First Course on Partial Defferential Equations – 1

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Lecture 24

Laplace and Poisson equations - 7

Good morning, welcome back so in the last class we were discussing about the uniqueness maximum principles and we have seen various cases we also introduce the concept of regularity and uniqueness as I said you have already obtain and we what we have introduce is that we have seen that if you though not complete prove you have seen if that u is harmonic then u is C infinity. And we also remarked that u is has much more regularity, in fact, u is analytic.

So let me see that what it all depends on the estimate on the derivative. So estimates that is what we are going to prove it, estimates.

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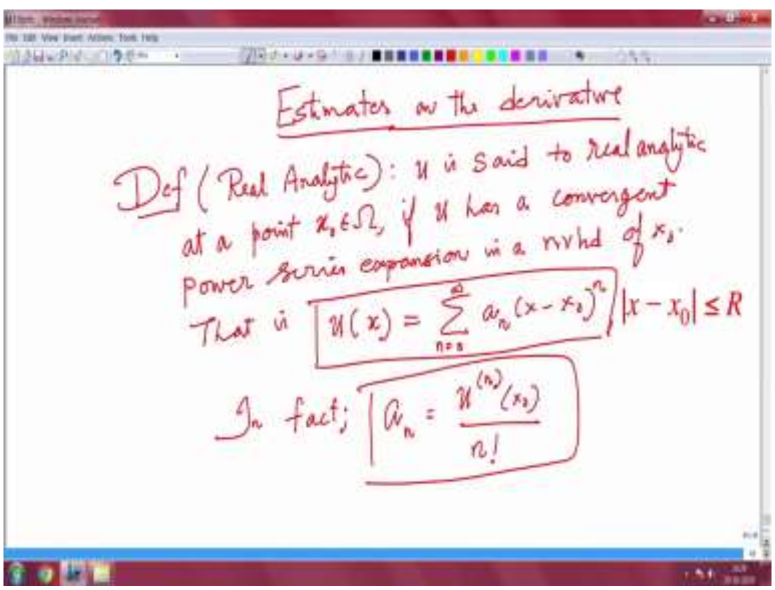

Let me make a remark there proving estimates before into getting solutions is a part of PDE analysis. What we call it a priory estimates that is very important in the study of PDEs. What you do is that assuming that there is a solution you try to estimate its various derivatives. So estimates on the derivative.

So before that I said I want to give you a definition first which all of you know that where real analytic, u is said to be real analytic, is said to be real analytic at a point x naught in omega if u has a convergent power series expansion u has a convergent power series expansion in a neighborhood of x naught that is u of x can be expanded as certain thing and this expansion is valid.

x minus x naught power n this a valid for mod x minus x naught less than equal to some number. So you can derived that, so I just given in the one variable case but other variable also you can do that analyticity definition and these series n equal to 0 to infinity. So in fact I have some in fact you can compute that in fact can writing it in one dimension for higher dimensions you can study merely an equal to u power n the nth derivative at x naught by n factorial.

So you can actually compute, so all the existence of that so you have to be very careful existence of nth derivative is enough to form the summation. That does not mean that their summation will represent the function back. You will be able to compute.

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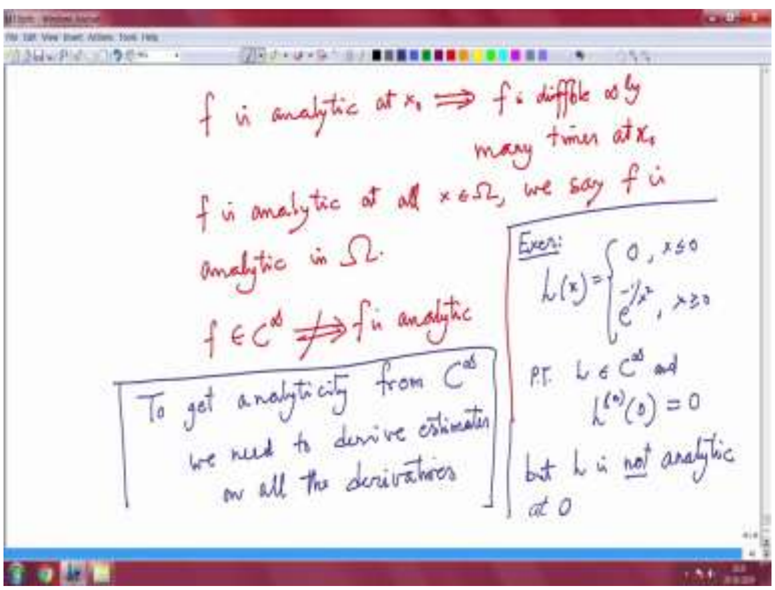

So f we see infinity f is so you let me make f is analytic at x naught implies immediately f is C infinity is differentiable infinitely many times at x naught. And if it is analytic at all points in a region then it is called analytic in omega. If f is analytic at all x in omega we say f is analytic in omega. But the interest is the f is C infinity does not imply f is analytic that is what I say you can have C infinity.

I will give you a very standard example if you are not seen this example this is a very standard example I am sure you would have seen it in the first basic course on analysis or complex analysis. So you define h of x is equal to 0 if x is in less than equal to 0 e power minus 1 over x square if x is greater than equal to 0. Prove that h is C infinity of course it is C infinity at sub that the origin is obvious because 0 at the power it is also C infinity at the origin.

And h prime any nth derivative at the origin is 0. That means the summation is always 0. So but h is not analytic h can because h is non-zero. So this gives you so if you form the summation based on h non-zero you get the series become 0 all the derivative. But then h is non-zero so let me not do it but h is not analytic, so that is an example.

So that is a important result you have it so get estimate the message is that to get analyticity from C infinity, to get analyticity this is what the important message you have to follow from C infinity we need to derived estimates on all the derivatives. That is what we are going to on all the derivatives so that is a message. And that is what the title of this section.

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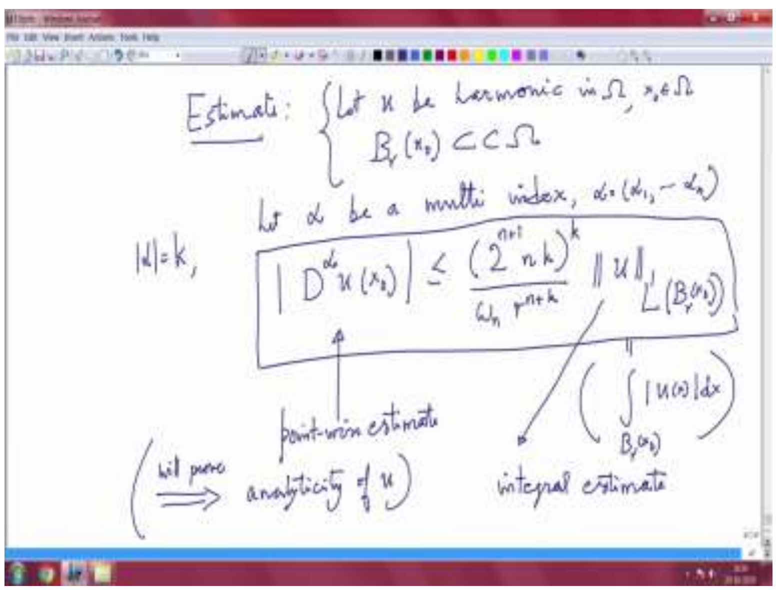

So we are going to derive some estimate, so we are going to state the estimates first. So and then as I said I am not going to prove it but I will give almost everything so that you can prove it because you have to do something to do this one. So the estimate which let me prove it. If we have to prove it for one variable estimate. For any multi index alpha, so it is all for harmonic function.

So probably do not forget about that one, so may be let me state their so we take it as. Let so this is our background let u be harmonic in omega. This the background it is all there. X naught is in omega and then you always have all the ball which you are considering whenever you have a ball this will come back in better here what have all the balls. So we are working only at that point. So these assumptions alright.

So the estimate and alpha be a multi in let alpha be a multi index. That means alpha is equal to alpha 1 up to alpha n. Then you can compute this is the point we estimate that is what we have to see that D power alpha of u you see estimate alpha to derivative where mod alpha is equal to k recall the multi index is not agent. Is less than equal to so let me precisely write down an estimate 2 power n plus 1 n k whole power k by omega n is the whole n of the unit ball r power n plus k into norm of u the integral value.

Which we call it L1 if you are not familiar with your notation just see that it is just integral value. This means integral of modulus of u of x dx over the ball of medias. The important thing in this estimate left side is a proper getting point wise estimate is a difficult thing. This is a point wise estimate you see point wise estimate derivation is generally a difficult job point wise estimate. Lp, L1 that estimate is slightly easier quite often integral estimate.

Once you have this estimates, so I will not be proving all this once you have this estimate you can prove analytic n. This estimate we will prove though I will not think prove it but that is it we will prove analyticity of u. You can show that the series is converges to u x analyticity of u. So you see is not just about the all differentiability of all orders. It is also important to have the correct estimates.

So that error terms converges properly in formula whatever it is to make it a function u of x. So to this a bit of analysis which you will be learning it. So forming series is not enough you have to understand what happens to the tail of the series. And this kind of good estimates good derivatives will give you this one. So let me tell you, so do not get worried about this complicated estimates you try to write down this estimate for as simple case.

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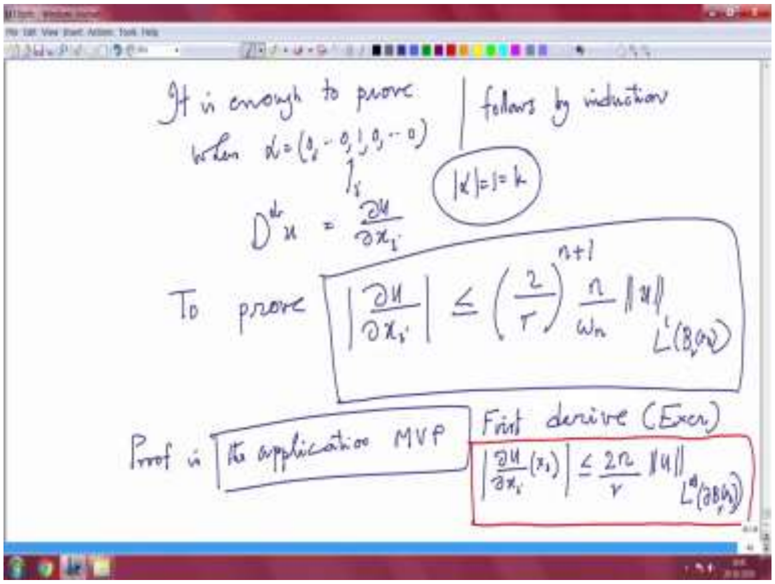

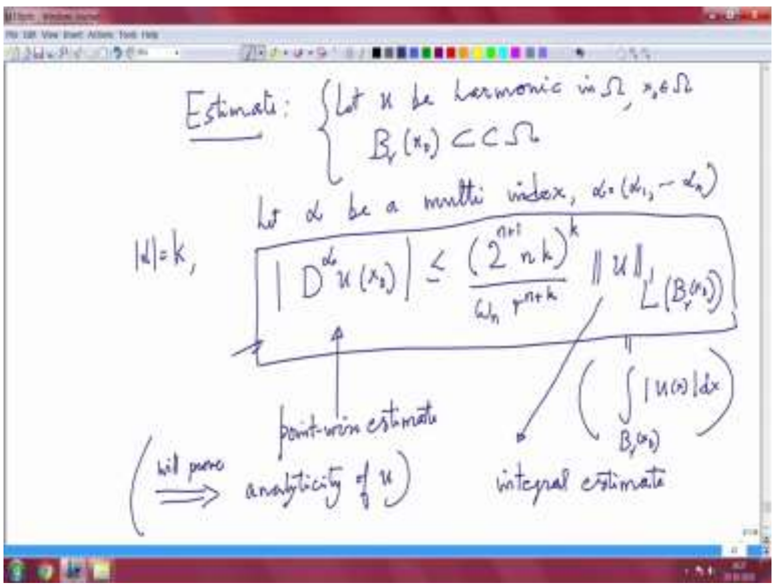

And if it is enough to prove there, then it is enough to prove enough to prove when alpha is a simple special a case 0 equal to 0 1 this is the ith case this is the ith case. So other thing will follows induction then you keep on apply. You can apply an induction argument to prove or any derivative. So you prove it only y in this case your D power alpha of u is nothing but du by dxi so it is enough to prove of this.

Once you prove it all for first order thing you can repeat it because you will be able to because if this is there you can repeatedly apply for du by dxi to get the other derivatives. And you can explain factor, you can get back the factor correctly. That is true when alpha is equal to this one your mode alpha is equal to 1. So in that case you have k equal to 1. When your k is equal to 1 you will get everything so you see 2 power n plus n k equal to 1 and this is one.

And then exactly 2 by a you will see what's estimate you want to prove it that is to prove basically you have to prove this to prove. So you which you will be able to do it that is what it is enough to prove mod you write down that estimate for the when alpha is equal to 1 is enough to prove this is equal to less than equal to I am rewriting that estimate in this place n plus 1 n by omega n this is the estimate you want to prove norm of u power L1 of Br of x naught is so this is what you want to prove.

Again the remaining thing is induction. So you see that one. So as I said what I will skip here is some argument you have to apply mean value theorem the proof is the application of mean value property and by Varignon's theorems. So you may have to apply mean value property basically so you see again the mean value property using this first derive this one. So this is an exercise I will leave it.

That you can leave it as derived exercise what I will prove is the first derive this it may be easy du by dr at x naught, du by dxi. So first derived du by dxi the less than equal to 2 n by r norm u at l infinity of boundary this you can write down the mean value property. This is by mean value property you see this estimate the difference between previous estimate and this estimate is different you see.

So write down du by you know using the mean value theorem you will be able to write down the value at point by the average for in his ball even for any ball inside. So you have to apply this mean value property to the right ball. So you have to apply this result for Br by 2 basically. So you can apply mean value property to any point you like it. So any point you take the any ball you like it. So you take a ball of radius r by 2 and then apply the mean value property you get this estimate. So that you can derive it. So once you derive that.

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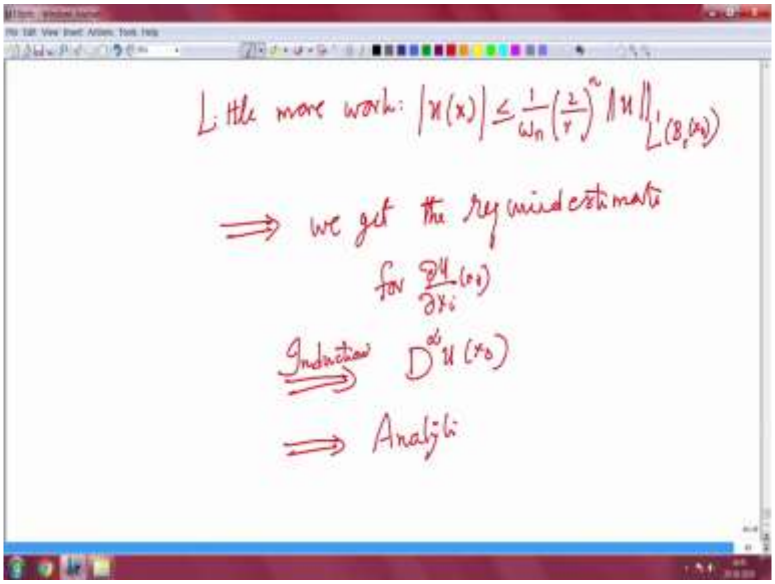

A little more work I given it in the book but it is better that you do it something independently du by dxi with that you can derive whatever estimate I we want it. Now with that you can you have to do one more estimate before proceeding work you prove you can get an estimate on u. your u of x less than equal to 1 over omega n I am skipping some few steps.

So you may feel that how it is done but you have to do it n norm of u L1 of Br of x naught. Using this these two you can also do this one come by n we get the required estimate it for du by dxi and at x naught and that by induction will give estimate for D power u of at x naught and that by some argument will give analyticity. So these are the various steps.

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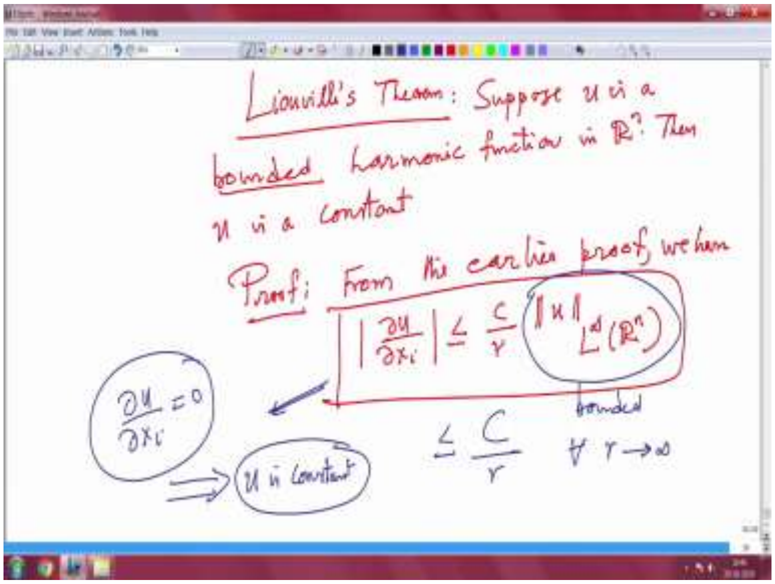

So let me derive an interesting result which you not in complex analysis what we call it a Liouvilli's theorem. So if you look at it complex analysis those who are familiar you know that the real and imaginary parts of an analytic complex and analytic function satisfies mean value property. Their also you prove this mean value property so the basically all this is part of your harmonicity.

Suppose u is a bounded this is what a bounded harmonic function in Rn. Boundedness is important you know that just like a bounded analytic function is there, is the basically the harmonic function property then u is a constant. This prove is now from the previous estimate in the prove from the earlier prove we have du by dxi modulus of du by dxi is less than equal to constant into C by r into now u is given to be a harmonic in Rn.

So you can take any ball you like it in particular you can take Rn and you get L infinity. So you have this, this you could write because u is given to be harmonic and bounded in here and this is a bounded quantity. So this is less than equal to some another constant by r and this is true for all r tends to infinity for all r. So that implies, so this implies immediately du by dxi is equal to 0 that implies u is constant you see.

So the first estimate in the beginning estimate will give you, so you do not need the entire thing you need only the first derivative estimate that is the first estimate which you get. The one thing is two things you are applied u is harmonic. So you get L infinity of Rn and that is a bounded quantity then L infinity is the supremum essentials supremum we call it. So let me not get into that one for continuous functions it is a supreme normal hence maximum or whatever it is and that is finite. That is given to be because it is a bounded function. So you have your harmonic function.

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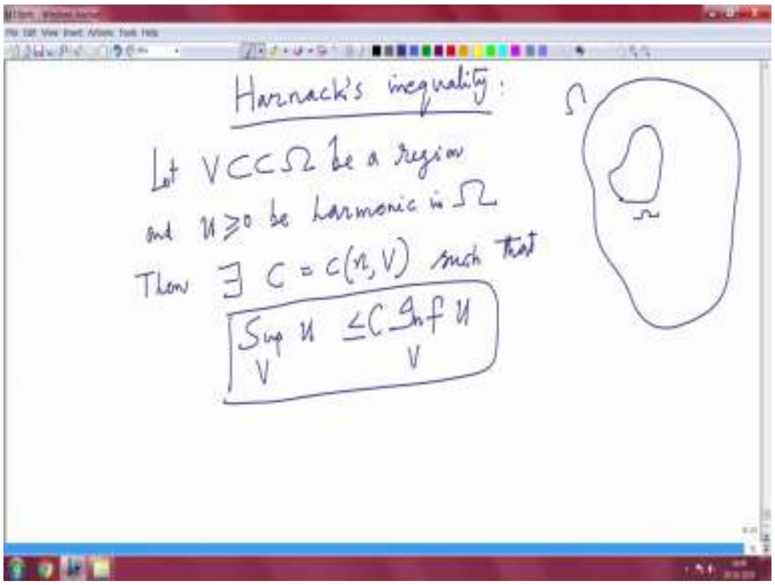

So what do I want to, I want to state before completing today's lecture and then we will go to another new topic what we call it an important result, what we call it a Harnack's inequality. So I am deriving many things again I am as is said all these all the proves I will not be able to give it. But then I am giving you some complete prove. Some idea that is what we do it in course and some we skip completely.

So what is Harnack's inequality? Let v compactly contained in omega. So you have a domain omega. And you have a v whatever be the domain v be a domain in omega v a region and u greater than equal to 0 be harmonic in omega. This is kind of again a very point wise something comparison between the values of two different points. That is a very crucial thing. So you are comparing the values, so earlier your estimate the value now you are comparing the values omega.

Then there exist a constant see there exist a constant see there constant will depend only on the dimension n and v nothing else. Such that so it is a kind of a it is a constant v such that supremum of u estimating v is less than equal to infimum of u. yeah that is a very interesting and this is true for any harmonic function is a very so just see that since the supremum that means this gives u your u of x is less than equal to u of y for all x in and y in v.

You take any x and y you will see that u of x is less than equal to u of y some constant of course what is it that constant. So you see you can estimate that way this is the famous Harnack's inequality. So let me not try to give the entire prove.

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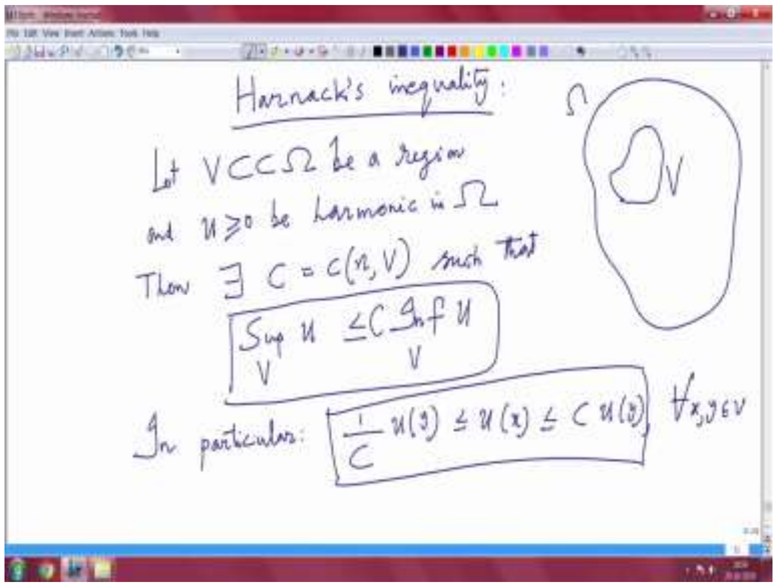

You should go and read the entire prove so in this case so let me go to that in particular that is what I said in particular I already mention. So let me write it. In particular, 1 over C u of y less than equal to u of x less than equal to C into u of y this is true for all x, y in v. So it is a comparison you are comparing the double and there is a somewhat I given v the and the dimension of the problem you have a universal constant. Of course when v to v change here I write omega here. This is not omega this is your v. So you have your v.

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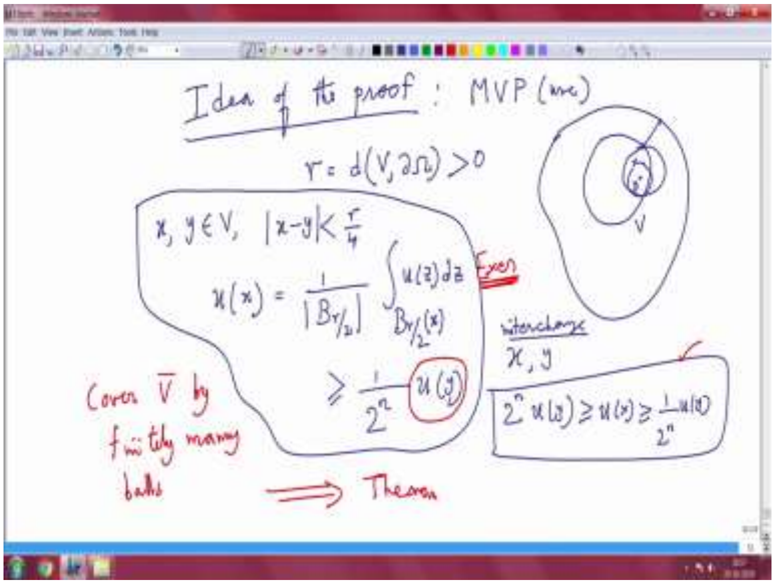

So let me give you the idea of the prove, idea of the prove so that I can idea of the prove, again mean value theorem property you have to use mean value property cleverly. So what you do is that for the mean value property the whole idea is that you look for your domain omega and v is contain there so you take a v here so in v is compactly contain it will have a distance. So you call that distance r to be the boundary the distance between v and boundary of omega.

And this happen to be positive because I am going to construct more balls. Then the next step, so you have a, so if I consider two points x and y in v. So if you start with any two-point x and y v and I want to find this constant. First I will prove an estimate first I will prove a similar estimate when x and y are close enough. So the main idea is like that you prove this estimate when x and y are close enough then try to cover your view cover v with a small balls whose distance is small.

And since V bar is compact view will get a finitely many things and then you will be able to prove you take the minimum, infimum and supremum. That is the basic idea so try to get this estimate when x and y are close enough with uniform y and this constant coming should not depend on anything else it should depend on only v eventually of better distance nothing else. So that is what we are going to write here.

If you take x and y v such that x and y is less than r by 4 the reason I am taking this smaller ball. So if you consider x a balls with center x or small center y it will be within the domain. So that is also the that is a whole idea behind that v. So you can whenever you take a ball here x here and y so the distance between x and y should be less than r by y. So you should be able construct balls around each to apply your mean value theorem.

So you can write basically your u x is equal to I can construct a ball of radius Br by 2 here. you see if this is n you intersect of Br by 2 of my u z dz. Then I can do something I there are some steps to be done with this you will be able to estimate this so I will change this ball centered at t x right now it is centered a t x. But then I will be able to make an estimate I will keep this here I will be able to make an estimate because x and y are close enough I can change the center to y.

And then I can see that that will be smaller so you will be able to prove something and end of it you will be able to prove this inequality. So this exercise I will leave it, so you should play with it. That you saw these are all basic analysis. Now PDE is coming here. And only mean value theorem because x and y are thing you are integrating if the ball of radius r by 2. Because x and y are thing you are integrating if the ball of radius r by 2.

So if you have x you are considering a ball of radius r by 2. So if this is x and this is y I will be able to construct another ball here. And that will have small so I will be able so you are integrating over a slightly bigger ball but then with centered x which I can change it to a ball with centered y but with a smaller ball. And by this will x and y are close enough I will be able to do this one and then you just calculate the value you will do this one.

Now x and y are symmetric right? x and y you can interchange x and y. You can get the other way inequality interchange because x and y is separate does imply instead of x you start with y. I can do the same thing then I will get an x here. So what I will eventually proved is that 2 power n u of y is greater than equal to 2 power n u of x greater than equal to 1 by 2 power n u of y. So I got this inequality with a constant 2 power n that depends only on n not even on v right?

Now whatever you do is that, so next step is what again I so do this, this is one exercise you can work it out. How do I change it the ball from centered x to a smaller ball of centered y to get this you want to have a ball around y. So that is what you have to do it, to get this. But that ball should be smaller so that you can estimate from below. And then just interchange x and y you get this estimate.

So for I x and y are small enough you get this one. So now what you do is that you can cover your v by finitely many balls, finitely many v bar rather cover v bar by finitely many balls open sets. And then each time you can estimate from x to y that is very nice way of doing that you can I almost given the theorem proves a theorem almost. So two steps you have to think first you do it for x and y are small enough which I almost given change this integral to an integral over with centered y.

By if necessary do it with a ball may be you consider the ball of radius r by 4 you will get this. And then interchange you get it that is a trivial thing from here. So x and y are small enough you can do that one. Now if a x and y are any arbitrary points in v you can reach x from x to y with finitely many steps. That is what we basically tell. So you can go one by one from x to y1 to y2 like that to reach any two points.

So this you can do it uniformly by using a compactness argument v bar. So you prove this theorem. So I think I will stop here. Now what I want to really do is a about next class.

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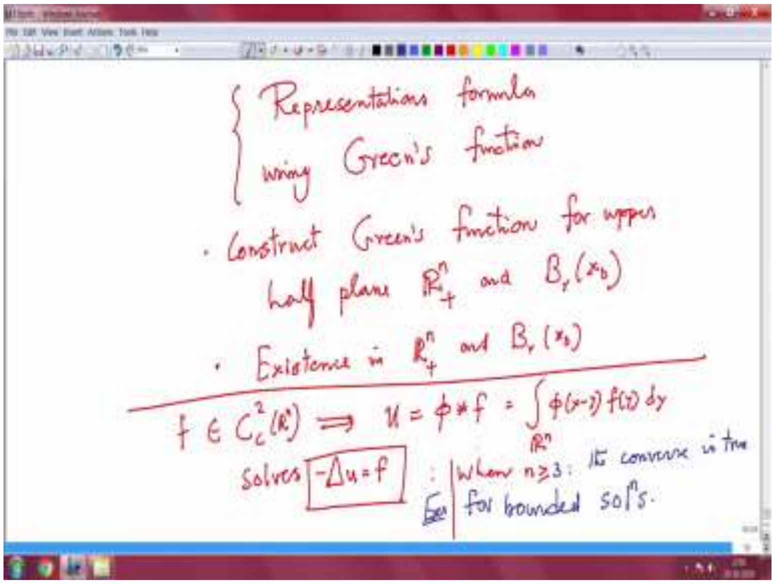

So let me prepare for the next class what I am going to representation formula. So the next class next couple of lecture not couple of lectures couple of hours I will prove representation formula using Green’s function. Green’s function is some sense is similar to fundamental solution but exactly fundamental solution. This has a singularity of the fundamental solution but it also take care of some boundary condition.

Using this we construct a from getting you prepared for the next classes construct Green's function for the upper half plain Rn plus and any ball of radius xt. Once you have a Green’s function you prove the existence some existence not in very generality existence for in these domains. Existence special not that also not full existence in Rn plus and Br of x. This representation formulize already familiar to you little bit.

So let me recall before I conclude you have already seen that if f is C infinity. So these are general mark for my next as I said thing. So please go through, so if you before you want to learn further so you had get prepared whatever we covered in this lecture. So f is cc 2 of Rn then you have seen that u is equal to phi star of f that is in nothing but integral of Rn phi of x minus y f of x y dy.

Solves minus Laplacian u equal to f this is but so you have a represent so whenever u is the solution you approved the existence of the solution. So what this is that when n greater than equal to 3 when n equal to 2 is result may be true that is a kind of again and exercise. So let me leave it as an exercise for u. Let say whether you will prove it n greater than equal to 3 the converse is also true for bounded solution. Converse is true for bounded solution.

May be I will explain to you since the time is up let me stop here. For n greater than equal to the converse is true for bounded solution. What I am trying that if u is bounded solution of this one, then u has this representation. So I will stop here and we will continue next time and we will finish as I said none of the four lectures that is 2 hours. This Green's representation and Liouvilli’s other formulas we derive for upper half plain and balls of radius. Thank you.