First Course on Partial Differential Equations -1
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Lecture No - 3
Preliminaries - 1

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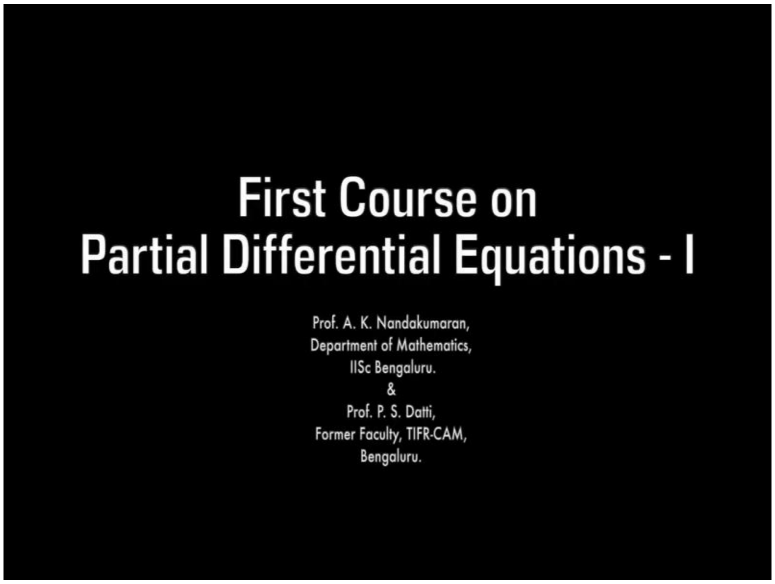

Welcome back, to the lecture on partial differential equations. In the first two lectures that is one-hour lecture, two half an hour lecture, we have given brief about the whole program and we also introduce some notations, some remarks in last two couple of lectures. Now, in the next two hours essentially four lectures we will present some preliminaries required for this course. As I mentioned or introduced already giving preliminaries in detail is not possible because the theory of partial differential equations you need to know plenty of preliminaries which you have studied and I advise or suggest you go through it.

So, in the couple of hours I will describe two aspects there are many things like multi-variables calculus like inverse, implicit functions theorems you should know about it, another important aspect is the surface measure, surface integration and all that. All that requires some amount of study, some amount of if you are not familiar you have to spend some amount of time there.

Then there are something about the ordinary differential equations, Fourier transform, some quick function analysis and all that, as I said it is not possible to cover all the preliminaries. So in the first I mainly restricted to two preliminary things, one is about the multi-variable calculus that I will do it in one hour, I will just want to state inverse function theorem and implicit function theorem only stating but then I want to develop the reasons and the ideas our main motivation is to introduce to you the ideas behind implicit function theorem and inverse function theorem those who are listening for the first time.

But, in the second one hour two lectures we will introduce to you an intuitive idea of behind the surface measure and try to tell you about Green's theorem and the divergence theorem which we will be using it frequently throughout this course and also in the next course which we are proposing. So, let us do what we call it.

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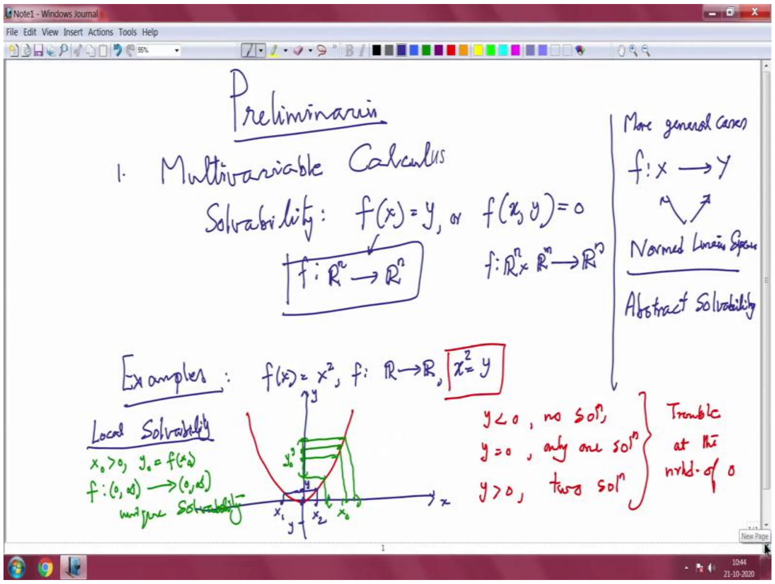

So, we call what is preliminaries so, in this preliminaries let me do some multi-variable calculus this is about the solvability we show and some of the preliminaries if necessary we will introduce to you as mean required. So, this is about solvability you want to solve the equations of the form f x equal to y or equations of the form in the implicative form, this is in the explicative form and f can be mapping here, in this case f can be a mapping from R to Rn and here f can be mapping from some Rn cross Rn, y will be m to Rn so you can have this kind of mappings.

So, though I will be restricting myself through this kind of equations let me also tell you the theory which you all developing actually can be used for more general cases which I will not do it, but I advise you to, you can have from f x to y and x and y can be normed linear spaces. Why this kind of abstraction and it can be infinite dimension and all that you can convert your p d to another problems to this abstract problem, abstract solvability and these theorems can be very handy when you want to solve it.

So, let me restrict myself to the case here so, let us do some start with some examples. The simplest example you can think of it my f x is R from R to R, f x equal to x square what f is from R to R, so you want to solve x square equal to y. So let us look at the figure here and this is your x, this is your y so start here.

So, let us look at let us draw this is familiar to you so let me draw my parabola so this is your parabola you see and then when you try to solve this you want to solve this one this equation very familiar to you. So, you know that Crick uses y less than 0 no solution, y equal to 0 only one solution and y greater than 0 two solutions.

So, there is some trouble at the origin that is what you see trouble at the neighborhood, neighborhood of the origin that is what you immediately see it. So let us do what is happening here. So, this is the case, so if you have this one so if you look at here for given y you have two solutions, two solutions are obtained. So, for this y1 you get this y, y here, you have two solutions x 1 and x 2 you have that two solutions there.

On the other hand, when you choose y equal to 0 then and y if you choose here no solutions, this is something like you are looking for a global solvability. But then there a concept of local solvability, what is local solvability tell you, what is this local solvability I am looking for a point x naught here so let me choose another color. So, look at here x naught and let me call it y naught is equal f of x naught, so that is my x naught you choose x naught positive and let y naught be f of x naught.

Then look at here this point, this is your y naught you can choose a neighborhood here and that neighborhood you can come up to the origin and but this point can go to up to infinity the neighborhood and this neighborhood we look at this inverse for every y here, y here you can have a so you get a neighborhood here, so you get a, so you have a neighborhood here, you have neighborhood here and for every y in this neighborhood, for this neighborhood you have unique x satisfy this.

So, that means in fact this neighborhood can up to 0 to infinity that means if you restrict your function f 0 to infinity, to 0 to infinity then unique solvability, unique solvability that is what you see that one, that mean you have a neighborhood of given an x naught there is a neighborhood of x naught and there is a neighborhood of y naught between that you have a 1-1 bijective mapping unique solvability means.

So, the unique solvability is you are looking for the bijectiveness of that mapping, so you have the so what is the trouble here so let me go to the. So, you understood the local solvability for a neighborhood you have neighborhood there give, there is neighborhood of x naught and there a neighborhood of y naught on that you have the unique solvability and you can put your inverse maps there, this is the same case. Now, let us look at the another example.

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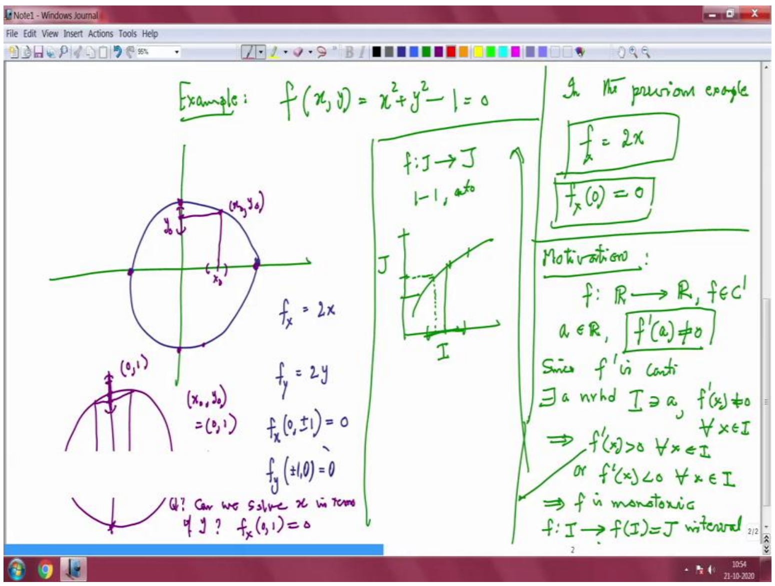

Another example I take f of x y is equal to x square plus y square minus 1 equal to 0, you know that this is the circle I am just trying to motivate you. In the previous example, in the previous example, you immediately see that f of x if you look at the differentiation this is nothing but 2 x so f x at 0 is equal to 0 so probably this could be the possible trouble I will come back to this one. So, before coming to that example let me give you one more motivation, we will come back to this example let me have one more motivation the kind of example familiar with that.

Suppose f is mapping from R to R you have already seen this example so I am recalling and trying to connect with that I choose a point here in R if I choose a point in R such that f prime of a is not equal to 0 and assume f is a c1 map, f is a c1 map means it is continuously differentiable function for which that means f is differentiable and derivative is also continuous it exists then since f prime is continuous, since f prime is continuous by continuity there exists a neighborhood u of a you can have a neighborhood and that can be interval, so let me write it in the interval form.

So, let me write it in the interval form there exists a neighborhood in fact it is a interval I, there exists in interval I containing a in which on that neighborhood your f prime of x is not equal to 0 for all x in I. That means there is neighborhood of a for which f prime is not equal to 0, that means this implies a prime of x is greater than 0 for all x in I or f prime of x is less than 0 for all x in I.

That immediately, so if assuming that if f prime of x, so that means immediately tells you that means implies f is monotonic that is the meaning either f prime is increasing or f prime is decreasing and when in that case f is increasing, you can take f from I to f of I since I am taking f of I it is immediate, it is on to immediately and it is monotone that means it is also 1-1 and I is interval f of I is also an interval, this is also an interval, that immediately tells you this is 1-1 on to.

So, it does not come here so I may need here once more so this here that imply f from I to J both are open is 1-1 on to you have a local solvability so if you have look at here what it says that if you take any point here for which f prime is positive it says that there is an interval here and then you look at its image, you look at its image and so interval here so you get an interval here J this is your interval I and here you can every point here you can solve it you have unique solution so you know the local solvability here you see. So f prime somehow that derivative non vanishing of derivative in one dimension you already know that there is solvability and that is what we observe in the various case in the previous example of this thing.

Now, let us come back to this equation, this equation, this is nothing (()) (16:06) equation of the circle, you see this is the equation of the circle here and look at here for this trouble you can immediately if look at its derivatives f x, f y, so you look at f x this is 2 x and this is 2 y and then the troubling points you can immediately see that if you take 0 plus or minus 1 this is equal to 0 and if you take f y that is plus or minus 1, 0 this is also equal to 0 so, these are the four points.

So, let me mark it with red color so you have four points here, four points here and four points you see. Other points if you choose any other points x naught here or anything x naught here and if you look at corresponding image you have a y naught here and you can find neighborhood here always and then corresponding neighborhood here and that happens in all points here for x naught on this path you look at it here.

So, you look at the point x naught, y naught here then with x naught here. Similarly, you look at a point here you can solve it but, now look at these points. So, let me draw once again the picture this is just for an example, so let us look at this point only one point I describe other points you does not look like a circle but then fine so look at this point, this point is 0, 1.

So, if I look at this point if I choose a neighborhood of point in the y axis so I am keeping x here the value, x naught value is 0 so your x naught, y naught value is equal to 0, 1 and I want to know so the question is that can I solve x, can we solve x in terms of y, terms of y that is the question in the neighborhood of y naught, y naught is 1 you have to understand that question.

So, we are looking for the solvability of x in terms of y, so if I choose y here I have no solution, if I choose y equal to 1 of course you have a solution, if I choose v I have two solutions you see so there is a lack of unique solvability but why that is because you have that point this is this point, so the derivative of f x at that point with respect to x equal to 0.

So, whenever the derivative of f with respect to x then solving x in terms of y has an issue but, then there is no issue to solve y in terms of x at that point so, these two points you have a difficulty of solving x in terms of y and there is no issue of solving y in terms of x because that does not vanish because this is two y. Similarly, these two point if you look at it you will have a problem other way, so I will like you to work out this kind of issues these are very simple examples and for this simple examples you do not have much difficult you can understand these examples are very familiar and is directly available to you.

So, look at all these points but what we sense here is that the derivative the vanishing of derivative creates an issue in the solvability we need to generalize these concepts in the high dimensions and eventually in infinite dimensions for that you require the concept of a derivative which is what we are going to do it now. So, let us go to the next page.

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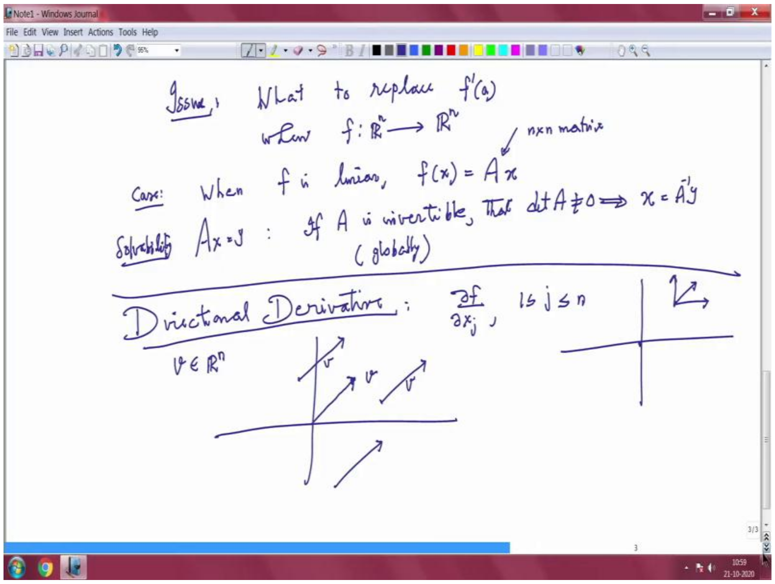

So, the problem is that major issue is that what to replace f prime of a when f is from say R n to R n. So, one particular case you know easily, you can see what it will be case, when f is linear, f is linear essentially means f is given by matrix A x where A is n by n matrix and you want to know the solvability of A x equal to y you know from linear algebra the complete analysis of it.

So, a one case is that if A is invertible, you see invertible that is the dominant of A not equal to 0 that immediately implies you can solve x uniquely A inverse of y and this is the global condition so you can solve globally, you can solve globally. So, one condition of determining of A is not equal to 0 you need to do this one, so you want to basically replace this n by n matrix into condition how do you do, what kind of conditions you can replace for f is form R n to R n which is not linear, we will be doing that.

So, I want to make a direction, first I will define what is called a directional derivative. So, you already know about partial derivatives d f by d x j for 1 less than equal to j less than equal to n. Now so that means for directional partial derivative if you want to define a derivative point or any point you make changes small changes in the coordinate axis but I can change any direction that is what I provide so you take V in Rn, here I want to tell you something you can view a point V either you can view it as a point or you can also view it as vector, the advantage of viewing it as vector you can replace this vector anywhere you like it, but it gives you only the direction and magnitude you can replace V.

So, (()) (24:38) V represent, so not this direction so you have, these are all V if you replace it. So when I write up vector here this not a point here it a vector and point may be here as a point view. So, this is useful you would have seen it when the motion of a particle, when the motion of particle you see the position view, view it as point but the velocity view it as a direction that is how you represent it, which we are familiar. So, given a function so I want to define let me go to the next page.

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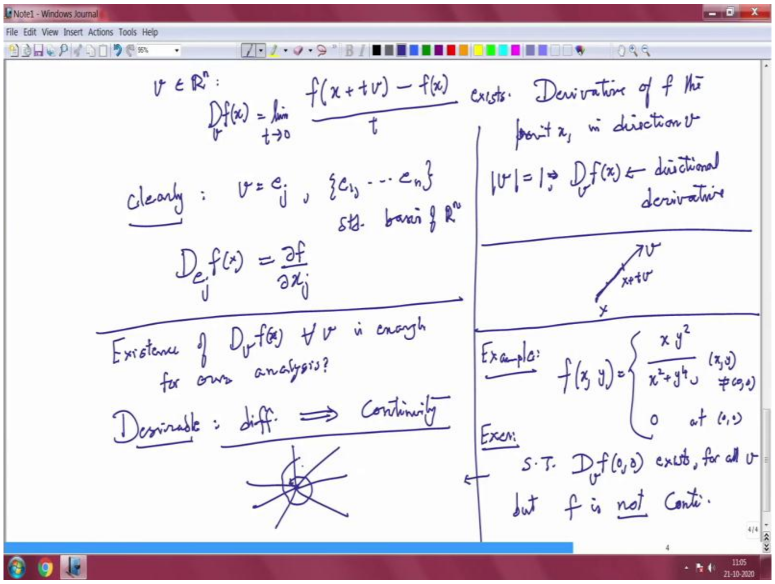

So, given a vector here v in R n I want to define what we call it a direction, so I want to look for the change in that direction t v minus f of x by t so that is what I try to do it I call this the if exists derivative of v up to point x to the direction v if the limit exists, limit t tends to 0 exists then you call it derivative of f, derivative of f at the point x in direction v and if mod v equal to 1 you know that is a unique vector then we call is the directional derivative and this is called direction of derivative otherwise, it will have a factor directional derivative.

So, if this exists you have it so clearly this is called the directional derivative clearly if you take v equal to the unique vector e j where e 1 etcetera is the standard basis which all of you know standard basis of R n. Then D v e j of f at x is also the nothing but your non-partial derivative so you say so, you have a derivatives defined in all directions, so if you have a point x if you have t here if this is your v your t v will x plus t v will be here this is the point.

So, when t changes this changes along this direction when t become smaller and smaller it will be in this direction you have to see domain (()) (27:51) so it moves here, so t becomes smaller and smaller the value of a changing along that direction. So, I do not want to elaborate and start giving example because I said I do not have time for all that so, what I want to tell you here one is this enough is the existence of, existence of D v f of x for all v is enough for over analysis, unfortunately so you may think that taking derivative in all directions and one of the desirable thing which you know in one dimension what is desirable, differentiability implies continuity this is the one minimum requirement we always look for.

So, whenever you are defining a concept of differentiation, you look for its continuity and fortunately the example which I am stating here and I request all of you or suggest all of you to this problem. So, I define f of x y this is a two-dimension example is equal to x y square by x square plus y power 4 so x, y is different from 0, at 0 you define 0, 0. Then you can show that so this is an exercise for you show that D v of f at 0, 0 exists for all v, but f is not continuous.

The issue probably those who already familiar why this is happening is that when you are dealing with a point you are taking derivatives in all directions you see you can take derivatives in all directions but the thing is that you can possibly approach this point not only in a linear direction you can approach this point like this also.

And that kind of approach of limiting, so in this particular f of x plus t y minus f of x by t you are approaching along the straight line, you are not approaching it is possible that you can take the limits not along the line you can also take along various curves and this is also the continuity, checking continuity that f of x y the continuity of f at the origin you have to show that f of x y goes to f of 0, 0 for all x y going to 0 not just along that one so you can have variations, ratio variations along arbitrary curves that is what.

So, you have to, when you are taking differentiation you have to incorporate that also, that is where the difficulty of defining the differentiation. So, we will go little more that we will do the definition and then will just give you the definition of the strong definition it is called the Frechit derivative.

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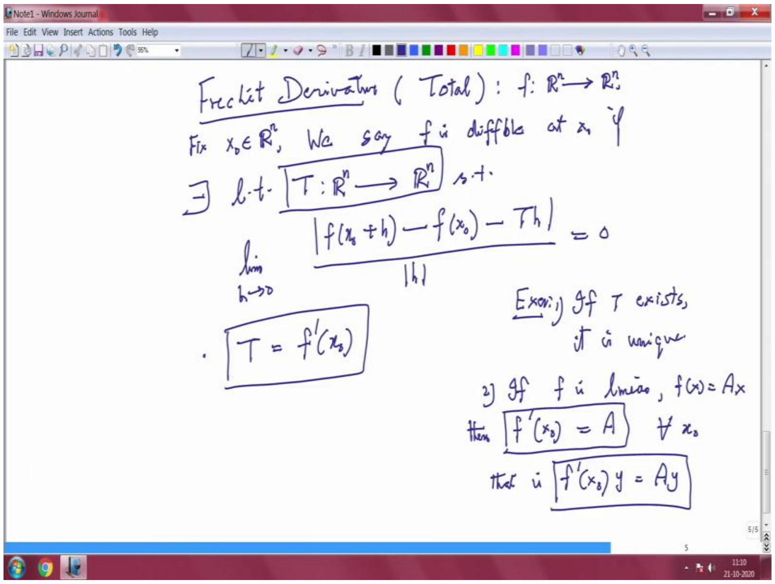

Frechit derivative and this is also called Frechit derivative or you also called total derivative. So, you have your function f from R n to R n and you fix some x naught in R n, fix x naught in R n and then we say f is differentiable at x naught if there exists a linear transformation, a linear transformation T from R n to R n such that f you look at the f of x naught plus h minus f of x naught, minus f of x naught minus T of h modulus by modulus of h its limit h tends to 0 is equal to 0.

So, the existence of the Frechit derivative is a linear transformation. So in this case, you denote T is equal to the derivative of f at x naught, so it is a linear transformation, it is not value as such, in one dimension you eventually represent value but it is linear transformation from R n to R n. So, you can prove some simple exercises for study more it if T exists it is unique.

One thing if T exists it is unique, the second condition you can also this replace one case, secondly if f is linear suppose if f is linear that is f of x equal to A x then you can see that, then you can easily see that f prime at x naught is A, you make sure that it is a linear transformations so it is matrix for all x naught that is f prime of x naught acting at y is equal to A y so that is the meaning of it.

So, I will do little more about it Frechit derivative, I will main aim, is one of the two results we want to present based on the Frechit derivative put the conditions on the Frechit derivative for the existence of your derivative and based on that we prove the solvability, we try to state two theorems mainly the inverse function theorem and implicit function theorem. So, we will stop here and continue in the next lecture. Thank you.