First Course on Partial Defferential Equations – 1
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Lecture 27
Laplace and Poisson equations - 10

Good morning and welcome back. So this may probably the last lecture in this Laplacian case let me recall a bit.

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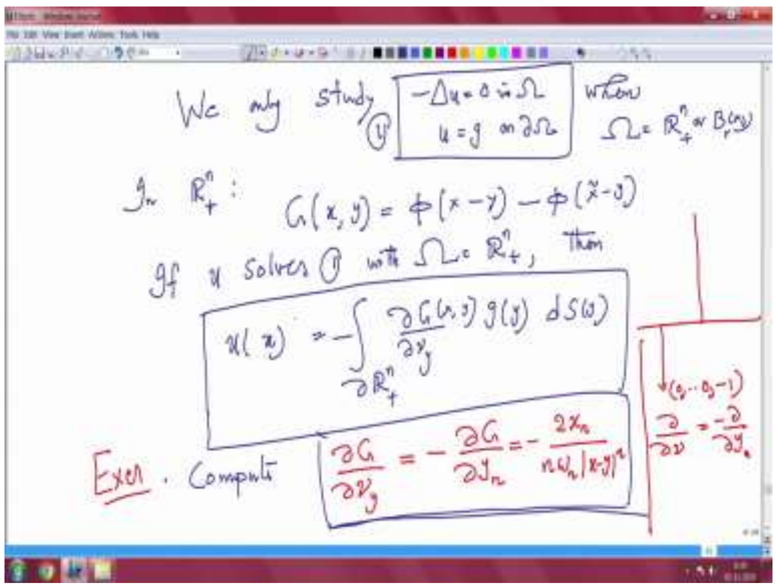

So what we have introduce is the let me back to one more thing. So we have introduced Green's function for the upper half plain now we will in and then using the Green's function for the upper half plain we precisely got your Green's function like you are d G by d nu and you got a formula for your solutions.

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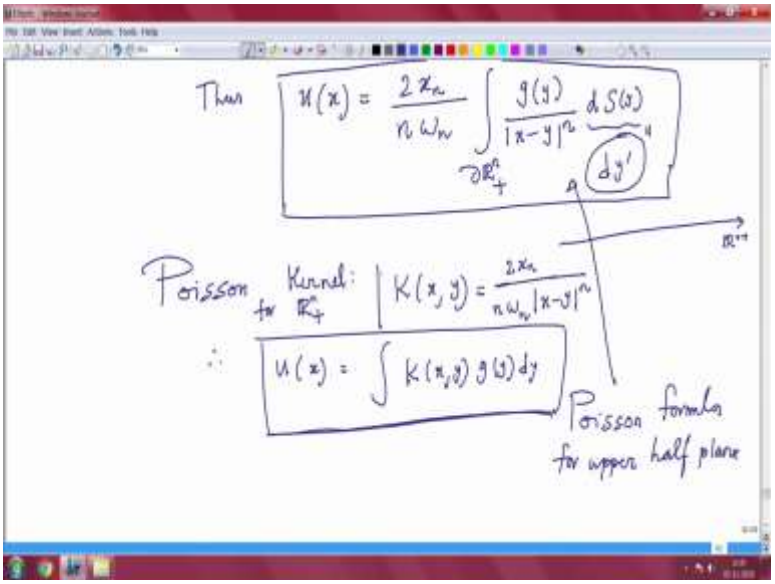

So this is the formula which is the Poisson formula we also introduce what is called the Poisson Kernel

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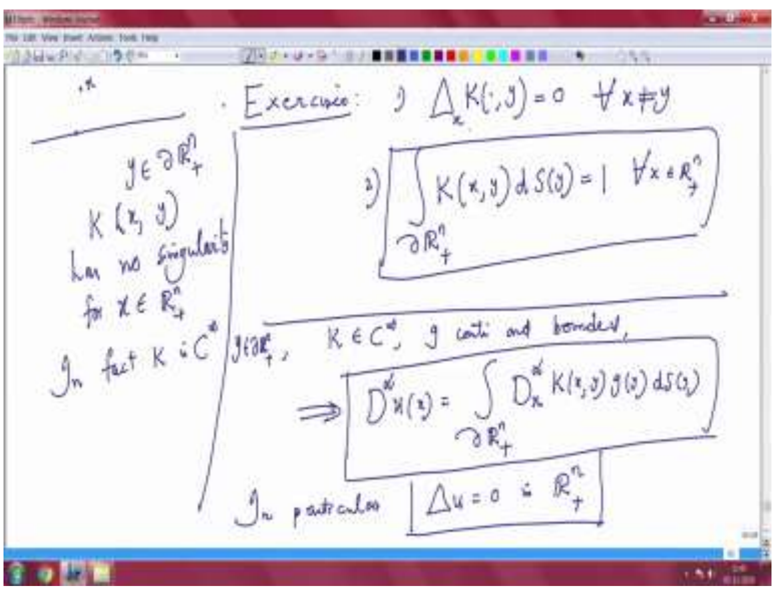

Using that Poisson Kernel you know that Kernel has satisfies certain properties.

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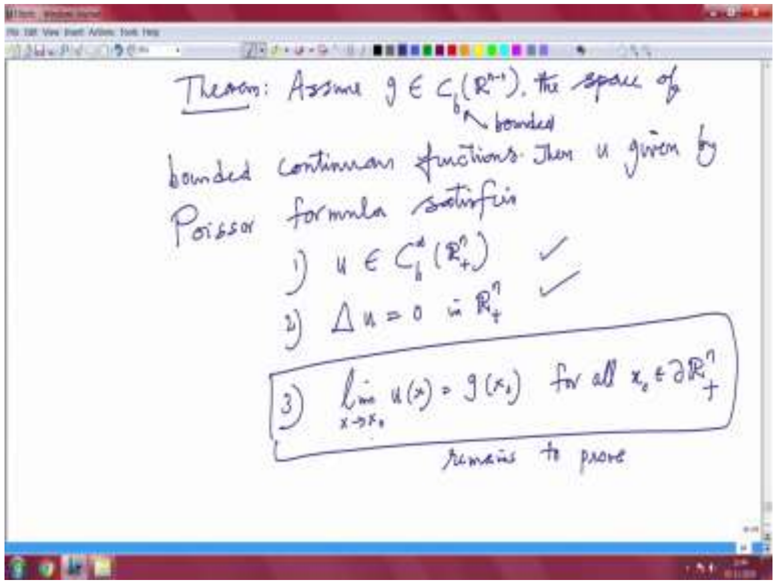

And using that we have actually solve the problem in an upper half plain where Laplacian u equal to 0 and u satisfies the boundary condition g in the sense of limit. Because u is defined only in the open upper half plain u is not defined the boundary. But then it has a limited the boundary and we are shown that u is infinity and Laplacian u equal to 0 in the previous class. So now we will show that the last part third part of that limit of u x equal to x naught. Let me give some ideas.

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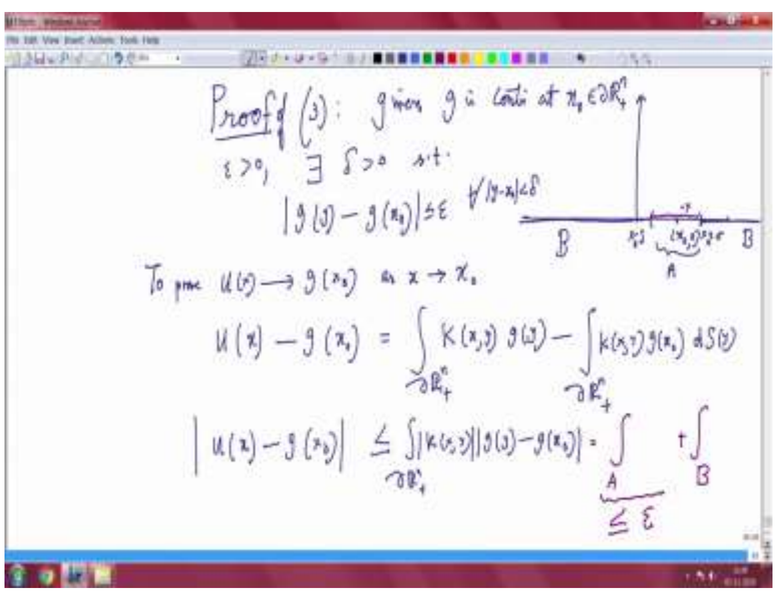

So proof of 3 so g is bounded continuous given that g is continuous at any point x naught at x naught fix x naught in Rn plus or Rn minus whatever it is. Rn boundary of Rn plus so your half plain so this is your x naught you can view x naught as x naught 0 or x naught. Whichever way you like it as a boundary. Now look at the point so first g is continuous so you have x naught here.

So that x naught here so given a epsilon positive you can immediately see that their x is delta positive such that modulus of g y minus g x naught is less than or equal to epsilon for all mod x minus y minus x not is less than delta. So you have a delta here and you have the I call this said to be A this interval. This is some x naught minus delta this is equal to x naught plus delta and this part complement of A this I call it B this is B together of that one.

So let me write down formula, so you have your u x what is your u x? So I want to show that I want to show x somewhere near. I want this is my x so when x tends to x naught in whatever weight is I want to show that to prove u x tends to g x naught as x naught tend to as x tends to x naught you see. That is what you, so we write down u x minus g x naught u x we use the properties of theorem.

So look at here you have the first one is K of x y g of y but then this is g x naught this is on the boundary of Rn plus and then minus u g x naught but integral of K x y is 1. So I use that fact to write down this is equal to Rn plus K of x y g x naught which is a constants what are they and this integration is with respect to d S y because integral of that everything is with respect to d S y which we are not which is nothing but d y prime in this case.

So if you compute this u x minus g x naught is less than or equal to I get this thing boundary of Rn plus K of x y modulus of so this is positive thing of g of y minus modulus so if you want you can put it g y minus g x naught on a in this neighborhood in this neighborhood it is thing and this is one so I can write I can split this into integral boundary is divided into A plus integral over B and this is small, this is less than or equal to epsilon.

Because g y minus g x naught is less than or equal to epsilon the other integral is equal to 1. So you do not have to worry. So only we have to bother far away.

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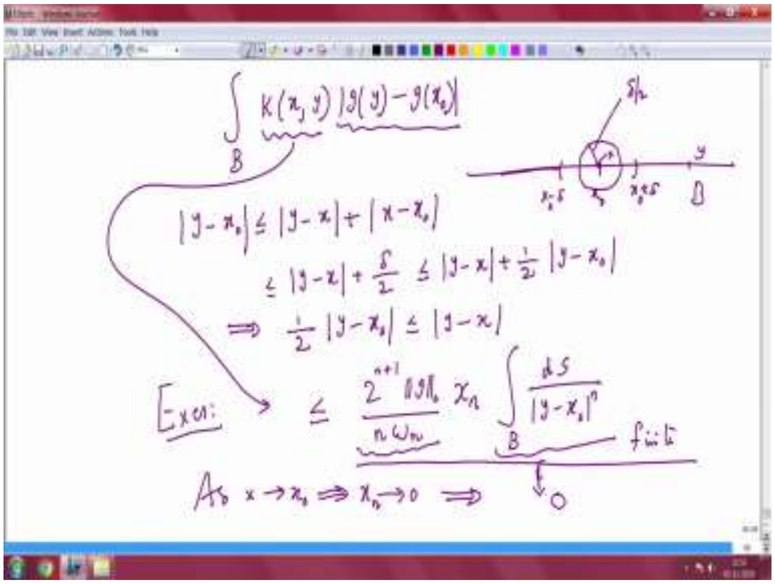

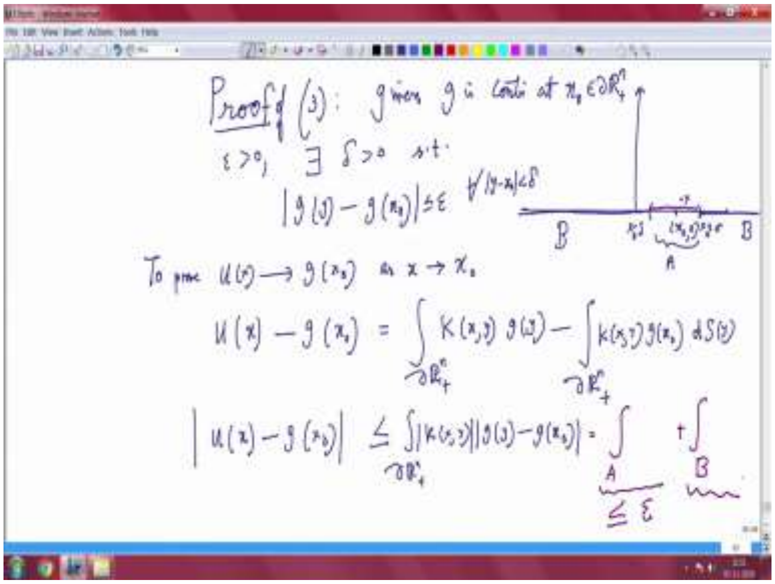

So you have to worry about integral over B K of x y modulus of g of y minus g of x naught. So let us now look at this one work. So you have your x naught here. You have your delta here x naught minus delta this is your x naught plus delta x naught plus delta. So this is your B part, so y varies only here now y is not here. So y is here this is your x naught. So y will be here x will be here x naught will be there.

So what we will do is that in that case, so you try to compute, so you take an x close to that one. in fact consider x with a ball of radius delta by 2 but you can do it delta by 2 a radius. So my x is because anyway I am taking s limit. So I can see within that ball. So what we can do that, our y minus x naught x naught here x here so you apply your triangle inequality it is modulus of y minus x plus x minus x naught.

This is less than or equal to modulus of y minus x x minus x naught is less than or equal to delta by 2 but that can be written modulus of y minus x this delta is y is here, so it is here so y minus x will be this will be delta will be half of y minus x naught. This delta is less than or equal to y minus x naught. So with this, this will give you half of modulus of y minus x naught if you take this to the left side is less than or equal to mod y minus x.

So with these computations so I will do a bit of computation as an exercise. So please do this you can show that thus I will be already have two terms. Let me recall this term, so this term we have to calculate it. That is what you have to do that one. So you can do that, so you can calculate this term this will be less than equal 2. This is important these terms are important so compute into norm of g which is bounded l infinity norm by n omega n. And you get a term here xn.

So to do the computation g is a bounded function so you can bound this. So only we are estimating K of x minus y this is bounded by 2 g you can do that 2 the norm of g that you can take it out. So you may not making this small you are trying to make this computation u xn you get modulus d S by mod y minus x naught there is no singularity be careful we note that. x is here y is away.

So there is no singularity or anything here, and this is also a finite quantity or anything here. And this is also a finite quantity. These are all finite quantity. So as x tends to x naught this immediately tells you that my xn tends to 0 you see that that is it when x tends to 0 x tends to x naught here my x tends to 0 that is very crucial. That implies this term tend to 0 that is it. That is the prove of the. So only thing is that you derive this estimate properly. So that proves it.

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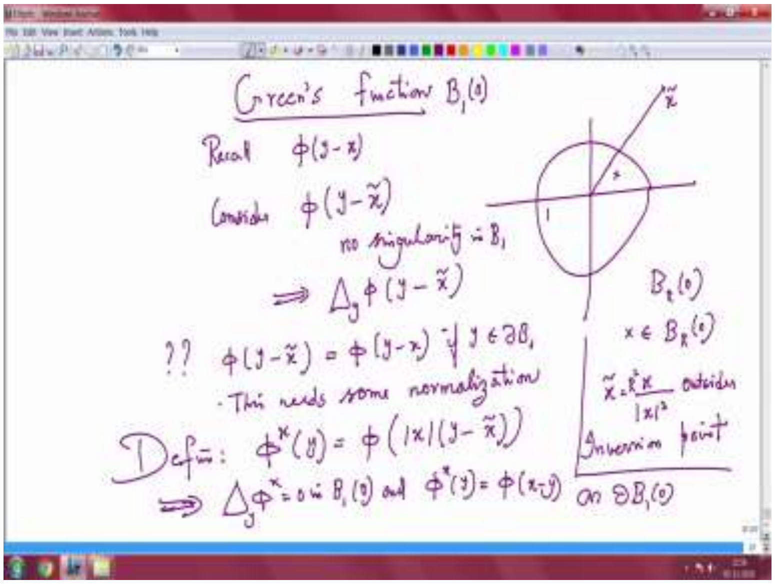

So now let me go to Green's function for the ball since all the computation done in the case you do the computations here for the Green's function again we use the symmetry. So going to so what is the, so let me start with a ball of radius 1 centered at origin the after that it is a dilation and a translation. So if you do it for this one the other things you can write it. So you have a ball of center this is one so I am going to do it for that one.

So I use the same symmetry argument, so this is symmetry what I saw what we call it inversion point. So you have a point here x you have an inversion point here. So for a general ball so let me call Br 0 which is a ball so if x is in Br 0. So you have point x tilde which we call it x by mod x square this is outside and call a inversion point. Yeah let me know why I do right. Similarly, you need r her so let me not write r here.

We are working with r1 naught B1 only with r you will have a parameter r here. So let us not bother about that you will have r square xi for example if you are working with r you will have r square here with we since we are working with 1 you will have x y that is a inversion point. So you use this is your x tilde now so I can still consider as a recall phi the fundamental solution phi you y minus x and then consider phi of y minus x tilde.

There is a slight difference compare to this one. So these singularities are x tilde just like previous case x tilde is outside. So now singularity no singularity in B1. Singularity in B1. That implies your Laplacian of y phi equal to y minus x tilde equal to 0. But the difficulty is that when y is on the boundary you want this to be, you want phi of y minus x tilde is equal to phi of y minus x if y it is on the boundary.

But this needs some normalization this needs you can compute this actually on the boundary then you see that this is not true this needs some normalization. So you have to add some constant, some parameter here. So I define, so define this is the definition phi x of y x is fixed only this with respect to y is equal to phi of modulus of x is fixes, so is a constant as far as y is concerned. Modulus of x is fixes so is a constant as far as y is concerned.

So y minus x tilde then this will imply it will not change the Laplacian, Laplacian of phi x with respect to y equal 0 in B1 of 0. And phi x of y you get to be phi of x minus y on boundary of B1. So you have this so you have your so still we use the thing symmetry we have used.

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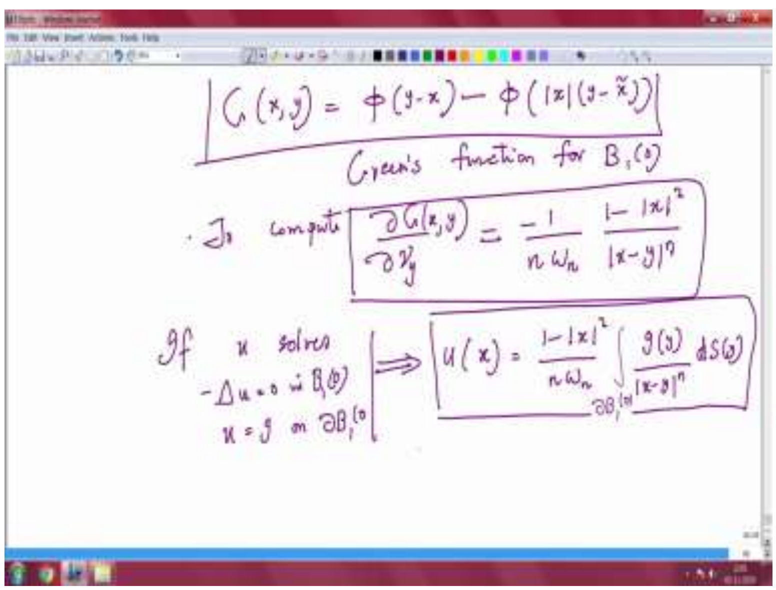

So with that so we can define G x y is equal to phi of x minus y minus phi modulus of x into so I am using this case x minus y or y minus x as I said let me do it in the other way so both are fine y minus x and y minus x tilde that is your Green's function. Green's function for B1 of 0. So you have to compute to compute which is exercise for the compute only thing take care of the normal properly.

Our compute d G d nu by this is what you want it at x y I would leave that as an exercise for you to compute you get this one 1 by n omega n 1 minus mod x square by n omega n you already have it mod x minus y power n. You compute this and then your representations formula you get it so that is Kernel coming into the picture.

So when so if u solves Laplacian of u is equal to 0 in B1 of 0 u equal to g on boundary of B1 B1 of 0 then the Green's representation formula u x equal to you get it 1 minus mod x square by n omega n. So take care of the science properly and g of y by mod x minus y power n d S of y. This is the representation formula. So what is your Kernel here, so if you work so let me go to next one.

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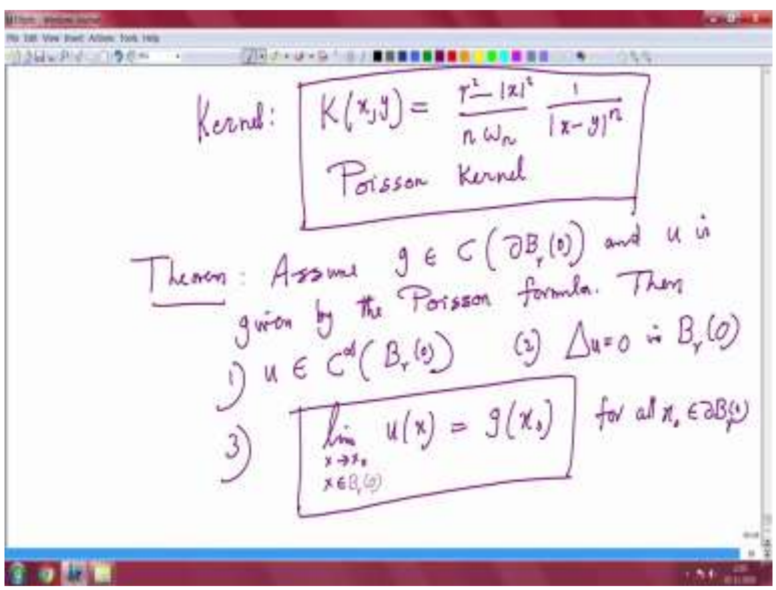

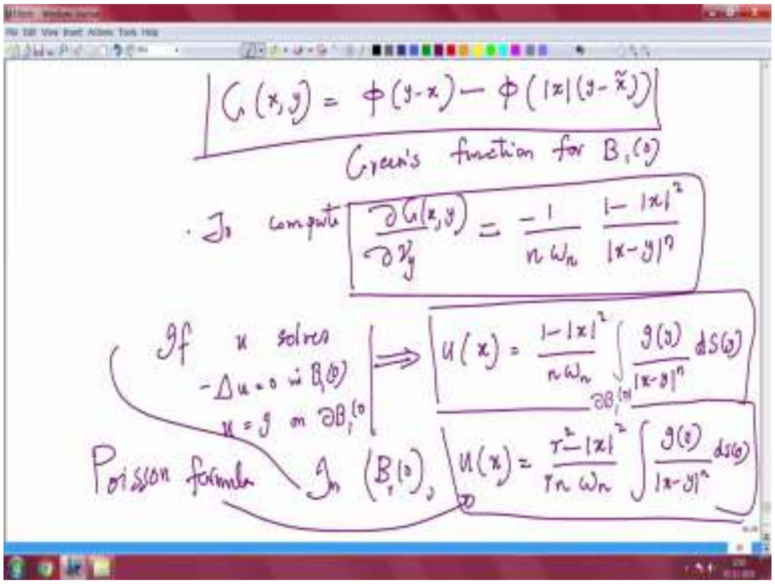

So you can do the same computation in if you are considering Br of 0 a ball of radius 0 you get your G x y, so you can introduce your G x y maybe. So let me go here, so we can rather in Br of 0 if you work in Br of 0 your u x will be, so you have r square minus mod of x square by rn omega n in the integrate g of y modulus of x minus y power n d S of y this is the Poisson formula.

So that is the Poisson in the other one I thing is called the so you do not need this so you have the Kernel in general, Kernel K of x y this equal to y be. So these are all immediate things r square minus mod x square by rn omega n and 1 minus mod x minus y power n. So you have that one. So that is the Poisson Kernel. So you have your Poisson Kernel. So you have the theorem, same theorem which I am not going to prove, in this case you do not need to assume G is bounded because g is defined on a bounded boundary.

It is a boundary, compact boundary so in a compact boundary with a continuous function gives you the thing, assume g is continuous on the boundary only B1 or Br 0 and u is define is given by the Poisson formula is given by the Poisson formula. Then 1 you have u is in C infinity of Br of 0, 2 Laplacian of u equal to 0 in Br of 0 in Br of 0 and 3 in the sense of limit, limit again u is define only inside the ball so x tends to x naught where x is inside the ball.

That is why you have to look at it but x naught on the boundary of u of x is equal to prove is similar for all x naught is in the boundary Br of 0. So you see you have so you understand that limit has boundary values as I limiting values and inside u given by the Poisson formula.

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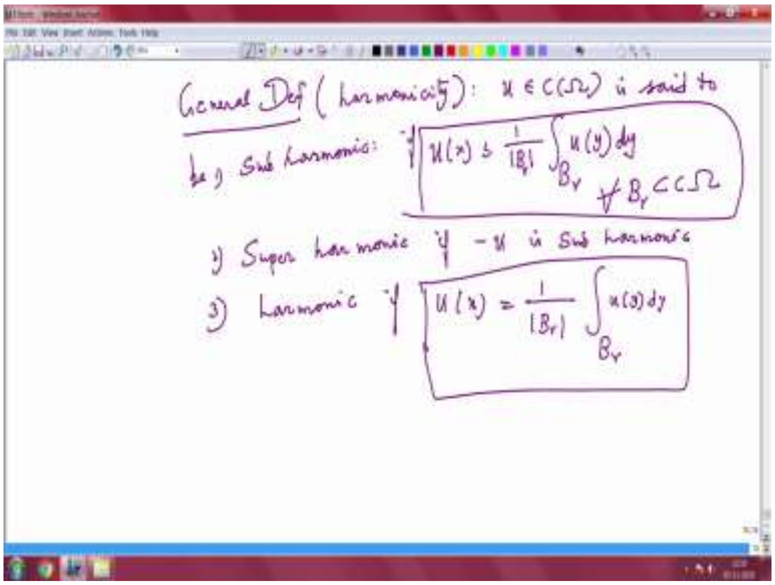

I think I more or less completed the topics which I planned to do to it in this course and anything further to do it we need more required more time and this particular first course does not permit me with enough time. But then I have promised one thing in the beginning so let me do it that one. If you recall one my earlier lecture do you have define the mean value property to recall that.

So if u is harmonic in the sense of whatever define Laplace in u equal to 0 then u satisfies the mean value property. And conversely we proved that if u satisfies the mean value property in a domain omega and u is c2 function then u then u is harmonic so for this c2 function level you have seen that the harmonicity in the definition and the mean value property are equivalent. So but what we have commented that time to define the mean value property you do not need C2.

So what happens in that case, so does the in the commented remarked that just starting with continuity you can define the mean value property and then you can actually prove it at time there is some prove based on what is called mollifiers and extra that now using our Poisson

Kernel I can have a look at different prove of that. So I can have a new general definition of harmonicity hence harmonicity you seeing the mean value property.

So let me do the general definition. Definition so I will call this is a general definition or a new definition of harmonicity. And doing this because this is where the starting point going to be the starting point in the next PDE-2 course where we introduce the Kernel's method u belongs to C omega you see I am not taking C is a u thing is said to be harmonic in the earlier definition of harmonicity u.

So you need twice differentiability of course if you have twice differentiability it will coincide with this definition that is why its call on general it is not different is said to be sub harmonic I use the inequality mean value inequality if u x less than or equal to 1 by modulus of B integral of may be any ball or Br integral of Br u of y u of y dy this is true for all Br compactly contained in omega.

So that is general harmonicity. 1 is said to be equal 2 super harmonic if minus u is sub harmonic and the reverse inequality you will get it that is all. And 3 so I have new definition of harmonicity via mean value property for continuous functions if u x equal to 1 over Br integral of Br u of y. So I am able to define my harmonic definition using the mean value property. So what is the advantage of this? If you look at my earlier results, if you look at my.

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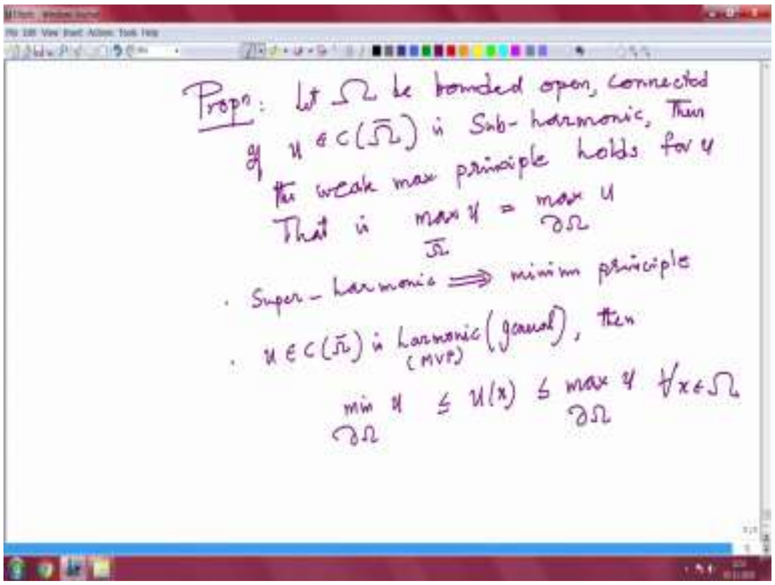

So let me go to the next slide if you recall my earlier results of weak maximum principle we actually used mean value property. So I do not prove that here now so you can follow the similar prove. If you look at the harmonic function satisfy the maximum principles or weak maximum principles if you look at it the main ingredients were mean value property. So for a continuous function now we have a mean value property.

So you have the maximum principle so let me write it as a proposition now. I am not going to prove because it is a similar thing proposition let omega be bounded open connected if u belongs to C omega bar we take up to the boundary that we have needed is sub harmonic when the weak maximum principle is true then the similar prove then the weak maximum principle is true holds for u that is maximum, maximum axis because u is continuous up to the boundary is obtain up the boundary.

You see d omega and similarly super harmonic minimum principle keep this in mind again and again that is for continuous function domain minimum principle. And if harmonic u belongs to C omega bar is harmonic in the general sense harmonic everything general I am just repeating general then you have the estimates from both side. I am writing both minimum and maximum principle is true so u harmonic means mean value property.

Please understand that now the harmonic property is mean value property then minimum of u over the boundary is less or equal to u of x less than or equal to maximum of u on the boundary for x in omega. So you have this maximum principle.

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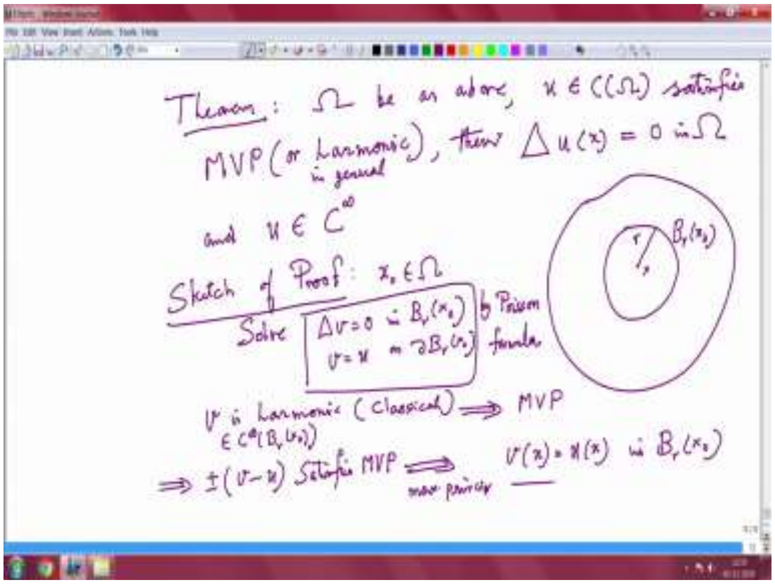

Now with that you have our final result which you are I do not prove it but you can quickly see that one. So you have your theorem omega be as above and u belongs to C omega now omega satisfies mean value property that means harmonicity in it. Mean value property let so what do we do that one mean value property or harmonic or harmonic in general sense harmonic both are same.

Harmonic then it is harmonic in the sense then Laplacian of u x equal to 0 this harmonic in general sense you see with mean value property in omega that mean and u is C infinity. This is one I skip it so its satisfies, so start with the continuous function and then you define the mean value property with holds for mean value property then u is C infinity and you get Laplacian of u equal to 0.

So you let me sketch the proof to complete. So you can work it out if anything is missing you choose a point x naught in omega. So you have domain so we choose a ball whatever be the ball inside Br of x naught. So you have that ball so what we have seen if your Poisson formula to define a inside functions so you recall that Poisson formula for a ball. Recall Poisson formula what does they saying? Using the boundaries on the, boundary of the ball you can define the harmonic function inside using just a boundary value.

So in other words you can solve the harmonic, classical harmonic function inside the ball using the continuous values of that one. So you solve you have to solve it. So that is the measure thing so solve Laplacian of v equal to 0 in Br of x x naught and v equal to u on d Br 0 x by Poisson formula so that is what we prove it now such a unique solution x is Poisson formula which we already done now in the earlier class.

So v is harmonic in the classical sense, so v is harmonic classical once it is harmonic in the classical sense in fact v is c infinity. That is what we proved v is C infinity we prove that Br of x naught this v but u is continuous and then once it is harmonic and C infinity it gives you mean value property. So that implies v minus u or plus or minus both you can apply v minus u satisfies mean value property. Now apply for that weak maximum principle that is what we just now proved there.

Principle implies and v equal to u on the boundary so the you can apply this v minus u the mean value property but v equal to u on the boundary these are all working we are applying the mean value properties in the Br of x naught because you can have that thing. So we are working only Br of x naught so that means you have the boundaries values 0 v minus u equal to B equal to u on the boundary the minimum is 0 and x1 is 0 so that implies v x equal to u x in Br of x naught.

That is the prove because now hence u is C infinity and satisfies the mean value theorem in that way Laplacian of u x equal to 0. So I think with this we will end this course on Laplacian you will see more on the you will see more results on the wave equations and head equations. So this is a very very little as far as Laplacian is itself is concerned. So we are not got into potential theory yet we have only given some basics of Laplacian mainly we about the mean value property and the maximum principle which gives you and the Green's function definition.

So you have the whole idea is to understand the fundamental solution and the Green's representation formula. And how the Green's representation formula is used to obtain the solution in at this in two cases with this for this Laplacian problem I will stop here. Thank you.