First Course on Partial Defferential Equations – 1
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Lecture 25
Laplace and Poisson equations - 8

Good morning and welcome back to the final part of my elliptic equation, namely the Laplace and Poisson equation. In the next couples of hours we will be doing Green’s function. We will introduce Green's function, Green’s formula and then Poisson Kernel and Poisson, Kernel and Poisson formula in the case of upper half plain and the balls, which we will use to establish some existence theorems.

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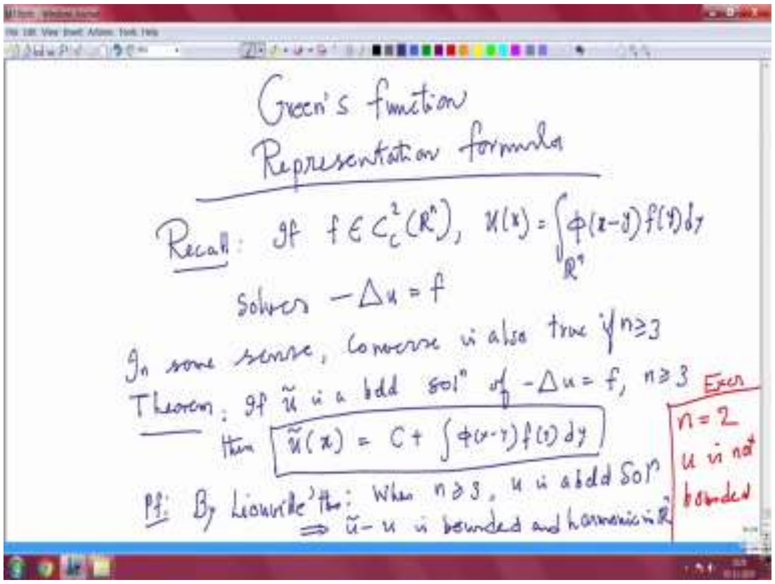

So we will, the first thing is that Green’s formula, Green’s theorem, Green’s function and representation formula. So let us give you, you have already seen some representation let me recall a result. Which we have done few lectures back. If f is in CC 2 of Rn you already seen that the u x equal to integral of phi is the fundamental solution f of y dy in Rn solves minus Laplacian u equal to f. This we are given a complete prove that is the beginning of the elliptic equations part in some sense converse is also true for bounded solutions.

Some sense converse is also true that is true if n is greater than or equal to 3. So you have this theorem that is the actually essay to prove. Theorem, if u tilde is a bounded solution bounded solution of minus Laplacian u equal to f then where n greater than or equal to 3 we are assuming I will tell you the reason but that is an exercise which you can prove it. A bounded solution of minus Laplacian u equal to f then u tilde has a representation.

I am telling you u tilde is equal to some constant plus u that means a phi of x minus y f of y dy you see that means every bounded solution of this equation as this form. The prove is trivial use due to prove is by Liouvilli's theorem, which we have seen already, Liouvilli's theorem. Because when n greater than equal to 3 u is a bounded solution. Yeah this is only when, so you are exercise is n equal to 2 u is not exercise u n u is not bounded.

You have to prove this actually of course you have one information the fundamental solution is bounded as mod x tends to infinity when n greater than or equal to 3 but then n equal to 2 the fundamental solution is not bounded. But a fundamental solution is not bounded is only one reason but that does not mean that it is integrally a bounded. So the exercise is you have to still use, you have to use the fact the fundamental solution is not bounded had mod x tends to infinity.

But then fundamental solutions are bounded function when mod x the singularity is at the origin this is I am talking about at infinity. So that is the fact which we are using it. So once this is there this implies u tilde minus u is bounded and harmonic in harmonic in Rn. So you use this Liouvilli's theorem to show u tilde minus u is a constant that is it. So what this case you have you already seen a kind of fundamental solution.

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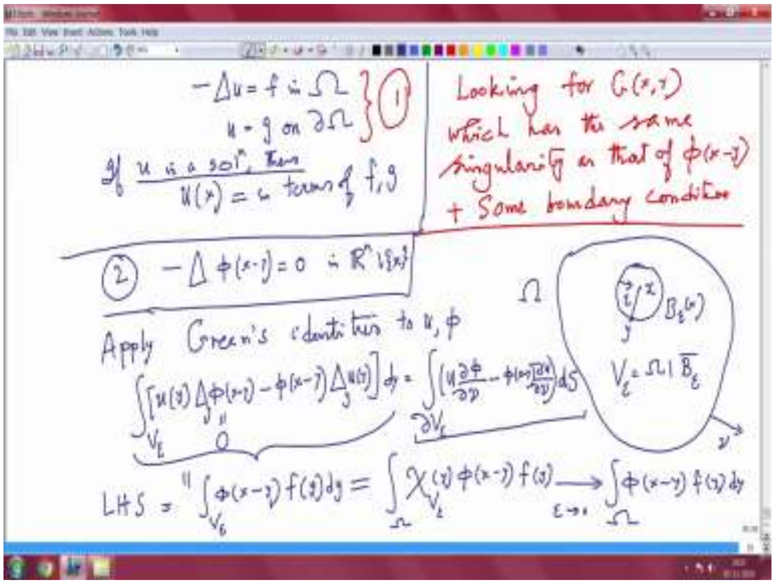

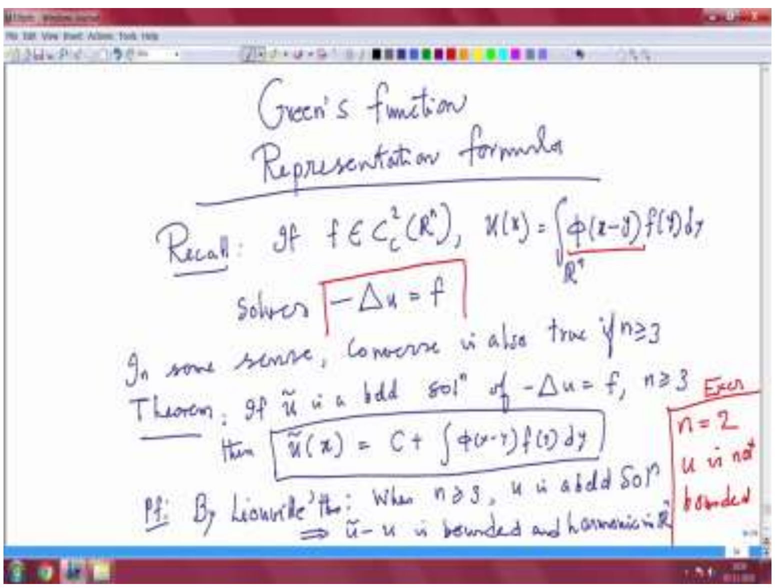

So now our aim is we want to study this equation minus Laplacian of u equal to f and in omega. So you are studying this in domains now in v bounded domain u equal to g on d omega. So you want to get a representation of your solution so you want to get if u is a solution that is what we call these are all some a priori information solution then you want to get right u in terms of f and g.

That is what we are looking at a representation but for that if you look at earlier, so what you are precisely looking if you look at the previous case this fundamental solution phi you look at the fundamental solution phi which has a singularity and that singularity the information stored in that singularity what eventually use to prove this equation if you recall.

So singularity will help you to get the equations, so you need a function, so you were looking for G at two variable function x and y, which has a same singularity, which has the same singularity as that of phi x minus y as that of phi x minus phi at then you have working in bounded domain and you also want to write the boundary condition. So u plus some boundary condition on g you will come back to this again, some boundary condition will come that one.

So let us look at that one so we try to prove it that one. So we have two equations one is equation is this one, so you always call this is equation one and you also recall the equation corresponding to Laplacian you have Laplacian phi x minus y of x minus y this is equal to 0. These are you have in this is true in Rn minus x keep that use these two factors. So you have other equation this is the equation we call it 2, we are using this.

Now we are going to apply Green's identities. So v has not singularity, so you choose y so y is in x and y are in omega. So you have a domain omega this is what domain omega and you have a point here x here and then it has singularity there. So I look at it this can be 0 with respect to x or with respect to y. So let me use with respect to y here because I am fixing x here. It does not matter which one fix it I have to fix one.

So I have a point so there is a singularity at that point. So I put small ball here and that is of radius epsilon I call this to be B epsilon. And this inside domain is your call V epsilon, V epsilon is nothing but omega minus B epsilon bar that is outside domain. So apply Green's identity apply here u of y doing it in v epsilon u of y Laplacian of phi of everything with respect to y x minus y minus phi of x minus y.

I am applying Green's identity to u and phi and Laplacian of y with respect u of y so this is integrator and then you have this is equal to by Green's identity with dv epsilon not d omega because your domain is that one. So that will give you u d phi by d nu of course nu is external outward union remainder, so the nu in this direction here for this then u will be in this direction. So to be carefully choose what is that nu.

That is all you have to, so the y is here when you are integrating. That part this portion is fine you have phi of course we have x minus y du by dv this is with respect to ds. Now we want to understand this look at this first term that we will come later this is equal to 0 because phi has singularity on this 1. So this equation and Laplacian y over u y is f. So this whole thing will be equal to this thing so left hand side will be integral over minus phi of x minus y f of y dy.

This is over V epsilon over dy. But this phi has a singularity but that singularity is but it is locally integrable you can apply some Lebesgue dominator convergent some theorem or not but if you want you a yeah f is replaced already by minus Laplacian if f so there is no psi here. So this if want it but there is a trivial thing from you can write it here if you want this you can write iV epsilon of y phi of x minus y f of y dy.

You have little smoothness we are already assuming so we are assuming that u is a solution of that, that we already assume. U is a smooth solution c2 solution that is what we will assume. So assume that it has a solution 1 has a solution smooth solution you are dying to derive this one. And you can apply because this will converge point wise to this quantity and you can bounded by an integrable function because phi is intergrable there omega this will go to epsilon tends to 0 integral over omega phi of x minus y f of y dy.

So that is a the left hand side is taken care of now look at the right hand side, right hand side it is dv epsilon dv epsilon has two boundaries one d omega, one is d b epsilon. But dealing with I am telling you what have to do either you have to be make sure that your normal derivative is this one. So we have four term two terms on the there are two terms here but the d v epsilon has two boundaries so totally there are four terms two terms on d omega that we will keep as it is you do not have to do anything because there is no epsilon.

But then there are two terms on d b epsilon we want to analyze what will happen to the thing which you will look that you will.

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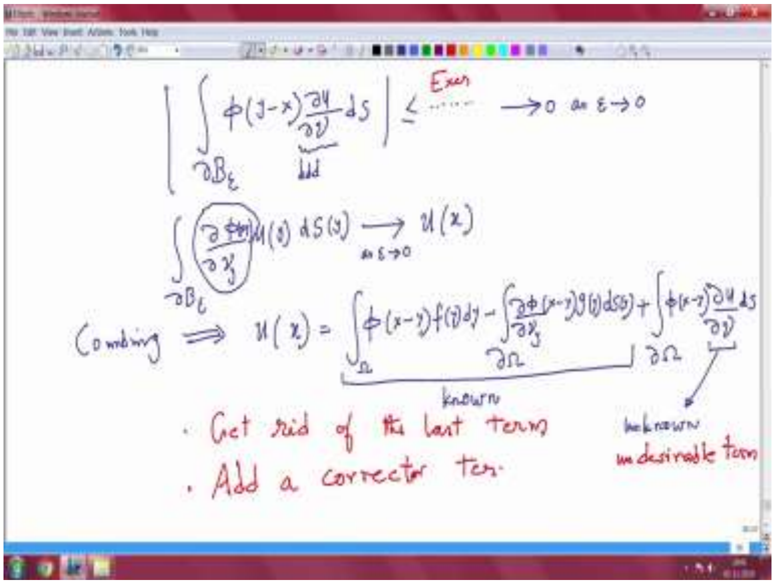

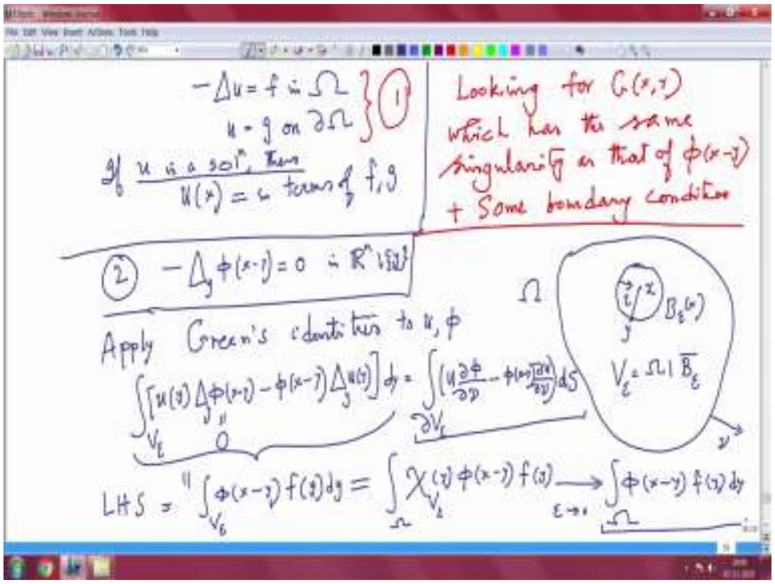

So let us do a take care of one by one, one term so one term let me write here. What you will get is phi of I am taking one term phi minus x du by d nu this is over d phi epsilon sorry this will be on d B epsilon and d s. Now look at this term this is a bounded quantity. It is a smooth thing and this we have already done in the beginning of the class and this has an estimate you can use this estimate on this small thing and you know all this measure is 0 actually you can precisely right estimate.

I will leave it as an exercise here. What x i we will come and then you can see that, that goes to 0 as epsilon goes to 0. So one epsilon term will go to 0. What about the other term other term is also you have worked out in the beginning? So once more you work out d phi by d nu this is all d B epsilon and then you have your u now. This is with respect to y you are doing it x minus y keep the parameters correctly. And this is u of y ds of y.

And if you recall and now we all would have gone through already my previous lecture you can see that this is gives you the basically the measure of that. So that exactly will go to the average and this will go this exactly will be the average of u y as epsilon goes to 0. This you get it to u x that is what the singularity which you have already done. This will go to zero as epsilon goes to 0. So combine your thing you have the next one.

So you see the left hand side goes to this quantity, you keep it as it is and then you have two terms on d omega that also take to the left hand side and then one term on d b epsilon goes to 0. The other term go this term goes to u of x. So we combine all these if you get a so combining will give you u of x equal to integral f of let me write phi first phi of x minus y f of y dy this is on omega minus you get d omega part phi of x minus d phi d nu with respect to y because we are integrating y g of y ds.

And there is one more boundary term and that is integral of d omega and phi x minus y and du d. here u is replace by g ds. Now look at these terms, so you have representation what is this representation is good enough phi is not f is not g is no, so you have to represent solution in this known. On the other hand, look at this, this is unknown and only it is an unknown unknown in the normal derivative of u on the boundary you can never impose that condition.

Because it is a boundary value problem this is a boundary value problem with u is already prescribe you cannot prescribe any more condition on du by d nu. So this a really undesirable term. So you have to the issue is that get rid of that, get rid of the last term. And how do you get rid of it there are good motivation again to do that how if you look at how did we arrive at that this formula you have applied your mean Green’s identities for u in phi both are Laplacian with phi having a singularity.

That singularity of phi produced u but now if you look for us you have to add corrected term it to get the get rid of the last term. You add a corrected term which should not have any singularity but it should have the value exactly as phi. So that motivates us to add a corrected term so we will do that. So add a corrected term that is what we are doing it.

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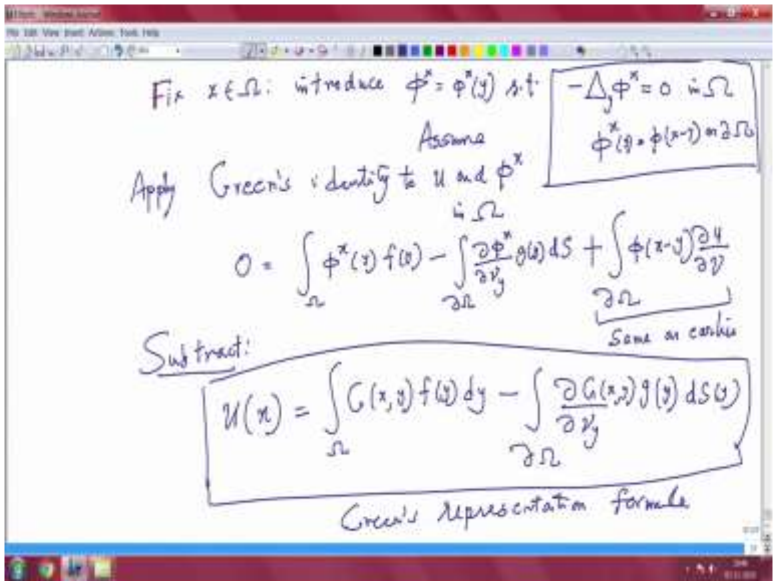

So we will go to that one so fix the x. so you fix the x so how do you add your corrected terms to do that fix the x in omega so you have to think x in omega introduce phi x so you do not need any singularity what we are introducing. So introduce phi x phi x is equal to phi x of y is not power it is just a notation to represent my x. Such that Laplacian phi x of course with respect to y that should be 0.

So you need this is in omega and then phi x equal to y equal to phi x of y minus y on d omega. Of course this availability a question because we do not know this all availability at so we are assuming, so this is introduce assume it such as solution x is. Once you do that now you apply again Green's identity apply Green's identity to not think to u n phi x. The only difference from earlier thing is that phi had a singularity so you have to find a ball around it.

Here you do not have to do it, this you have to do in omega because there is no singularity so you can straight away apply the inequality and Laplacian phi is equal to 0 follow there is not even need to follow it just now thing in equality so you get this equation 0 is equal to phi of y f of y dy this is in omega. Same term with phi of x minus y replaced by thing replaced by d phi x by d nu y this is over d boundary of omega g of y this is ds.

And then you will have exactly the term the if you look at it here you will have this boundary term instead of u of x minus y you will have phi x of y but phi of x of y is same as phi of x minus y that is what we have used and d nu by d. This exact the last term is same as earlier. Now subtract once you subtract you get u x u x equal to f y is common early a thing you have phi of x minus y.

So you are subtracting and introduce a new notation I will say so it will be omega you will have G of x y f of y dy and minus integral of d omega d G by d nu y of x y, this is not x minus y x y. I will tell you what is that one, g of y ds of y this is what is called the Green's formula, Green's representation formula. So please careful that since you get it we are not solved with in equation what we have done is that assuming that u is a solution and then you satisfies this formula. That is all we have done it nothing more than that.

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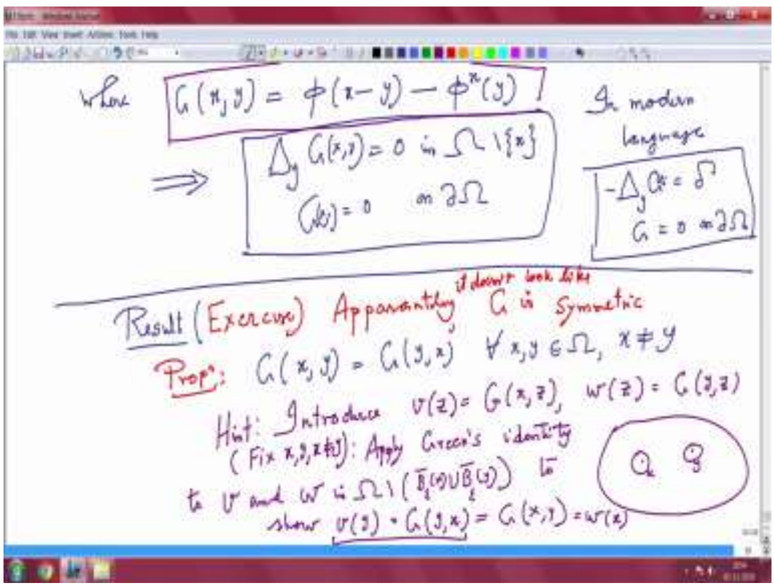

So what is G, where G of x y is equal to phi of x minus y minus phi x of y. So what are its property indeed both phi and phi x is Laplace psi this harmonic everywhere this is harmonic except that y equal to x. And now phi and phi x is the same as the boundary this implies Laplacian of y of G x y is equal to 0. So it has a same singularity as that phi omega minus x. And then you also have G is equal to 0 on d omega x is fixed, so G x dot equal to 0.

In modern language it is all that measure valued function minus Laplacian of G with respect to y equal to derive delta with G equal to 0 on d omega what e G x x dot you see this the modern language. Which we will do that. So that is what the Green's function. We will see more and more about the Green's function. So I want to make a remark this is one important one another important result I would like you to see, I will give you the hint here but try to prove it.

So it is an exercise for you. Apparently G is not symmetric apparently G is does not look no let me not write that is apparently G does not look G is apparently it does not look like it does not look like G is symmetric because we careful about that G is symmetric. But the result tells you the proposition which you can prove I would like you to prove it proposition tells you that G x y equal to G y x for all x y in omega with x not equal to y.

You see the way G is define you see the way G is define is like that. And there is no does not look like there is a symmetric of course this is radial, so there is no issue but this does not look like this. but it is true. So let me give you the hint which you can use it to prove that one, you introduce v z equal to introduce v z equal to G x x z and w z is equal to G y z fix x and y, fix x y everything in omega x naught equal to y and define that.

And of course so this is defined as so you have a domain omega, you have two points x and y. Now take small balls here B epsilon of x B epsilon y. So apply Green's identity to identity to v and w in this domain omega minus B epsilon bar of x union B epsilon of y bar to show v to show what you want to do, to show v of y is equal to v of y is equal to show that v of y is equal to this is actually G x where this is equal to you show that this is equal to G y x.

But then v of y is equal to G of y which will get in G x y you show this and then you can also show that w z is equal to g x y w x y this is up so equal to w x. So you can get all your results. So we will stop this lecture here. So what we are going to do is that you are going to construct Green's function for two special cases, so the last remark in this lecture is that though once you so these are the steps once you get a Green's function you have a Green's formula.

So even if you get a Green's function you have a Green's formula does not mean that you can prove it. The x says u is in detail solution which you have to do it. So getting a Green's function is one criteria and then you have to show that the u define by that formula well defined and satisfies the equation and solve the boundary value problem that is not an easy thing.

Secondly obtain G in generally is as difficult as solving our problem of Laplacian of u given Pfaffian differential equation because to solve G you have to solve this phi of y. You have to solve phi x but solving phi x is again a boundary value problem but then it is independent of given data f and G that is only thing difference from our problem the data f and G are not coming. This a kind of universal for f and G.

It depends only on the operator so what we will do is that we construct Green's function using some symmetry of the domain for upper half space and the lower and the ball. Use that to prove some existence theorems in the upper half space. That is the course of action in a next few lectures. Thank you.