First Course on Partial Defferential Equations – 1

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Lecture 29

One dimensional heat equation-2

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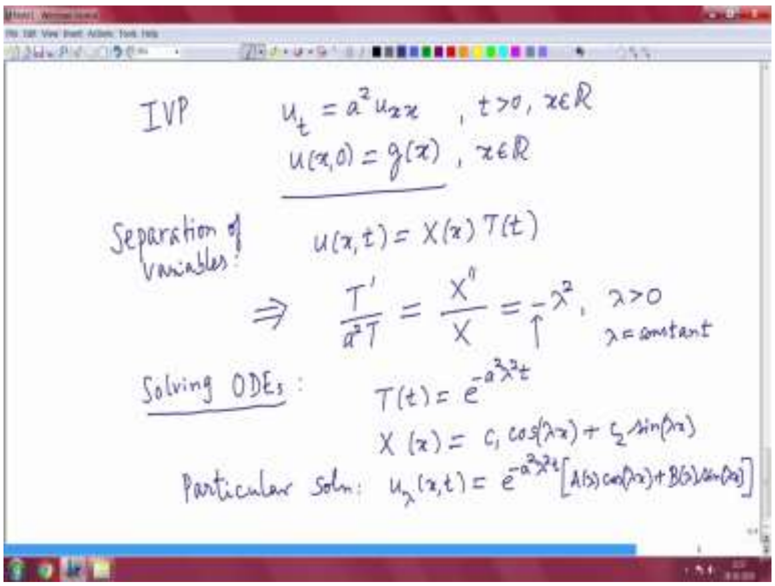

Hello everyone. Today we analyze the initial value problem for the heat equation and try to obtain a formula for the solution. So, this is in t positive x in R, so one dimensional heat equation. So our arguments are not very vigorous to begin with but once we come to the formula and we will concentrate on that formula and analyze that further. So we begin with a solution in the variable separable for variables.

So u x t is a function of x and function of t. You formally substitute this into the given equation namely that heat equation and that implies T prime by a square T is equal to x double prime. So there are two derivatives with respect to x. So, there will be two derivatives of this function x and only one derivative of the function T. So the left hand side is only a function of T and this is right hand side is a function of x.

So this ratio must be a constant and that we call it minus lambda square, lambda is positive. Why we have chosen this negative sign? So if you choose positive sign there then and if you look at the equation for T. So that shows T is exponentially growing physically if the rod is heated in a bounded fashion that means this bounded temperature, then we can expect the temperature to remain bounded for all future times.

So we cannot take positive sign there which makes this T exponentially grow and that makes the function the solution to grow exponentially large which is not physically meaningful. So that is why this negative so I that is important. And then so lambda is a constant, positive constant. And that results in two ordinary differential equations one for capital T and one for capital X. So that further implies solving the result in ODEs.

That gives us T of t is equal to exponential minus a square lambda square t we can always put a constant there but we are going to put constant with x and since they appear as product so that constant T is absorbed there. And X of x I will put some constant cosine of lambda x plus another constant that is solving the second order equation for capital X.

So in particular we obtain a particular solution which you denote by u lambda because that depends on lambda e to the minus a square lambda square t. So let me write instead of C1 so this also can depend on lambda cos lambda x plus B lambda sin lambda x and then using the linear super position principle.

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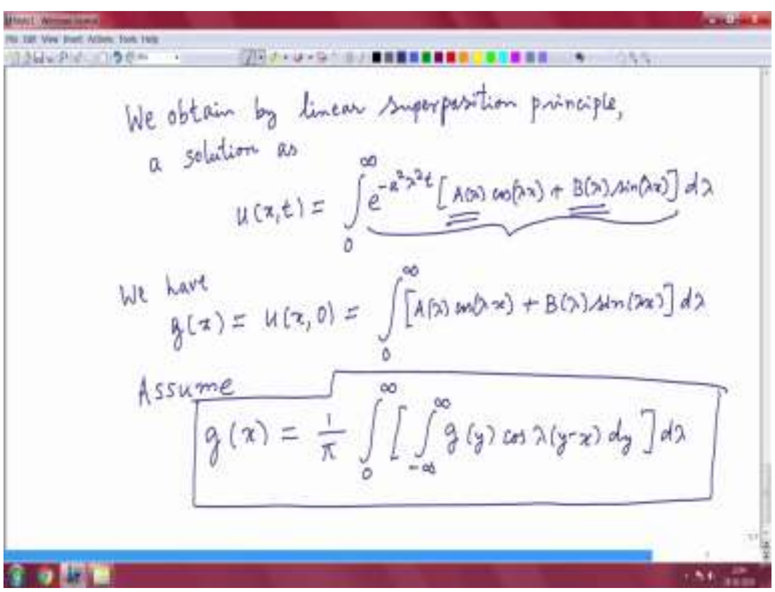

So since the equation is linear and we have obtained a particular solution for each lambda positive we can sum it but since lambda is a continuous variable here. So we have to replace the summation by an integral, so we obtain by linear super position principle a solution as u of x t zero to infinity e to the minus a square lambda square t A lambda cosine lambda x plus B lambda sin lambda x d lambda.

So in the begin I said that these arguments are not trigger us because we do not know whether this integral is finite or not but for the time being assume they all make sense and formally differentiating under the integral signs. So since this integrants satisfies the heat equation. You just say that u also satisfy the heat equation but this here this A lambda and B lambda are still to be determined. And that is where we use the initial condition.

So we have g of x is equal to u x 0. So this is the initial condition and formally substitute t equal to 0 in the integral. So those of u who are familiar with fourier transform technique will recognize we are doing something similar but without using that word. So in fact it is quite easy to explain using the fourier transform technique but since many of you may not be familiar with that technique.

So I am just putting forward some formal arguments. But eventually it will lead to the same thing. So g is a expressed as an integral with these unknowns A lambda and B lambda. So again those of you are familiar with fourier cosine formula fourier sine formula. You see that A lambda can be obtained as inverse fourier cosine formula for g and B lambda inverse sine formula from this. But we cannot directly do that so again make one more assumption.

So assume so these assumptions seem very restricted but when you arrive at the final formula all these things will make perfections. So assume g of x can be written as 1 by pi 0 to infinity minus infinity to infinity g of y cosine lambda y minus x dy and then you integrate with respect to d lambda. So in a some sense I am assuming that g can be written as the inverse of its cosine transform.

So this looks very strange assumptions but those of you are familiar with fourier transform that is not strange. Let me and now what you do is you expand this cosine and express the integral similar to this that is what we do. So just write this u the cosine addition formula and rearrange.

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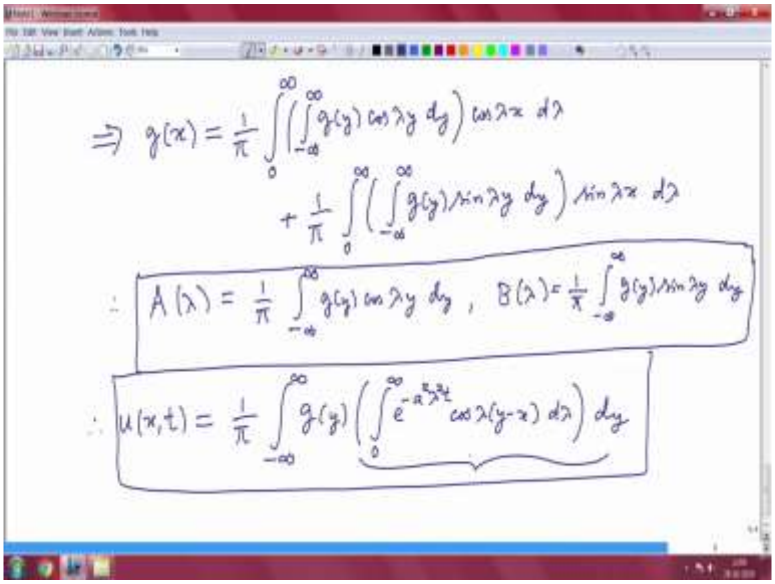

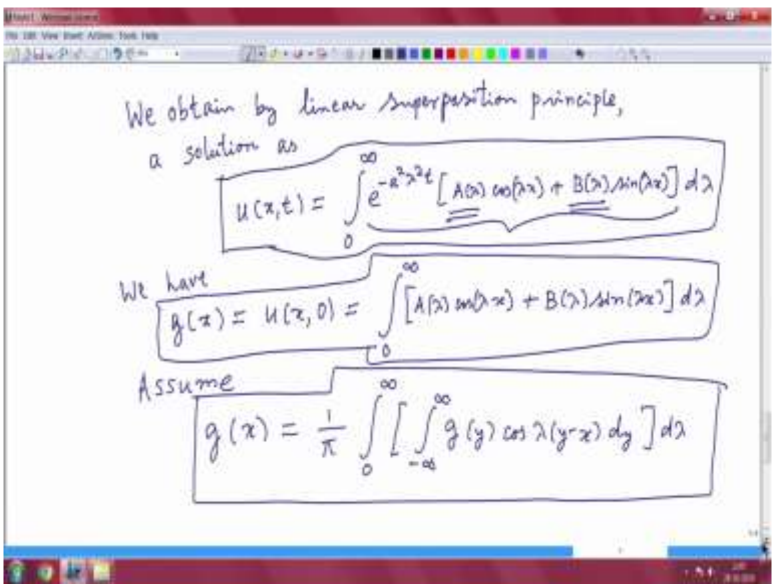

So that implies g of x, just I will write it here 1 by pi 0 to infinity, g of y cosine of lambda y dy multiplied by cos lambda x. And this with respect to d lambda plus 1 by pi same thing, now sin terms, dy sin lambda x d lambda, with this assumption on g we could write now g as sum of two integrals and just compare this with the one we obtain from the formal solution.

And you immediately recognize that so we take, so therefore you just compare them so you get a lambda is equal to 1 by pi minus infinity to infinity g y cosine of lambda y dy and B lambda g y. So this is nothing but fourier cosine transform of g and this is fourier sin transform of g. So at least we have though we have not justified any step but still we are able to obtain expressions for A lambda and B lambda and now we plug in, in this go back again.

So our u x t is sitting here you just substitute there and again make some simplifications. So let me, so therefore so I am skipping few steps here and you also do there are two double integrals, so you also interchange the integral so we obtain this 0 to infinity e to the minus a square lambda square t cosine of lambda y minus x d lambda and then there is dy.

So, it from substituting A lambda B lambda in this expression and doing some computations and interchanging the integral results in this formula. If you look at the integral there is no g at all its involves only the unknown functions namely exponential and cosine so there is a possibility of simplifying that integral. So now we concentrate on that integral.

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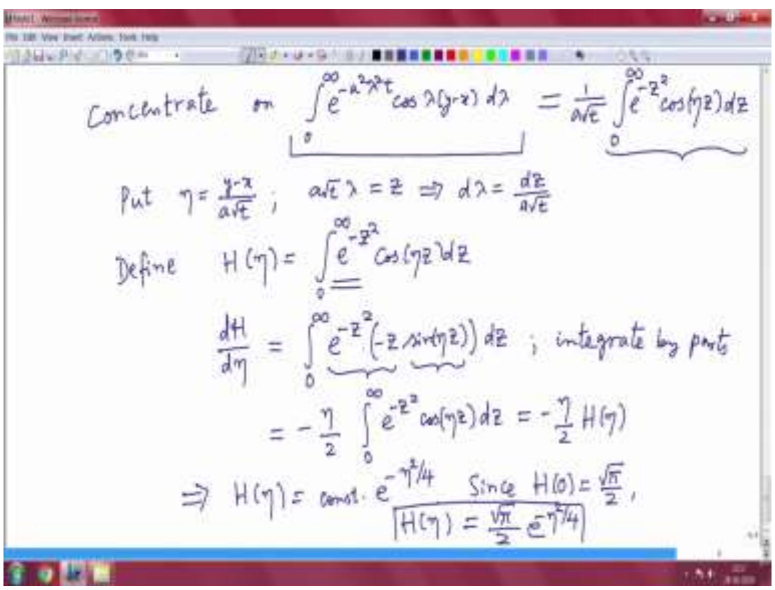

So just so concentrate on so let me write it separately 0 to infinity a square lambda square t cosine lambda y minus x d lambda. So here that initial value, initial function g is not at all involved it is an integral only in terms of known functions the exponential and cosine let see what simplification we can do. So, to write some change of variable, so put eta is equal to y minus x by a root t.

And you make the change of variable this see integration is with respect to lambda. So what I make is so t is any root positive. So a root is t lambda I put it as z a new variable. So that implies d lambda is dz by a root t. So if you make those change of variables. Now define this integral, so that integral becomes so this becomes let me write it here itself 1 by a root t that is coming from this change of variable and then you have integral 0 to infinity e to the minus z square cos eta z dz.

And this integral I am going to call it as 0 to infinity cos eta z dz. So this is very nice function and this is a bounded function we know so this a function is differentiable with respect to eta. I just so recall the results from calculus where we can perform differentiation under the integral sin and exponential function is going to help us e to the minus z square this is minus z sin eta z. So this is differentiation under the integral sign.

So this now you integrate by parts, so you take this these two terms together and this separately and you integrates by parts. So you differentiate sin and you integrate this one because of z we can integrate that one, so it is again e to the minus z square. So I will just you do that and what you get is let me write it, once you differentiate this sin eta z, so we are going to get an eta there so we get minus eta by 2 0 to infinity. So integrate by parts. And that is precisely H n.

So the function H satisfies a first order ODE, variable coefficients and that implies H of eta is a constant times e to the minus eta square by 4. Since, H 0, put eta equal to 0, this cos 0 is 1, so this is a well-known integral and whose value is pi by 2, root pi by 2, we obtain H of eta. So this constant is root pi by 2. So we stop here and in the next lecture we continue from this point.

So after plug in this H eta into the solution we will rewrite the solution and see its connection with the fundamental solution of the heat operator. And then we will verify the formula indeed is a solution of the heat equation so that we will do in next class. Thank you.