First Course on Partial Defferential Equations – 1

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Lecture 23

Laplace and Poisson equations - 6

Welcome back again in the last of lecture we had discussed about the maximum principle in fact strong maximum principle, which says that if you have a bounded subharmonic function it cannot have an interior maximum. Similarly, if super harmonic, it cannot have an interior non-constant, interior minimum. If it is a harmonic it cannot achieve more interior and maximum and minimum. So from that we can immediately derive some what we call it a weak maximum principle.

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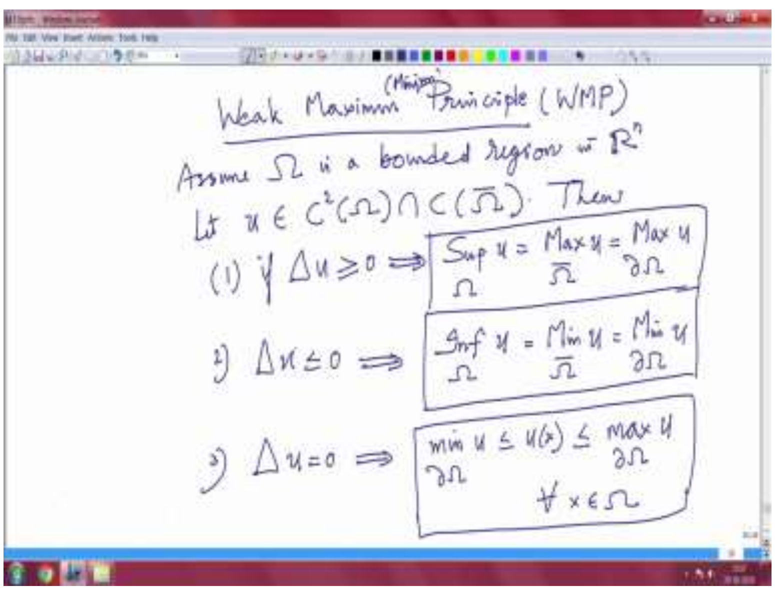

So it is a kind a division assume so let me write the statement. So as again assume this is trivial, so there is nothing much to prove it follows immediately omega is a bounded region in Rn and let u belongs to C2 of omega so you have twice differentiability in interior but you have continuity up to the boundary. Then you have 1 if u is subharmonic that imply what we have proved is that if u is subharmonic and C2 in this non constant it can have an interior maximum.

But if we have bounded if a continuous function in a closed bounded domain has to have an maximum. So that maxima I has to be have the boundary so that is what you have so supremum of u over the domain omega and the it will become a maximum omega is open. So that may not have that one so on the omega bar it will have a maximum that is true and that maximum has to be achieved on the boundary. That is what it will be. So to understood the theorem properly.

So the prove is over that is the prove. So if you have an interior maximum so I am also giving you the prove this an obvious thing. So supremum of u should be maximum of u and that maxima has to be achieved on the boundary. The strong maximum principle says that it cannot be in the interior. So then since it has to have a maximum including the boundary it has to be the maximum up the boundary achieved there.

So it is a similar result if Laplacian of u is less than or equal to 0. That implies infimum of u over omega is equal to minimum over, minimum and you write it infimum is achieved u that is equal to minimum has to be achieved on the boundary that is what the strong maximum principle strong maximum principle says you cannot have interior but then it has to have since it is a counter closed domain.

And the third one if Laplacian of u equal to 0 that implies the u will be bounded by minimum and boundary. So that implies from that it to you can reduce that minimum over u over the boundary less than or equal to u of x. Because minimum is thing it has to be on the boundary achieved less than or equal to maximum of u on the boundary so this should be for x in omega. So you see, so the maximum principle the power of maximum principle you have a point wise estimate.

If you have a harmonic function you can estimate by its boundary values. And in PED the boundary values are given to you. So you are, we get a even if you cannot solve the equation that immediately tells you about that the bound lower bound and then upper bound about a solution of like harmonic function. So any harmonic function you take a bounded domain it will be estimated by its boundary value in a PED it is concern boundaries values are given to you.

So it will be estimated by the non-values people solving it, it is a kind of appropriate estimates which you are deriving it. So that is the prove is complete already. So these are called weak maximum and, it is not only weak maximum and minimum both are there weak maximum and minimum principles you do that one.

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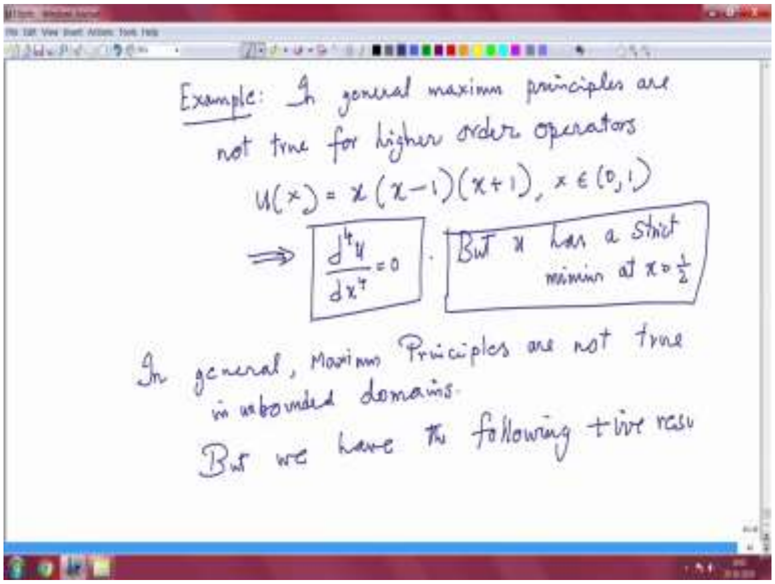

And as I say there are such maximum minimum principles can also be derived for general second order elliptical and other operators also. But as I said we are not going you anything but then in our reference book we have a given some nice examples of a thing. So if I get time I will come back and do something but I do not think we will have a sufficient time this 20 hour course to derived all the proves and theorems.

But when you are studying this course it is important that you also study something more than what we present here about other second order elliptical operators some cases are given our book and there are many interesting things in our, in the other books like I mentioned in Gilbert Trading or many other books here. So with this I want to make start with the some examples now, two examples one example requires more prove I will sketch you the prove but you complete or read the prove here.

In general maximum principles are not true for higher order equation or true for in general there may be some special case there true for higher order operators. So you look at this example consider this function u x equal to x into x minus 1 into x plus 1 for x in 0 1. Since this is a third degree polynomial this immediately implies that is this a d power 4 u by d x power 4. This is equivalent to fourth order equation.

But is the u satisfies this fourth order operator d power 4 by d x power 4. But u has a strict minimum at x equals to half it is an interior minimum. So you see event the first order, one variable k is this maximum principle is not true. The other result is not really an example this also an example but it is more like a theorem.

So let me also write in general, maximum principles are not true for unbounded domains, principles are not true in unbounded domains. But there is a very interesting example but we have the following example following positive result. In general, it is not true.

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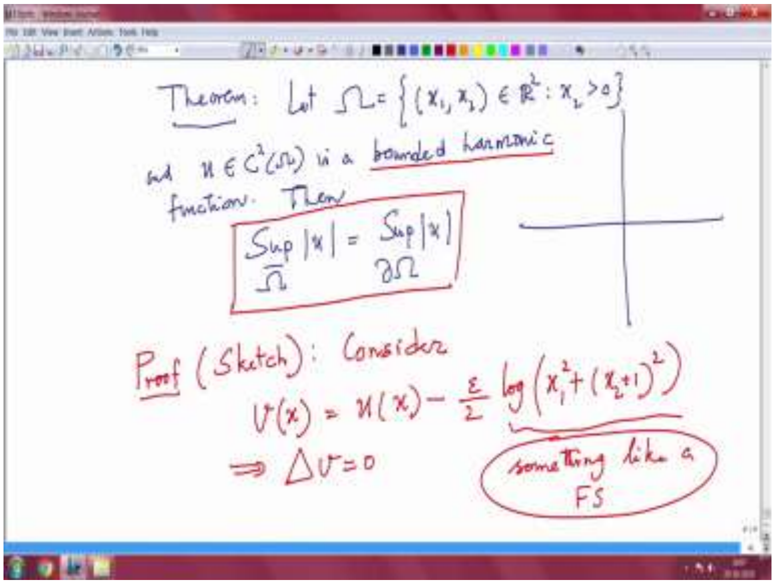

but there are some domain which I want to tell you a result, so this is a theorem which as I said now I am going to present few theorem without prove in this short course it is not possible to prove several theorems. So we will make some of them will make only the statement but then it is up to you to go see the syllabus. Since I am not covering the prove here that does not mean that you the first time learners should skip the prove is important that you study the prove.

So let me give you an upper half plain, this is upper half plain x1, x2 as used in problem. And u belongs to C2 of omega is a bounded harmonic functions. So this is a positive results in the upper half plain bounded harmonic function. Then supremum of a u if a result is mod u over omega bar then entire upper half plain this only tells that the supremum is attained up the boundary. So there in the previous case let me go to a previous case let me probably state one result probably it will be helpful to you.

So in example why in unbounded domain it is true. So you again consider omega upper half plain. This is an example for upper half plain. And you look at u of x1, x2 equal to x2. Then u is harmonic then u harmonic and u equal to 0 on the boundary what is the boundary in this case this is the boundary. This is your d omega. So the maximum is not on the boundary. You see u x1, x2 x2 as increases, it increases.

So u is not the it is the maximum is not attend maximum is not on the boundary. Maximum is not on the boundary. You have an unbounded so why are the other case center you have the maximum attended up the boundary. So unbounded domains are generally true. But we have at the result which you are doing is that you see you have anything and have if you happen to your bounded harmonic function the previous things is that you are u is not bounded.

So if you have u C2 of omega in upper half plain this is a very special case then the maximum is attained here as I say that I am not going to prove it I want to just sketch the prove here. I will sketch give a sketch of the prove here. So the idea is that you consider another function this is not is very consider another function it, this is a need 2 dimensional case v x equal to u x. So you work out the details.

So I will only ask you to work out the details here. So u x equal to minus epsilon by 2 log x1 square plus x2 plus 1 whole square. If you look at it you know that v is harmonic that implies v is actually harmonic. Why it is harmonic? Of course you can verify the harmonicity but if you look at it this is something like fundamental solution.

Something like not exactly fundamental solution you see so you do not have to worry but you can guess it but you can actually prove this is a harmonic function. So epsilon is a harmonic function close to it. So what I will do now so the sketch of the prove is so you define a domain is also a you have to write down it is there in the book. So you write down not exactly the semi-circle I will put minus a here, a here something is less than semi-circle.

So you look at it. I call this is your omega a. When you have this one the thing is that and you consider your v in this domain. So you consider v here. v is defined everywhere so consider v in omega a. And which is a bounded domain. You see bounded domain. Therefore, by our earlier maximum principle the maximum is obtain.

Maximum of v is attended on the omega a but d omega a has to two boundary one here gamma a, here gamma a1 and the boundary of omega is this, this is your boundary of omega a it consist of gamma 1 a and gamma 2 a. gamma 1 a union gamma 2 a because so the maximum is attended there. So the important step in this week I will not prove it that is what you have to look into the book or you prove yourself is not difficult to prove by yourself.

Show that the maximum of v is not attend on curved boundary is not attend on gamma to a. Hence it is attend on gamma 1 a1 and then take a tends to infinity or a large you can this is proved some sort of the contradiction. If there is a maximum you should derive a corner if there is a maximum attended on gamma 2 a. And take a large and derive a contradiction to show that the maximum of e is not attend in omega a.

And then when epsilon tends to 0 show that so the maximum will be attend as epsilon times to 0 show that the maximum of u is attend on the x axis. So that is the theorem which you want to prove it. Please go through it carefully. So that prove is have not accept the claim I have proved everything. Only the claim you want to see that the maximum of v is not attend on this example. And then once you prove that is you just to do a small calculation.

So that is not difficult thing but you prove it. And take epsilon times to 0 and do that one with this I can prove the uniqueness of,

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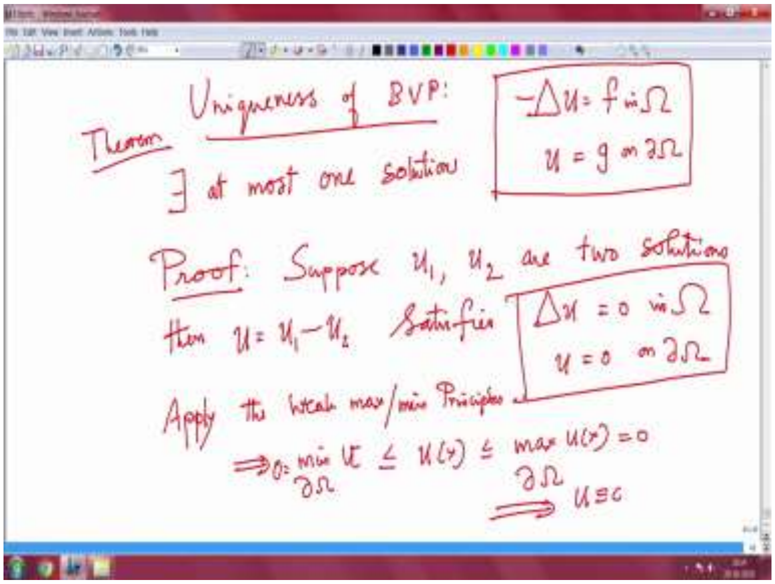

So what are we trying to prove it? Uniqueness this is the thing uniqueness of boundary value problem. What is our boundary value problem? We have your minus Laplacian of u is equal to f in omega that is our problem u is equal to g on d when I say uniqueness we are not claiming there is a solution that means there does not exist. So the theorem is there exist at most one solution. That is the theorem. There exist at most one solution.

So how do you prove with that prove is trivial. How do you suppose u1, u2 are two solutions. Then u equal to u1 minus u2 satisfies Laplacian of v equal to 0 in omega and v equal to 0 on the boundary of omega. That is it now apply maximum principle apply the weak maximum principle, minimum principles. What that does tell you this will imply the minimum is 0 what is the minimum of v over the boundary not v sorry.

We have to write u equal to 0. So minimum of u is less than or equal to u v x this is equal to 0 less than or equal to maximum of u x over the boundary x equal to 0 implies u v x identically 0 in omega so you have immediate uniqueness theorem.

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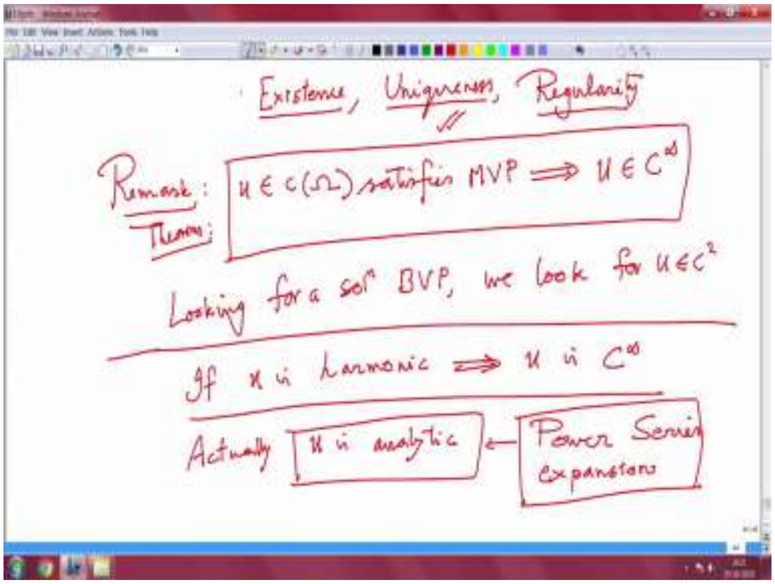

So typically when you study partial differential equations there are three questions the existence uniqueness and then there is a third thing what we call it regularity. Which will be discuss in something so as I said we will also discuss existence in becoming lectures. Uniqueness is what we have done now. And also I have remarked that remarked earlier though as I said now let me recall again remarked you proved if u is in C omega this is not proved u is since omega satisfies for this is regular theorem.

As I said we will not prove it here right now satisfies mean value property that implies u is equal to is C infinity. So the main existence theorem you may be demanding some regularity, so when you are looking for a solution, looking for a solution of your boundary value problem we look for u is in C2 not more that because your operator is a second order operator. So when you have a second order operator.

And look for a solution inside to because you are looking for a solution of Laplacian u equal to f and u equal to g. But after obtaining your solution u, you ask questions whether u is C3, u is C4 and u is C infinity. And these such questions so after proving you look for higher regularity in the obtain solution such results are called regularity results and this all the more important when we study solutions in the Hilbert space approach which you will not see it here. For example, you study weak solutions and you look for weak solutions in some Hilbert spaces.

So initially you start with the weak formulation so this a just general mark before we go further. When you have start with different formulation you look for a solution of your thing is not in the regular spaces like C2, C3, C4 et cetera or holder spaces which you may see in the next course you look for solutions certain Hilbert spaces and then look for after obtaining solutions in a certain Hilbert spaces try to say something about we call it sub low spaces you look for solutions and try to prove that solutions are in bigger better spaces rather using that results try to prove that u is actually S booth solution.

So the regularity is also implied to prove the classical solution from weak solutions. So here what has mentioned is u is since C omega what we have proved is that you assuming u is in C2 of omega twice differentiability hence satisfies the mean value property and then u is a C infinity u prove that u is harmonic.

This is something more than that so there is a nice prove using a convulsions and mollifiers that if without the assumption of u is C2 just u is continuous and mean value property. Because they define mean value property continuity is enough because they are just integral of u. You can actually prove that u is C2 which can actually prove it.

And hence it is harmonic, so this implies that every harmonic function even though you start with a C2 solution end of it you is a C infinity function. So this operator Laplacian is provide, so these are all some sort of smoothing operator and you see this is in the introduction at all u electric operator and heat operators have this smoothness.

In smoothing operator but you will see in the way study of wave equation it takes you the singularities of propagator you cannot expect this operator to produce smooth solution in fact sometimes you lose your regularity that is what we see when we have studied the Burgers equation. It can generate singularities; it can produce physically relevant non smooth solutions.

This is something like kind of two different things for elliptic equations and the hyperbolic equations. So let me proceed further. After proving the uniqueness, so I want to discuss something again I will not be able to prove many of the results. So I just want to talk to you about what is a, so what I have just stated in maybe I will not will not have a time here now so what you have with all these results with remarks if u is harmonic.

Which gives you implies u is C infinity, so you can talk about u prime et cetera, et cetera here but what in the next class what I am going to prove is that actually u is analytic. Analyticity is a stronger property which I am going to take analytic that means it has a power series expansion. I will tell you what is a real analytic functions and then I will move on to again as I said I will not prove each and every step but I will try to give you the important steps in the analysis.

So the since we have few minutes left but I think we start fresh by giving you the definition of a proper analytic functions and give you what exactly you have to prove it. So you see starting with an analytic continuous functions satisfies the mean value property it becomes a harmonic function, it becomes C infinity and then it becomes a analytic so it is a powerful thing. So with these I will stop this lecture and we will continue. Thank you.