**First Course on Partial Defferential Equations – 1**

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**Lecture 30**

**One dimensional heat equation-3**

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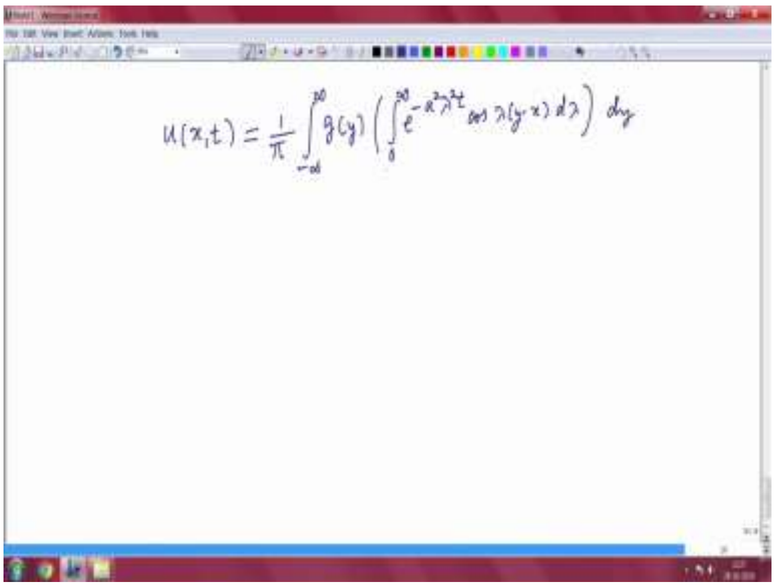


So welcome back, we will continue discussion with the solution of the heat equation we started in the previous lecture. Now we complete the formula, so we are in the process of analyzing this integral. So again let me write it, so we had obtained the solution in this form g y e to the minus a square lambda square t cosine of lambda y minus x d lambda then dy. So we analyze this thing separately.

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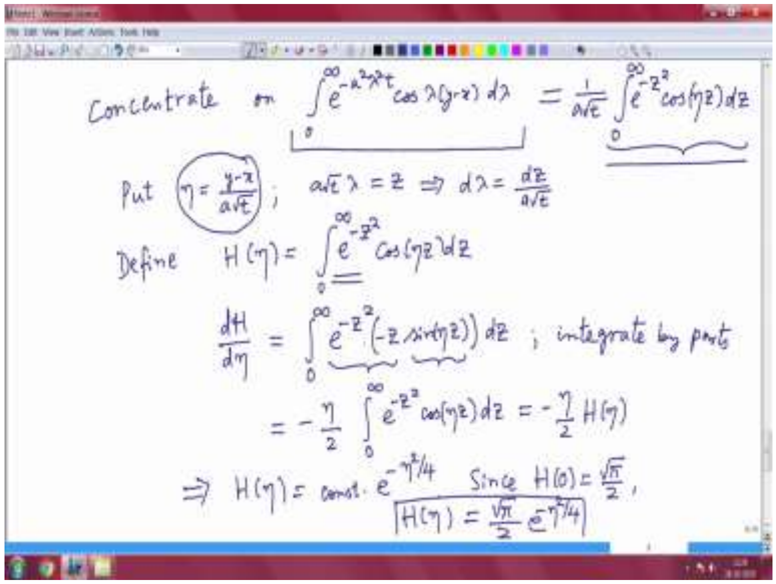


And we obtained this function H eta and there is 1 by a root t and again we substitute back this variable eta. In the original variables

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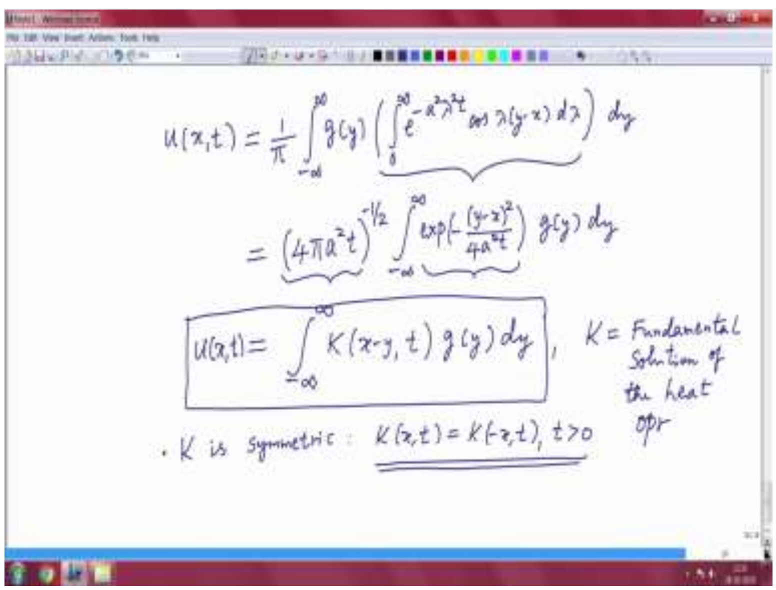


So that is simplifies into so let me write 1 by 4 pi a square t to the minus half minus infinity to infinity exponential, so since expression is little big. So, let me write exponential so y minus x square divided by 4 a square t into g y dy.

And you recognize this multiplied by this is precisely the fundamental solution of the heat equation. So this is a, write it here K of x minus y t g y dy. So K is the fundamental solution of the heat operator which we first obtain as a special solution to the heat equation. And now it also makes appearance in the formula we are trying to derive for the heat equation. So just let me write u x t.

So we also see that K is apart from the properties I listed earlier, K is also symmetry meaning K of x t is equal to K of minus x t t positive. So if t is positive so that you can, because in the exponential term this sits as an x square so x and x square equal to minus x square so there is no change there. And now you look at for just a few comments on this formula. So let me just introduce some notations.

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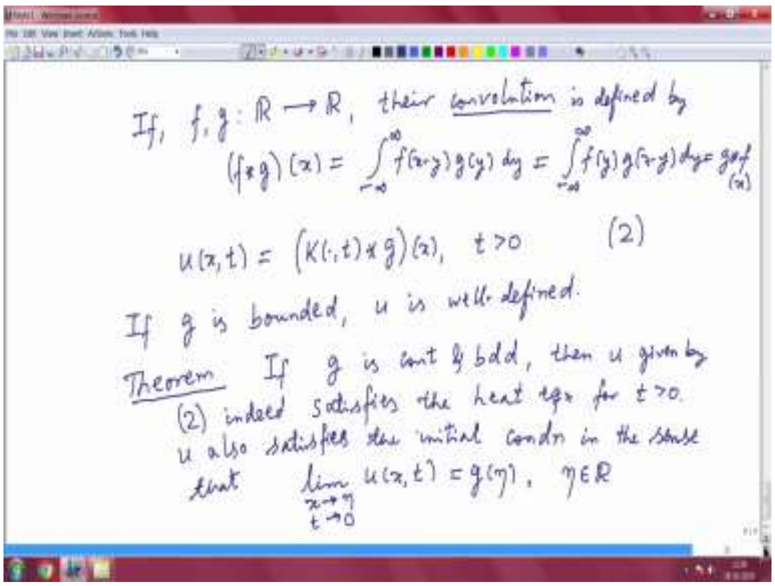


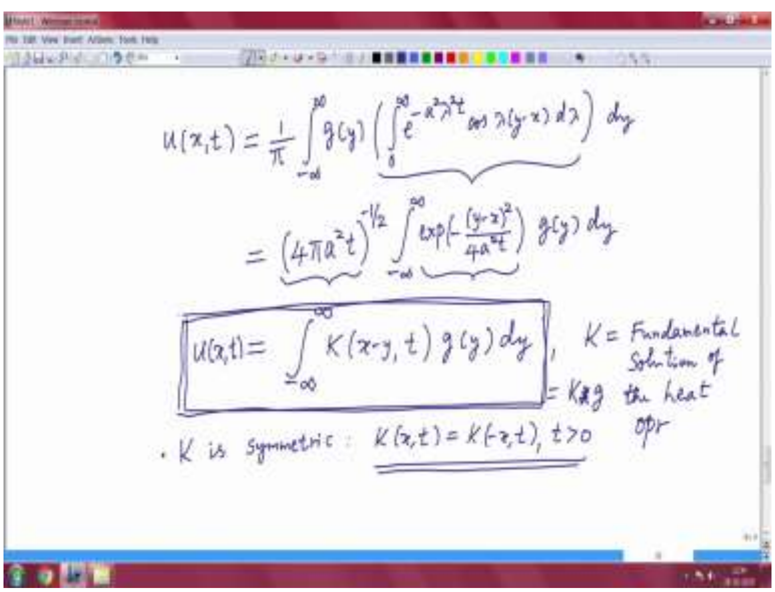

So if f and g are functions from R to R, R R to C their convolution is defined by so convolution of two functions is again a function so defined by f star g of x minus infinity to infinity f of x minus y g of y dy which we also can written as by changing the variable. So this is f of y g of x minus y dy, so this g star f. So provided the integral is finite and other so this f the star operation the convolution operator is a commutative operator.

So in this sense we can write with this notation u x t as K star so I will put the function here t star g of x. So the u we obtain by heuristic arguments can be written as convolution of the fundamental solution and g. That is what I want to comment. I want to stress that. So now so we obtain this formula by assuming several steps in between but the final formula is very neat, very neat.

And it make sense for example if g is bounded because of the exponential factor in K u is well defined so there is no problem at all. So the integral adjust for all t positive. And now try to show that u is indeed the solution of the heat equation. We cannot claim uniqueness so you just say so let me state it this as a theorem so if g is continuous and bounded we have to assume continuity does not imply boundedness.

Because we are on the real line that is not a compact check and bounded continuity is also not require if you are using Lebesgue integral let me just state that g is continuous and bounded then so let me put this as 2 then u given by 2 indeed satisfies the heat equation for t positive u also satisfies the initial condition in the sense that so limit u of x t as x tends to eta and t tends to 0 is equal to g of eta so eta is real variable.

So in general this limit is satisfied at all the points of continuity of g. So if you are not assuming g is continuous everywhere. So this limit exists wherever g is continuous so the prove of this theorem depends on one more important property of the fundamental solution.

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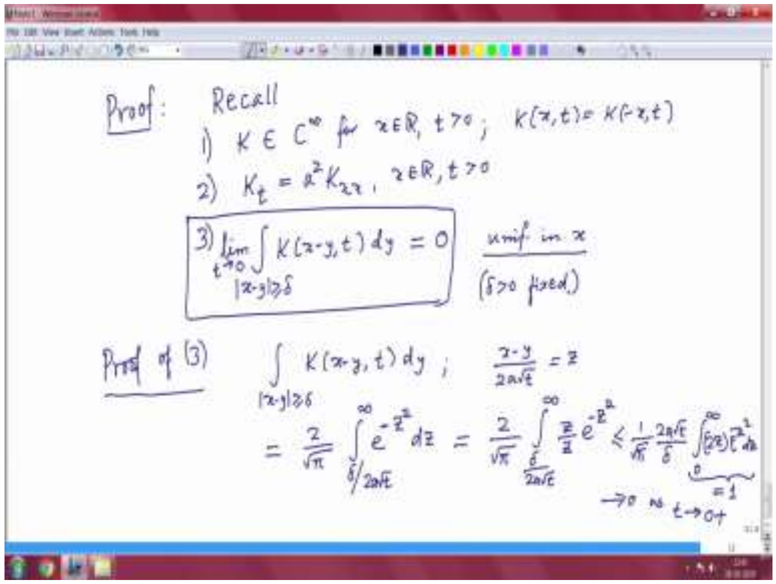


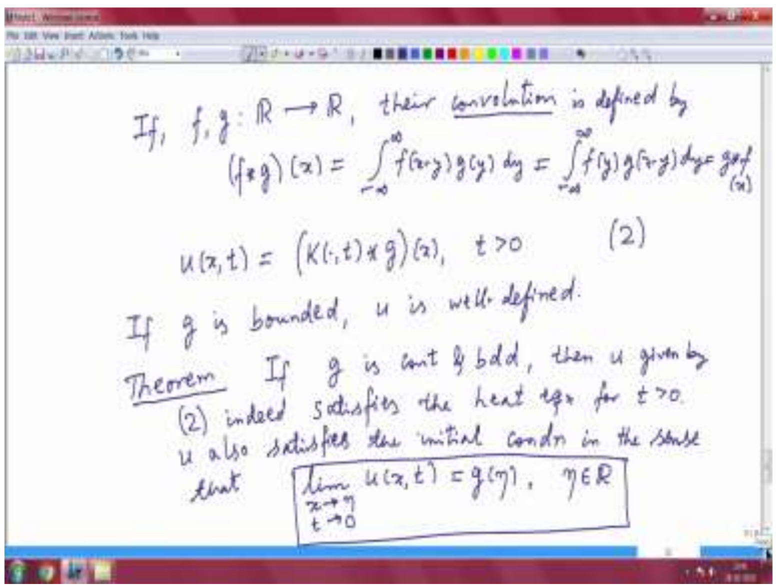


So let me just so proof, so recall the again properties of the fundamental solution. K is a C infinity function for x in R and t positive K is symmetric we saw that and K satisfy the heat equation so this is we also saw this Kxx for x in R and t positive. So one more important property so I write it here so if you integrate K x minus y t with respect to y not on the whole real line but you just leave stay away from this x at a positive distance.

And then you take this limit as t tends to 0 this is 0 uniformly in x that is important uniformly in x. So usual epsilon delta if you take that definition of the uniformity means that delta does not depend on x. So this is crucial in verifying this part of the theorem. This part so we will come to that just so proof of three that is not very difficult. So again let us write that what is in terms of K. So you consider this x minus y bigger than or equal to delta, delta is positive.

So for any delta positive so if you stay away from B x and then this limit is 0. So make the substitution x minus y by 2 a root t is equal to z. And then this integral is changed to 2 by root pi delta to infinity, so there is a portion coming from minus infinity to minus delta and delta to infinity but because of symmetry we can convert that minus infinity to minus delta integral to delta to infinity integral and that is why that 2 comes.

So it is very simple one, there is no complication at all. And sorry the delta so this become delta by 2 a root t because we are making the substitution so if x minus y is in absolute value bigger than delta then mod z is bigger than or equal to delta by 2 a t 2 a square root t. So what you do here is now its simple thing 2 by pi, now let me write that you add a z and divide by z e to minus z square.

And this we do can do because z is away from the origin and for 1 by z you replace it by z is bigger than or equal del do I but delta divided by 2 a root t. So 1 by z is less than or equal to 1 by root pi 2 a root t by delta and the remaining integral you can just write it as 0 to infinity now there is no problem 2 z e to minus z square dz.

And this integral is just 1 and since root t sits in the numerator so that goes to 0 as t goes to 0. Of course we are always considering t positive so let me stress that also there is a square root there. So there is this prove property 3, so now it is very easy to prove the stated theorem.

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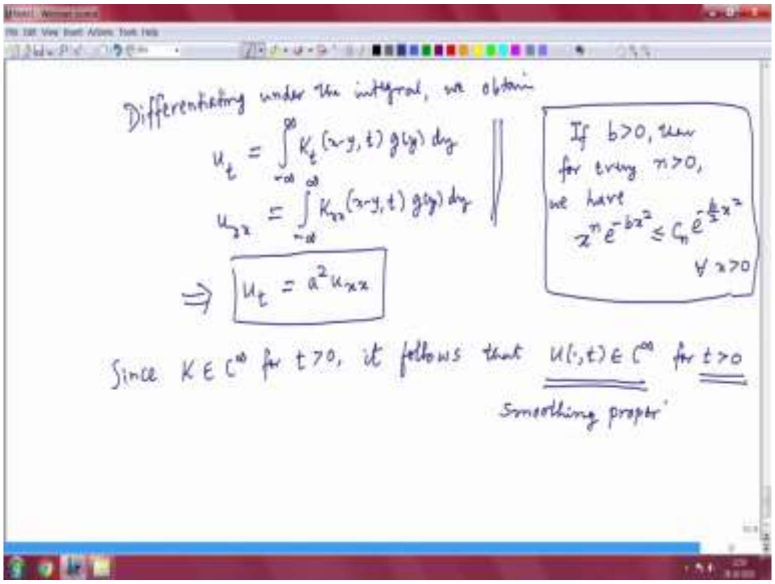


So again differentiating under the integral sign so the this is let me make a comment here differentiating under the integral sign. So we obtain u of t same as minus infinity to infinity K of t x minus y t g y dy and uxx is Kxx x minus y t g y dy. So the justification of taking the differentiation under the integral sign the important thing we use here for the exponential functions.

So if b is positive so this is a small result in analysis then for every n positive can be integer it can be anything we have x to n e to the minus b x square is less than or equal to some constant which depends on n e to minus b by 2 x square for all x positive. So this crucial property helps us in taking the differentiation under the integral sign and that in the immediate implies because the fundamental solution satisfies the heat equation uxx.

So the third part is proving the initial condition is also satisfied. So, before doing that thing since K is a C infinity function for t positive and just like we did it here taking the differentiation under the integral sign it follow that u dot t is also a function, as a function of x for t positive. So though initial at t equal to 0 g may be just a bounded function but this heat Kernel makes the solution as c infinity function for any t positive. And this is refer to as smoothing effect smoothing property.

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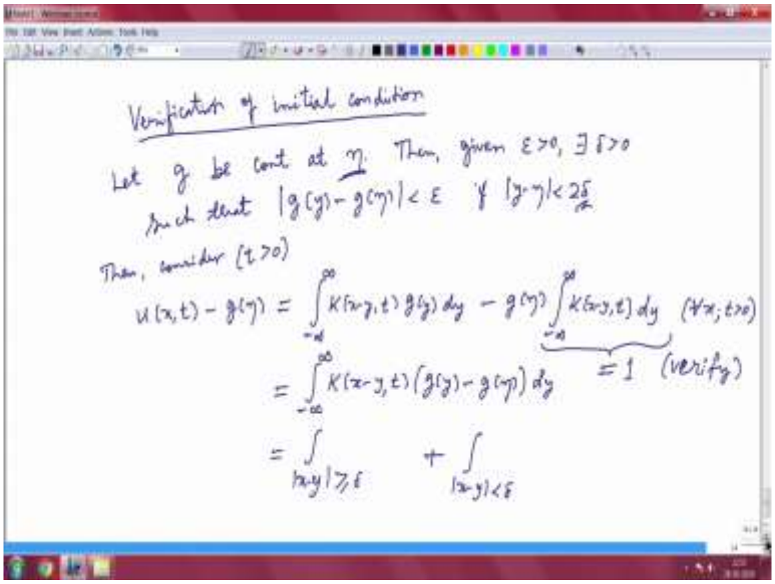


Now we will come to the verification of initial condition. So let g be continuous at eta then given so this is just definition of the continuity epsilon positive there exist delta positive such that g of y minus g of eta is less than epsilon if y minus eta is less than 2 delta for technical reasons otherwise you simply write delta there then consider u of x t minus g of eta.

So let us simplify this so I forgot to mention one more property which I am writing now K x minus y t dt t  dy sorry minuis g eta. So it is so this is for t positive here is the property, so this minus infinity to infinity K of x minus y t dt. This is an easy verification this is simply 1. And you just go back and see the expression for K you this is sorry making mistakes. So this is for all x and t positive.

So there is x here so this is just constant one. So verify this, so this is one more property of one more important property. So this we combing the two integrals and write it as x minus y t so g y minus g eta, eta is fixed. And now you break the integral into two parts. So, this mod x minus y bigger than or equal to delta and mod x minus y less than delta. So delta is coming from here, so the same delta we take we break this integral into two parts. So now we rewrite so let me just again write it.

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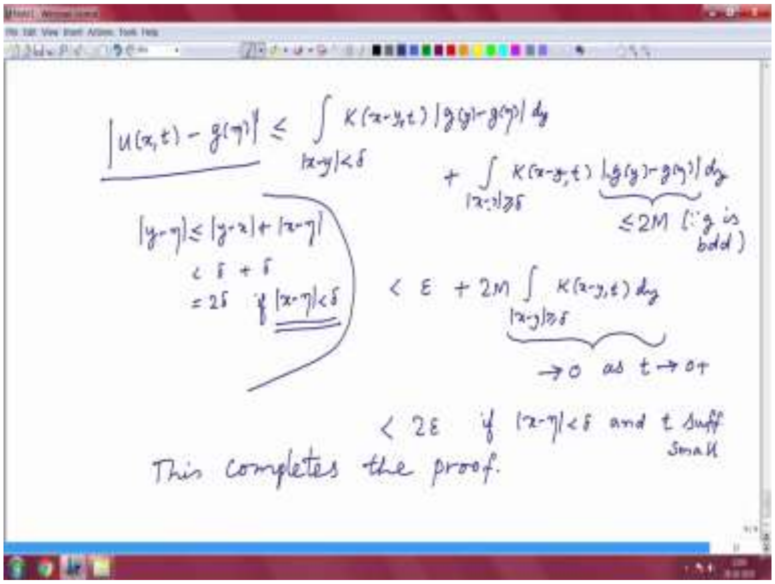


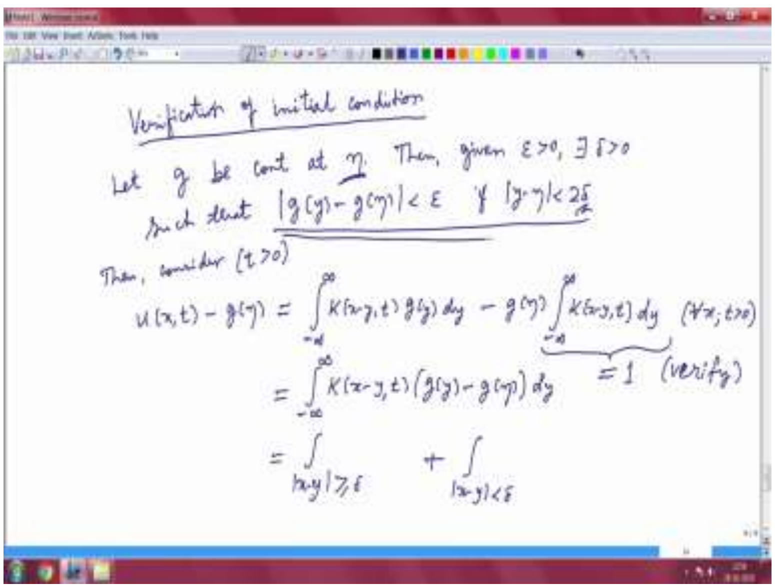

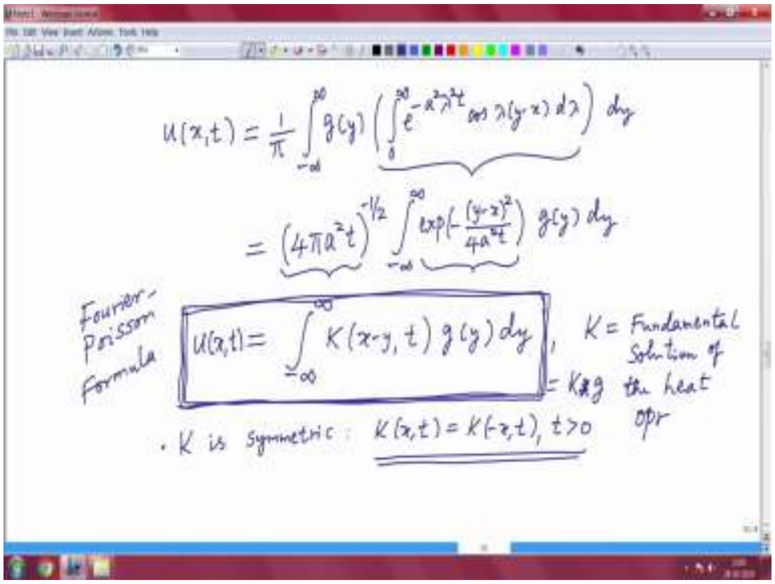


So u x t minus g eta absolute value now we take absolute value here less than or equal to mod x minus y less than delta K x minus, K is a positive function so that is also, K is a positive function because it is exponential, so g of y minus g eta dy plus x minus y bigger than or equal to delta, K of x minus y t g of y minus g of eta dy. Now concentrate on the first integral, so here x minus y is less than delta.

So, what about y minus eta, so y minus eta by triangle inequality is y minus x plus x minus eta. So if I also take x within delta neighborhood eta. So this is less than delta another delta so that will be 2 delta. If x minus eta is less than delta. And that is anyhow we are going to take the limit as tending to eta. So this assumption is fine. So that makes y minus eta less than 2 delta.

And by the assumed continuity property g of y minus g of eta is less than epsilon so the first integral. So it just this is separate thing so this is less than epsilon. Because the integral of K is 1 so the first integral the second one so we are assuming g is bounded so this is less than or equal to say 2M because g is bounded. So M bound is the bound on g. So this is 2M and x minus y bigger than equal to delta K x minus y t dy.

And the third property of the fundamental solution since delta is positive say that this goes to 0 as t goes to 0. So we make it less than 2 epsilon if x minus eta is less than delta and t sufficiently small. And that prove the required so finally what we have assume that absolute value of x t minus g eta is less than 2 epsilon if x minus eta is less than delta and t is sufficiently small. And this completes the prove.

So though is started with some heuristic arguments we have arrived at a formula for a function u and which we are able to show that it is a solution of the heat equation and also satisfying the initial condition as described in the theorem. So with we will stop here and in the next lecture we continue from here and discuss uniqueness and some other results. So just remember this and this is refer to as Fourier Poisson formula.

So I conclude here this talk with so though we started with some heuristic arguments but to arrive at this Fourier Poisson formula and under boundedness and continuity assumption on g we did verify that it satisfies the heat equation and the initial condition. And in the next lecture we will discuss uniqueness and non-uniqueness results. And also heat equation in a finite rod. Thank you.