First Course on Partial Defferential Equations – 1
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Lecture 28
One dimensional heat equation-1

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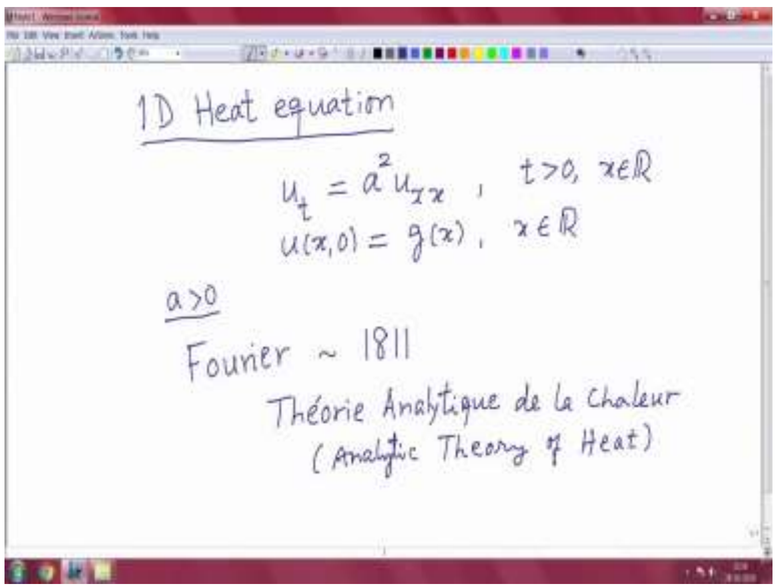

Hello everyone, today we will study the one D heat equation. So this is a equation given by ut equal to a square uxx again t is positive x is in real line and initial value g of x. So here is a is positive a constant. So physically this represents the evolution of temperature in an in a finite rod. So though not physically meaningful but we will see that just like wave equation. This also helps us in analyzing later the temperature evolution in a finite rod.

So, before we go further few historical remarks. So this equation first appears in the work of Josef Fourier a French scientist. So, somewhere in the beginning of 19 century. There he derives this equation using Newton's law of cooling. So most of the things we study now like fourier series fourier transform they all appear in this works and this is called is in French, so let me just write it Theorie Analytique, this is in French, de la chaleur.

So simply translated this is analytic theory of heat. So whatever he fourier in this work it influence a lot especially in the 19 century mathematical physicist, mathematicians so and in fact, now there is separate branch called harmonic analysis which originated in this work Fourier. So he was also successful in deciding to construct chaleur to escape extreme weathers, so extreme cold condition would like to remain warm and in extreme heat conditions would like to remain somewhat cool.

So he was successful in determining so remember this is in 19 century so of course now we have air conditioners heaters and what not but in those, so this is more than 200 years. So he was successful in determining the depth at which chaleur must be constructed in order to escape extreme weather. So and there he exploited the linearity of this heat equation and this linear super imposition that we all aware of now.

But it in this work he so there will be temperature variation daily temperature variations and yearly temperature variations. So he took all that into account and successful arrived at the depth at which a chaleur must be constructed. So a brief account of this I will give you reference.

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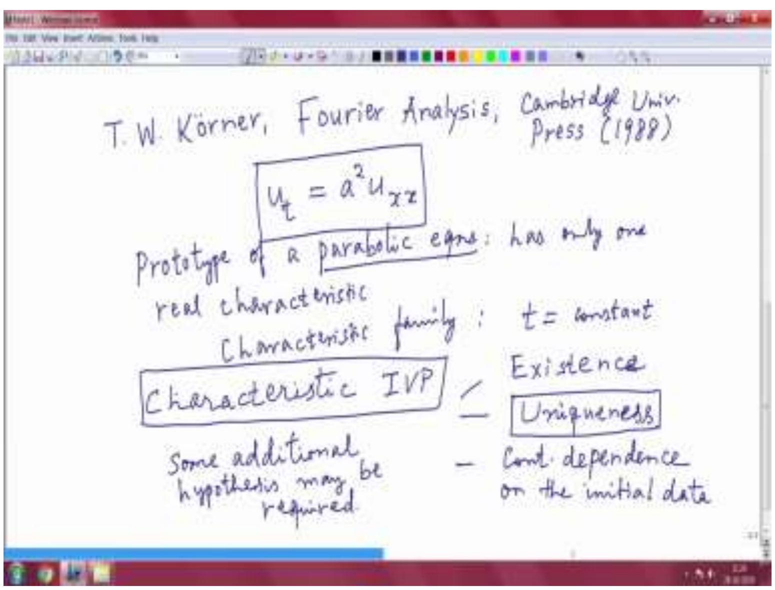

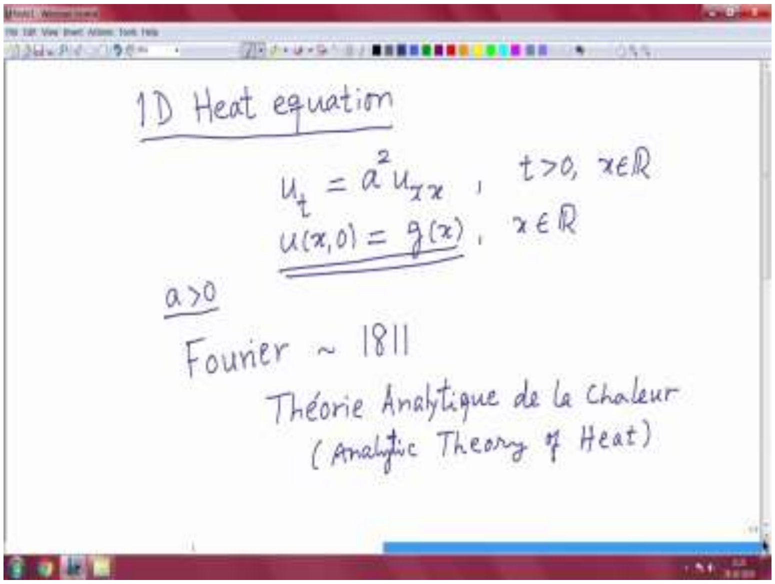

So you can just there is a book by T.W.K Korner title is Fourier Analysis so you will also find a good account of fourier series fourier transform which we are not covering in this course. So this is published by Cambridge University Press now there are many books. So I am just mentioning one. So today what we will do is this try to find a formula for the solution again in the fourier spirit.

So before going further let me mention one thing about this heat equation. So it is a prototype of parabolic equation. So it has only one real characteristic, has only one family when I say one and you can easily see that these characteristics, the characteristic family are given by, characteristic family t equal to constant. And remember we are giving prescribing initial condition at t equal to 0 though physically meaning full.

So at some t equal to 0 initial time u prescribe the temperature in a rod and then you see how it evolves though physically meaningful but as far as mathematician is concerned. So what we are dealing here is so characteristic initial value problem. So I just press this thing characteristic initial value. Because we are prescribing the initial values on a characteristic t equal to 0 is also characteristic.

So in mathematically such characteristic initial value problem they may fail. So there are three things we should be interested existence of a solution and then uniqueness this is again important as far as the physics is concerned, uniqueness and then the continuous dependence on the initial value. So together they are known as Hard mass well pose conditions initially. So when we are dealing with characteristic initial value problem there could be problem with any of these three things one or two.

So that means so even for existence or uniqueness some additional conditions may be prescribe. So some additional hypothesis may be required. So in fact there is no uniqueness in general but if you restrict the initial value to certain class then there is uniqueness. As we see existence will be there again this continuous depended on the initial data is there only if you restrict the initial data to a certain class.

So it is a very difficult to do all this in this set of lectures. So we will just concentrate on certain things but this is an important equation which makes it appearance in several branches of mathematics. So and it is also a typical parabolic equation and it has many properties interesting properties some of them will be discussion.

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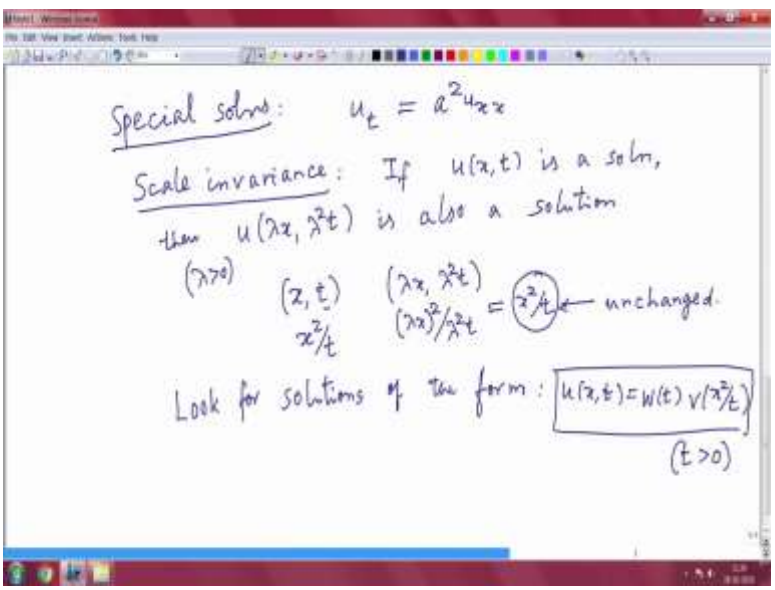

So first we will obtain some special solutions and you just let me write it. So we see first you observe that there is a scale in variance. What does this mean? Let me explain that so if u x t is a solution then u of lambda x lambda square t is also a solution. This easily follows by direct differentiation. So if we scale the variable x to lambda x so lambda is a positive number any positive number. And t by lambda square t you see that. So what does this do?

So this x t there and this lambda x lambda square t there. So what is common there, so if you look at x square by there and again I take x square by t here so that means lambda x square by lambda square t and that just give me. So the scaling leave this x square by t is unchanged. This is unchanged so that is fixed.

This suggest to look for a solution of heat equation which is of the form a function of x square by t only because we have seen that this heat had this scale invariance and that suggest as to look for a solution that may not be there. But just we can certainly try that. So, look for solutions of the form u x t so we can even put additional thing here.

So just put v t and w x square by t. You do some I have used, let me write that, w is the function of t only and v is a function of x square by t of course this requires, we have to only deal with t post because we are dividing by t.

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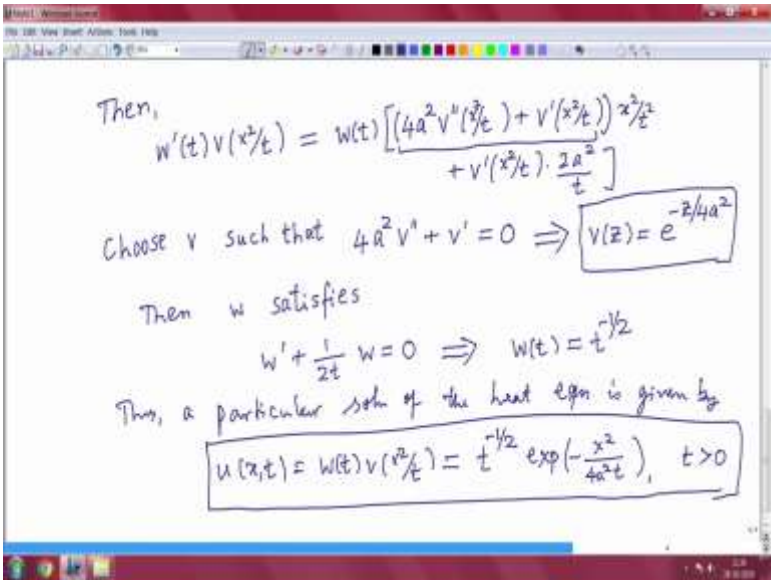

Now you just make the necessary differentiations and plug in in the heat equation and so this w and v they satisfy, so I am skipping few computations. So, I am just using prime, since they are w and v are functions of only one variable I am just writing prime, prime is derivative with respect to the that variable. So w t 4 a square v double prime x by t square no, sorry, x square by t plus v prime x square by t bracket mod x square why mod, mod is not require we are in the one variable so just x square by t square plus v prime x square by t 2 a square by t.

So remember w and v are at our choice so we make first choice, so choose v such that this term in the bracket is 0. And that immediately gives, so one solution, so that is a though it is a second order equation for v. It is a first order equation for v prime, so you do two integrations and you get one particular solution, which is so v z is equal to e to minus z by 4 a square that is one solution. And now you plug in this solution v in this thing and you obtain a equation for w prime.

So this is gone and now you differentiate this one and plug in here. So what you get is, so let me just then w satisfies, so you have to work it out. I am just writing w prime plus 1 by 2t w equal to 0 and again a particular solution of this first order ODE. Remember this is ODE, so this is w t is t to the minus half. So thus we have obtain a particular solution of the heat equation.

So thus a particular solution of the heat equation is given by u of x t is equal to w t v of x square by t and this one. Now you replace z by x square by t, so let me write this is expression in the exponential is big so let me write it as exponential. So this is x square by 4 a square t. A constant multiple of this solution is going to play an important role. So that is called so let us put some name.

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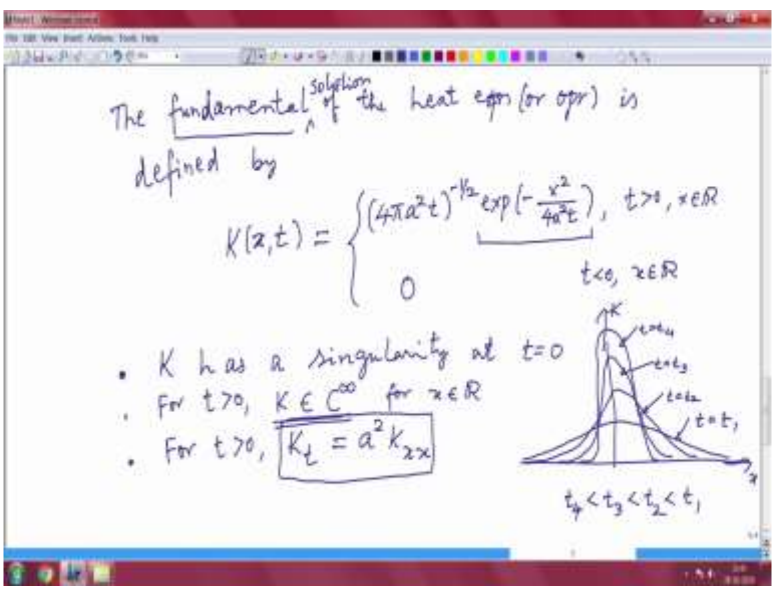

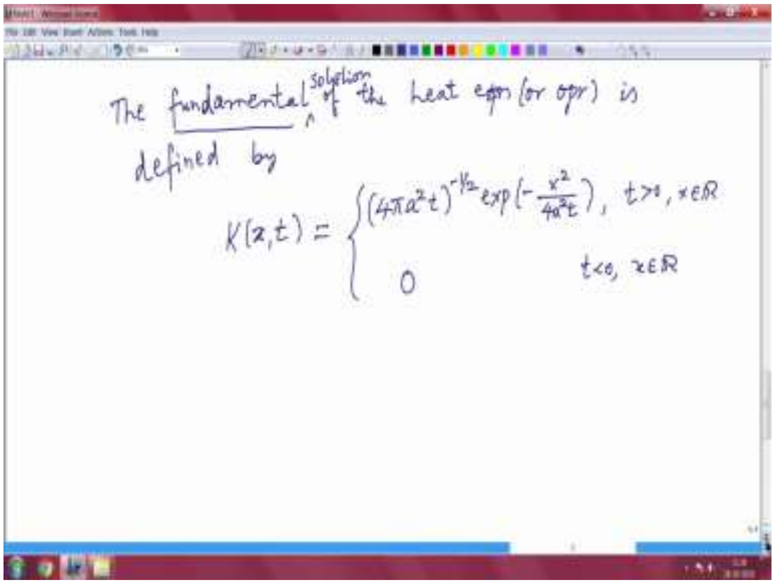

So, the fundamental solution of the heat equation or heat operator fundamental solution is defined by or operator is defined by denoted by K x t. So there is a constant multiple here. So 4 pie a square t to the minus half exponential so this is t positive x in R and 0 where a t less than 0. So what we obtain here, so I am just adding putting a constant term there. So this 4 pie a square we will see the reason for putting that particular constant how that plays an important role.

So immediate observations about this K. So for t positive exponential is infinitely differentiable also this t to the minus half. So immediately just by looking at the expression K has a singularity at t equal to 0 for t positive K is an infinitely differentiable function for x in R. And just now we have seen that, so again for t positive K satisfies Kt equal to a square Kxx.

So K satisfies the heat equation in the region t positive and it is a very smooth function it has only singularity at t equal to 0. Some of them, some profiles of K I can show you, so this is also a since this a Gaussian curve so it is also refer to as Gaussian. So it is, so yes t becomes smaller and smaller you will notice that the exponential term becomes closer to 1. Because t is infinity so this will become 0.

So exponential become 1 but there is a t to the minus half factor so as t becomes larger and larger and we will see that space is less here, so this safe, so t equal to t4 equal to t3 sometimes. So this is t equal to t2 and t equal to t1 and you drawing x and K. So here t4 is less than t3 less than t2. So important thing we will notice that that we are going to use that the fundamental solutions satisfy the heat equation in the region t post that is important and we will also see that this K is a very smooth function. So it is infinitely differentiable.

So these two things will help us in finding a formula for the initial value problem. So this occurs in the formula for the solution of the initial value problem. So anything else yeah with this we close this lecture and in the next lecture we will obtain the solution of the initial value problem for the heat equation. And there again we see the role played this fundamental solution. Thank you.