So there is a result due to Widder so we can see these characteristic Cauchy problems pose so many problems one has to work out for example here regarding the uniqueness of the solution these ones says if the solution of the heat equation so instead of both condition here is sin condition solution u of the heat equation any solution heat equal is non negative so this is the condition Widder puts is non negative than uniqueness holds.

So if you consider the solutions which are non-negative then we do have uniqueness. So much for this uniqueness and non-uniqueness and in the next class we will be studying one simple case again where uniqueness holds and that is somewhat easier to prove that is just uses the ideas of the calculus. So I will do that I will do in the next class so the details of Tychonov example and the prove of the theorem though it is hard. So they will be provided in the notes so you can go through them in leisurely. Thank you.