First Course on Partial Defferential Equations – 1
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Lecture 26
Laplace and Poisson equations - 9

Good morning and welcome back to the lecture. So we will continue what we, where we have stopped. We have introduced the Green's function in general and that we have derived the Green's representation formula if there is a solution.

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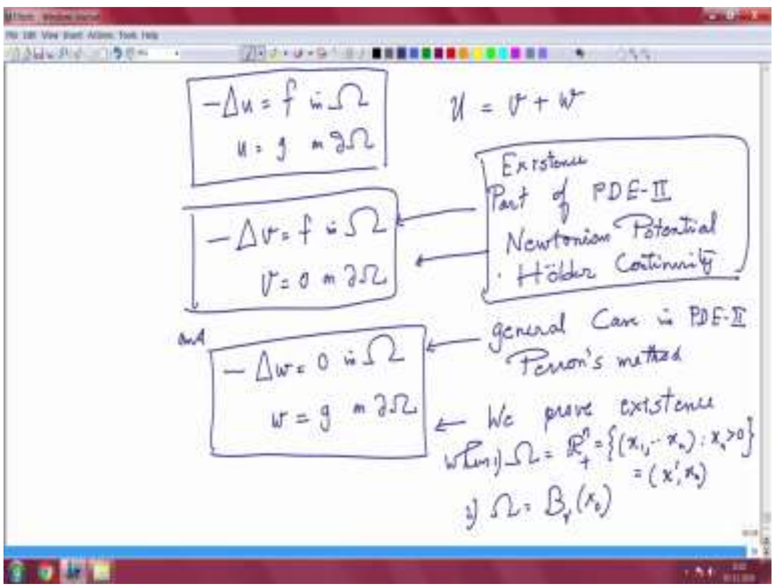

So let me start with some comments. We want to solve this problem our aim is to solve this problem f study the existence in omega so you will see how difficult even that such a simple operator on d omega. Of course we are not going to do this in this course at all completely in the generality, so let me complete what am I going to do it. So this solution you can write it as u equal to v plus w and then what is v, v is Laplacian of v is equal to v f.

So v had separating your internal source and term and then external by linearity you can do it. And minus Laplacian of w equal to 0 in omega. And w equal to g on d omega this is done with linearity. So we will do this one. So what are we going to do? We are not going to do this problem, this is part of PDE-2, existence part of PDE-2 we will study there. And this is called what we called the Newtonian potential.

And you need to introduce what a called a Holder continuity all these we will do it in the part-2 that is the main part remains which we will do it in the. For this case also general case in PDE-2 general case, general domain in PDE-2. That is, you we prove existence what is called Perron's method we have to introduce mode general super and sub solutions and there are some difficulties you have to introduce the regularity of the domain and all that.

But what we do it now in the remaining lectures of this elliptic this Poisson equation we prove existence we prove now we existence when omega the upper half space that is equal to Rn plus what is Rn plus? Set of all x1 et cetera xn which we also write it as x prime x1 xn for the convenience where xn is positive at your Rn plus that is upper half plain and we also prove this is a case 1 omega and 2 you prove in omega is ball of radius r x naught. So these two cases we are going to prove it the existence theorems.

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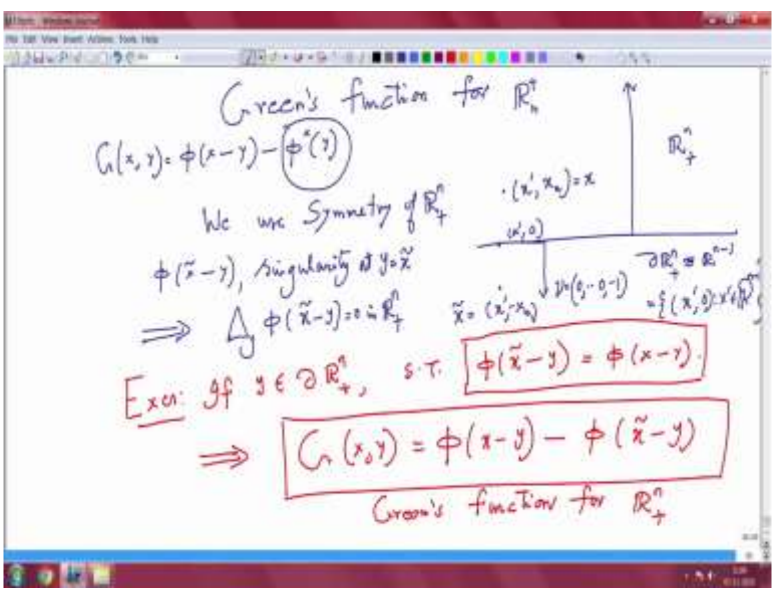

So that is what our plan now. So we will use the Green's function for Rn plus. So Rn plus is in two dimension you can precisely write it this open set. This is Rn plus, this is your Rn plus this is your boundary of Rn plus. Keep all this in mind Rn plus you can view it as a Rn minus 1 you can view whatever you like it or you can view it as x prime 0. So these are all identifications with x prime in Rn minus 1.

Sorry the last it is okay. So and its normal derivative is this one you see the normal nu is equal to 0 et cetera minus 1 you need all this to compute on the boundary with all that we use the so you want to introduce so you already have phi x minus y. Let me recall the Green's function G x minus y is equal to minus phi x of y. So you want to find this should be harmonic. So we use symmetry we use symmetry of Rn plus.

So if you take a point here, so let me extend it, so if you take a point x prime xn here. If you look at its symmetric point here, this will be x prime minus xn and when you come here it will be x prime 0 it will comes 0. So you take your same phi, so this is your x if you have it. And I call this to be x tilde. So if you take so this is symmetry if I am using if I use phi of x y is, so y is on the x and y are different. G x y is different with x and y.

If I take x here and if I take phi of x minus y and if I consider this domain, upper half plain, in the upper half plain, it has a singularity, but if I use x tilde is my singularity in the upper half plain it will be. So look at here this function phi x minus y x tilde minus y, so you have to use that one. y minus x tilde both are fine. So if you use x tilde minus y satisfies, the singularity is at y equal to singularity, treating it as a function of y, singularity at y equal to x tilde.

That implies immediately Laplacian with respect to y phi of x tilde minus y whether you use x tilde minus y or y minus x tilde all that does not matter because phi is modulus so its defined in terms of the modulus so it is fine. This is equal to 0 in Rn plus. So you immediately go the only problem which you have to verify so this is a small exercise for you to verify. Exercise you want to satisfies the condition when y is on the boundary.

If y is on axis internal x tilde is internal, if y is in the boundary show that this trivial just write down that is all but is better for the first time if you are learning is better to 2 it 1 then phi of you have to careful x tilde is the thing y the yn term means 0. This will be same as phi of x minus y.

So you have your already, so is the just use the symmetry to do this. So this implies your Green's function for the upper half plain is equal to phi into x minus y minus phi of x tilde minus y.

Where x tilde is the reflection point. So you see Green's function for Rn plus 1 that is what. So you have immediately your Green's function.

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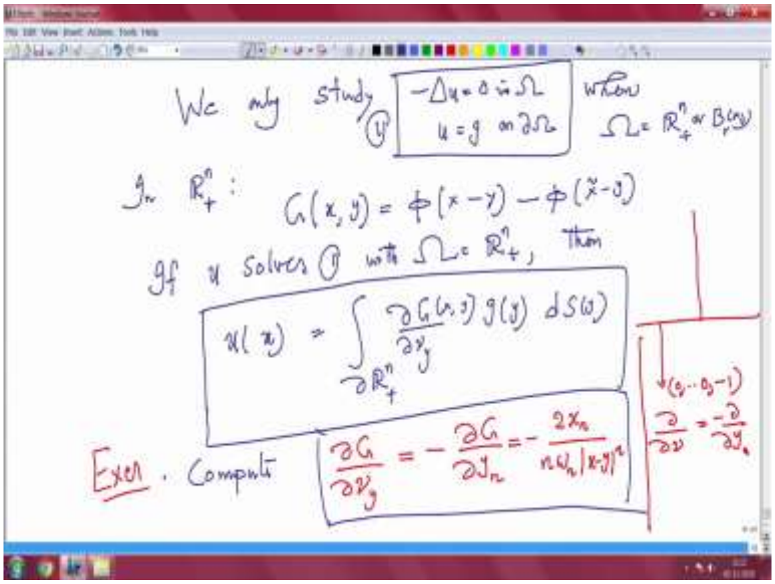

So this implies immediately you can write down the Green's formula. So your u x, so if u yes so let me before that let me show so I am only go to study this problem. So we only study this problem, I told you what we are going to we only study in this course minus Laplacian u equal to 0 in omega and u equal to 0 g this is the problem which we are going to study, when only two cases when omega equal to Rn plus o Br of x naught.

So first we are considering Rn plus, so you have other so in Rn plus, so you first do it Rn plus and for Br of x naught we will do it in the next class. So we will give more details here the similar things can be computed by the students when they study. In Rn plus the Green's function G x y is already done, we are repeating actually. But let me do it minus phi x tilde minus y we already done is not necessary.

So if u solves 1, so I call this to be 1 solves 1 with omega equal to Rn plus we are doing it then the solves smoothly then u x equal to this is the Green's formula integral G of x y is this one. You have to compute that one, you have to do all that G of x y that would not term on come f is

0 because if we are not considering so we do not have anything. So u of x y is equal to f is 0 in this case that is what we are study the other thing is more this is in boundary of Rn plus d G by d nu y and now you know how to compute d nu.

Because it is d phi both are d phi only thing you have to take the normal correctly. So you compute the normal derivative. So I will leave that as an exercise g of y d S y I am giving a more computations. So you have this one so your job is to compute, so I want you to compute so I will leave it as a this is exercise which we are keep on doing it. Now you are doing it compute d G d nu y when you compute that so I will do the computational result.

You know this is already equal to d G by d y n because the normal here is 0 et cetera 0 minus 1. So, d by d nu the derivative will be minus d by d y n the nth derivative. So it is easy to do that one so you compute this and you get what you get here is this is an important formula which you know minus 2 x n y n omega n mod x minus y power n. So this is important, so keep this formula in your mind. So you have can precisely calculate your d G by d nu.

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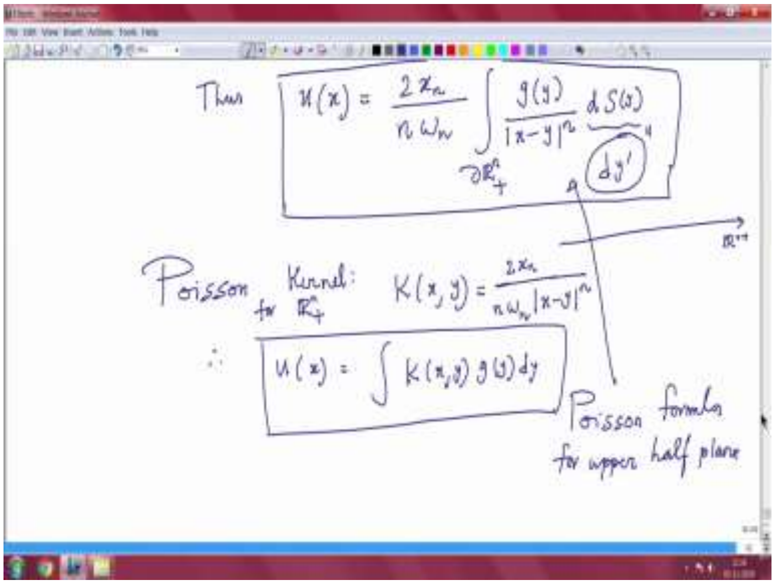

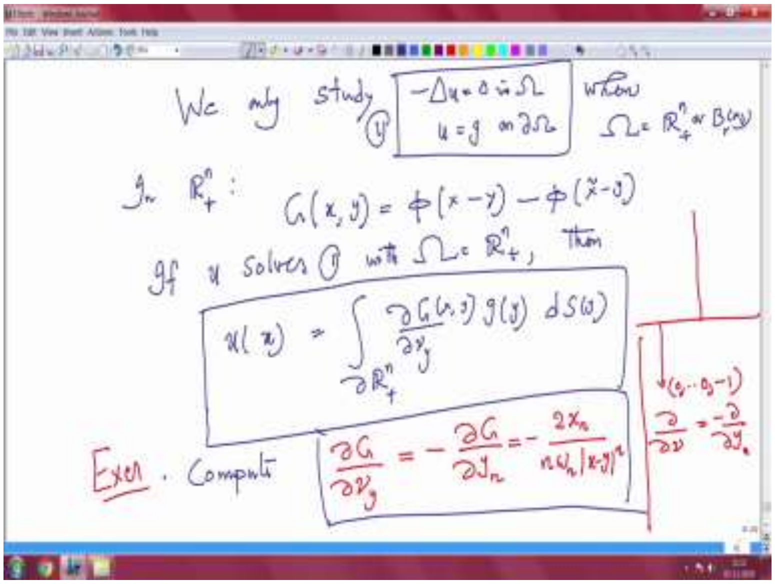

Therefore, thus u x equal to the formula if you solve u x equal to so let me so thus exactly what u x if u x solves this equation 1 then if u thus u x equal to 2 x n. So you have a minus sign in there 2 x n there is minus sign yeah there is a minus here I did write that. So that minus minus gets cancel you will have n omega n integral of g of y by mod x minus y this is on the boundary. So x n can take it out that is why it is coming.

This is in Rn minus 1, so x n not coming to picture. The Rn plus so d S y keep that in this case your d S y is nothing but d y prime keep that in mind because it is in this Rn minus 1 direction. So your d S y is same (())(15:53) because it is already flatten domain. So this is so you can easily integrate. So you have your formula but let me use a notation d S y because you cannot do it this all the time d y prime.

So we introduce what is called Poisson Kernel, therefore definition Poisson Kernel introduce K x y is equal to 2 minus 2 x n whatever it is. 2 x n I am using, write down everything for this case, later you will see x minus y power n, this is your Poisson Kernel, that is therefore your u x is a compact notation I am giving integral of K of x y g of y with a Kernel integral form, you see this is very important to understand, getting an integral formula is precisely when you have it a function, these are kind of mod generalized agents.

When you have a derivative and if you want to solve the inverse problem d G by d x equal to f you write g is equal to integral. Similarly, u solves as a kind of operator Laplacian then you are trying to write u in the integral form that is what. So you have to see that these are all kind of fundamental results. If you have one dimension, if you want to solve d G by d x equal to f how do you write?

Then integration theories developed and you write G is equal to as in integral of f of x d x. Here you have an operator L, which is Laplacian in this case and you have Laplacian of u equal to something the including the boundary condition. So you are trying invert that operator Laplacian in some sense you are writing your minus Laplacian inverse. Can it write that into minus Laplacian of inverse, can it you have an integral representation?

And what these results tells you is that you can indeed to do that, but not as symbol but with the Kernel. That is the importance of this, you have to see all that result in that context, in that generalization of all that. So these are all general comment. So you have this Kernel and then this is called the Poisson formula for upper half plain. This is also Poisson Kernel for Rn plus, everything in that and then this is Poisson formula for upper half plain.

But as I said this you are getting, assuming that u is a solution, but now we want to prove that this u is interior solution. What this integral is meaningful and does this all everything? So let me try to do that one and state a theorem may be in this thing and try to prove that theorem in the next class if we get time we will do here.

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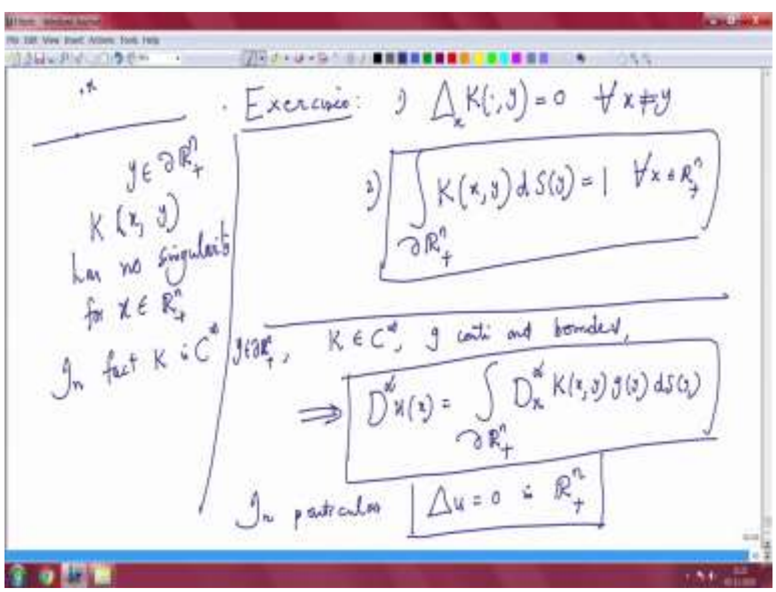

Observe now so Kernel, so you can prove some results which I will not prove everything but I would like first time is would like you to do this thing. So I will leave it as some exercises here. It maybe some computations but first time you do it 1 yeah before that you will say that, if y is in Rn plus on the boundary before this exercise let me have this comment.

If y is in Rn plus, so you see here. The singularity of K is when x equal to y. If x is here x minus, so you y is only the thing, K has now singularity, K x y has now singularity for x in Rn it is in the upper half pain that is the open set that is fine. In fact, K is C infinity for each x so you can see that K is a C infinity function which is thing for y in Rn K is a C infinity function K is the infinity. So there is no singularity at all here.

So the exercises which you can prove with that notion Laplacian of x of K x y K dot of y, y is on the boundary that in fact this you can prove it 0 for it is x not equal to y not even have to be boundary why I say that when y is on the boundary there is no singularity for x in inside but to understand how this singularity is B x.

So you because K is coming exactly from the Green’s function and Green’s function coming from the fundamental solution so all these are thing. What the second thing which you may need to do some work integrate this one on the boundary Rn plus you integrate K of x y d S y is equal to 1 for all x. So you have a same thing is very crucial to prove this one. All these will be used and another thing which you have to understand since K is C infinity.

So if K is C infinity so let me not write that is K is C infinity for y in Rn y on the boundary. Yeah understand this Kernel very nicely K is C infinity and assuming g is bounded g say continuous so assume is a g is continuous and bounded you can actually show that u is differentiable so these things together tells you can take differentiation in say so you can compute will give you, you can take as differentiation inside.

Because it is C infinity and you can take do this integral over boundary of Rn plus D power alpha x of K you see K of x y g of y d S y. So there is no problem in taking inside the intake with all this smoothness and all this bounded you can. In particular Laplacian of u equal to 0 that is what it shows. Laplacian of u equal to 0 in Rn plus this is what you want to solve it. So only this boundary condition you have to solve it.

So one part we already got of the solution you see. So this is in Rn plus so that immediately you take alpha D square u by D x I square and do that alpha equal to 1, 0, 0 in some of it, you get Laplacian K. Because Laplacian of u inside you get Laplacian so there is no problem of taking the Laplacian there. Since K is infinity and you are working with boundary. So you can prove all this thing. So you have your Laplacian of the first part of the result is proved. So what you had done, so that implies the following theorem.

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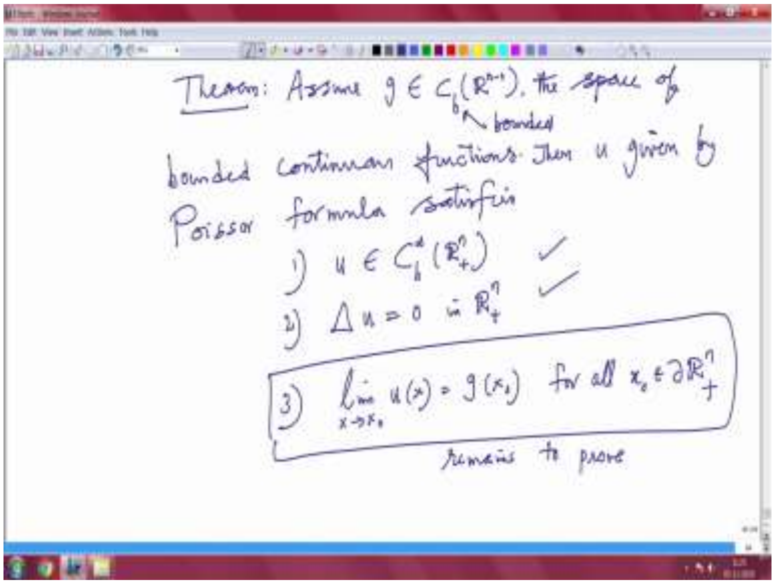

We will start the prove may be we will continue in the thing. So I will try to give the idea so let me write the theorem, may be prove I may not able to compute, so I will stating so assume g is in C b of Rn minus 1, Rn minus 1 this means it is bounded. So you have a bounded continuous function so this is the space of bounded continuous function. Then you defined by the Poisson formula define by or rather given by u given by Poisson formula satisfies 1 u is also a bounded continuous function which we already seen essentially.

We know do that you can do that. 2 Laplacian of u equal to 0 in Rn plus and what we have not proved, so this is proved this is done and what we remains to prove it satisfies the boundary condition in the sense of limit. You have to understand because u boundary you have to understand and g is just continuity. So you are not writing g because u is defined only on the interior, u is not defined on the boundary.

So you have to be careful the Poisson formula defines u in the interior, in the Rn plus, not on the boundary. So how do you interpret the boundary value? So you interpret the boundary value of u as a limit going to 0 and that is what we need to prove it. Limit of x tends to x naught where x naught equal to of u of x is equal to g x naught, for all x naught in Rn boundary of Rn plus.

So what remains to prove is only remains to prove this fact, remains to prove. So you have one existence theorem now remains to prove which we will do it in the next class and then we will do the Green's function identity Green's function introduction of the Green's function given in the context of a ball. Thank you. So we will continue in the next lecture.