First Course on Partial Defferential Equations – 1
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Lecture-21
Laplace and Poisson Equations-4

So we are going to discuss the mean value theorem or mean value property. This is what I have mentioned there. So mean value property you will see soon, that is an equivalent property of harmonicity.

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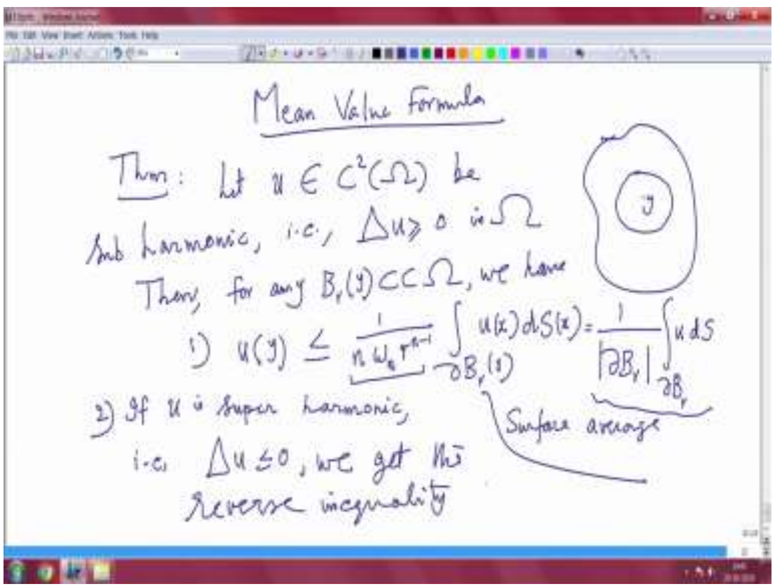

So let us begin with what is this Mean Value Formula, so rather a theorem. So let us make, start a theorem and then try to prove it. Theorem, so you have your domain omega that is there and you have your, let u belongs to C2 of omega be harmonic, that is, Laplacian of u equal to 0 in omega.

Then there are three components so let me compute. Then for any ball, you look at any ball at the point so you choose a point y here, you look at a ball which is completely contained in omega. That means compactly embedded that any ball of radius r centered at y that means compactly contained in omega for all bar that any we have u of y.

Let me do it in three steps. So I do it more general thing, means sub harmonic. So let me do it the sub harmonic that is Laplacian of u greater than or equal to 0. Last lecture we have introduced these, u is sub harmonic and Laplacian of u had. Then you have the inequality, less than or equal to, is nothing but 1 by n, omega n, r power n minus 1, integral over the boundary of the ball, B r of y and then u of y, ds of y, so ds of x.

So let me use, y is used here so let me use x here around that part. This is nothing but the surface area. So this is nothing but the Rh, 1 over modulus of d b r or integral of d b r of u ds, you see. So this is nothing but the average, surface average. This is nothing but surface average. This is one. Two, if u is super harmonic that is equivalent to saying that minus u is sub harmonic. So you get the reverse inequality. Super harmonic, that is, Laplacian of u less than or equal to we get the reverse inequality.

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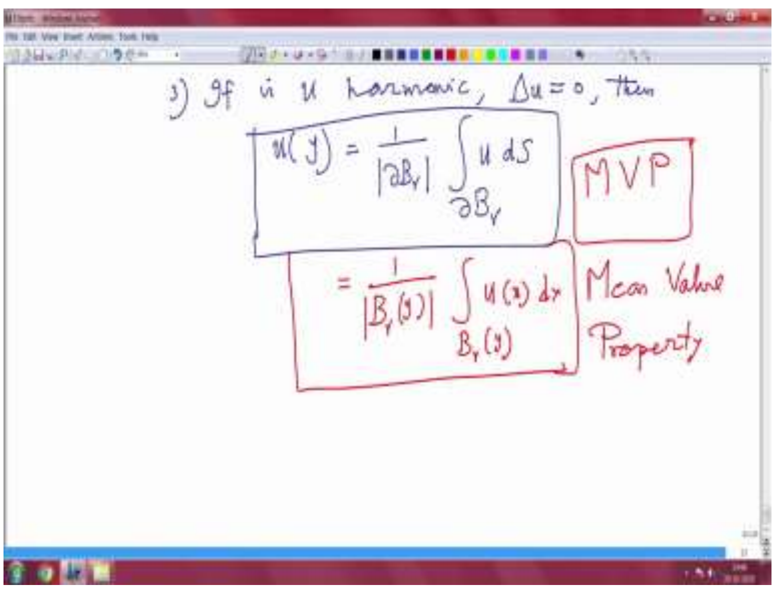

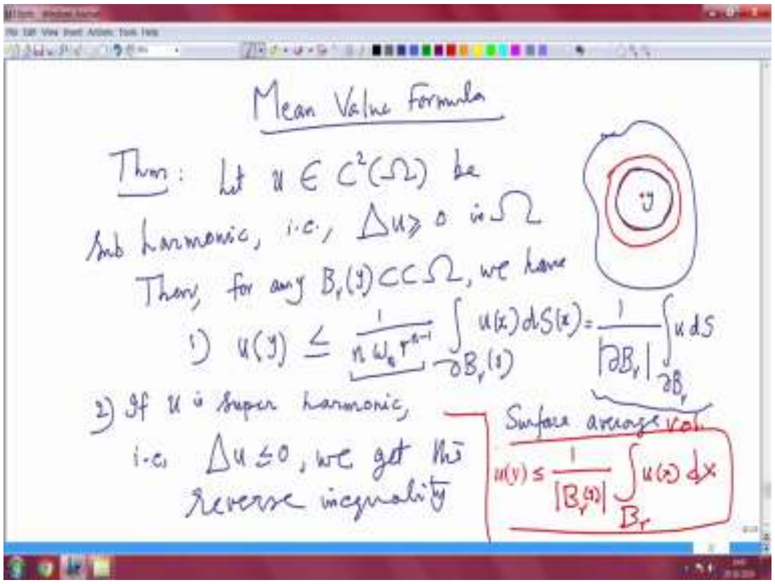

And the third one is if u is harmonic, this both sub harmonic and that is Laplacian of u equal to 0 then you have the equality. Then you get u of y is equal to the equality, 1 by modulus of d b r into integral of d b r, u ds. So you see this is called the mean. In fact you can also, this is all the, so let me go to the thing, you can also get, so let me write down that one.

So you can also get not only the surface average, you can also get u of y less than or equal to, so you can also get the volume average. That means b r of y modulus, integral of b r and u of x dx. So you see, you have both. This is the volume average. So you have both the surface average. So that is very interesting.

So if you want to find the value here at one point, you average, you can average it in anything, any ball as long as the ball is inside. So either you can take the surface average or you can take the internal average, surface average or the internal volume average. That is what you will be getting and that is true here also.

This is also equal to 1 by b r of x, modulus its average, integral of b r of y, this should be b r of x only, b r of y, modulus of b r of y, into u of x, dx. This is called the mean value property. This is also called mean value property. So we will be using this. So whenever any harmonic function suggests the mean value property in every open ball, that one. So I am going to give a proof of this. So let me try to give a proof of it. So let me start the next page of the proof.

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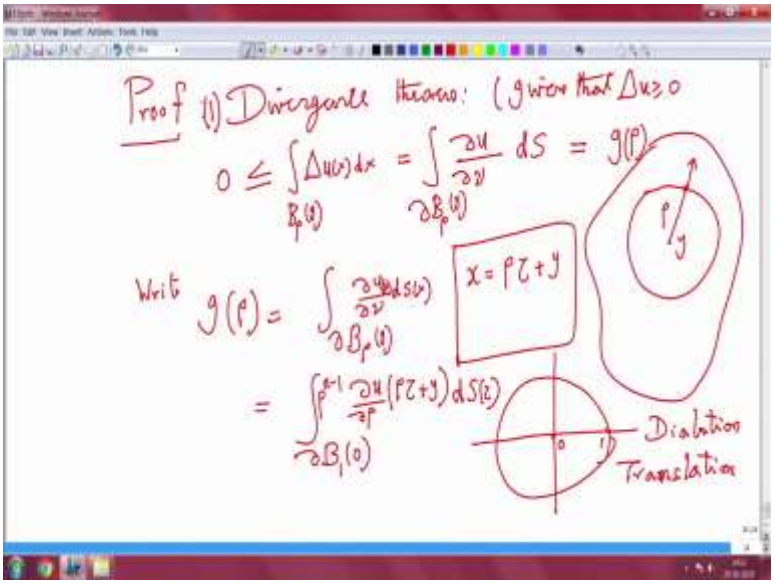

So use the divergence theorem first. Divergence theorem, so we are going to prove one. The other things we can deduce it. Divergence theorem tells you integral of Laplacian of u on any internal ball, so b r of y, dx is equal to integral of the boundary b r of y, du by d Nu. This is a normal derivative with respect to dx.

And you are given, given that Laplacian of u, u is sub harmonic, that Laplacian of u greater than or equal to 0. So this is less than or equal to 0, this quantity is that. So I call this to be, say this is equal to, in fact I can define for any ball. So if you have instead of r I can define for any ball. So you have a ball contained in omega, this is your y, I can take any ball.

This formula is true for any ball of radius Rho because our r I will reserve it for something else. So I will integrate with respect to b Rho, r also you can use it. So this is true and this depends on Rho so I call this to be 0. So this is my definition of 0 and 0 is positive. So I want to write, do something now, 0 is equal to, so I use this 0, integral of d b Rho of y, du by dn, I may not write each time ds.

Now I want to take this derivative outside but d by d Nu depends on Rho because d by d Nu is the normal derivative along this direction, this depends on Rho. So whenever you want to take a derivative outside whose integral is also depends on that variable, you cannot directly do it. So one of the way we want to do it is that you translate to, you make it an integration, you try to make it in a fixed domain.

So whenever you have an integral which is varying with respect to your parameter and if you want that derivative to take inside or outside, the one procedure is that you convert that integral to a fixed thing. So now this is on x d x of x, so doing it. So you want to translate to this to a fixed thing which is centered at the origin and you want to translate this y here and then the radius you want to make it.

So you have make a dilation and translation. So you make a change of variable that is what you want to precisely do it. So you make a change of variable, x is equal to Rho, Tau, plus y. Then when Tau varies from x varies from 0 to Rho in that ball this Tau will vary from 0 to 1, you see it will vary.

So it will become a change of variable, so this will become d, b1 with respect to the 0 and then the normal derivative it will become d by d Rho because it is a normal derivate, is Rho here so get it, d u thing and x is changed to Rho Tau plus y. Keep it in mind, Rho and y are fixed. Only Tau is now varying, so the integration with respect to ds of Tau. So once this is there, now this is independent of the boundary integral.

The surface you are integrating so you can take it outside and then there will be a Rho power n minus 1 here because you are making a small Rho. These things we have already seen it in our, in fact we have seen it yesterday in the previous class. You will have, whenever you are changing a ball or surface of radius Rho to a radius 1 there will be a Rho empowering minus 1 parameter coming. This variation we have seen it and the integration is with respect to Tau.

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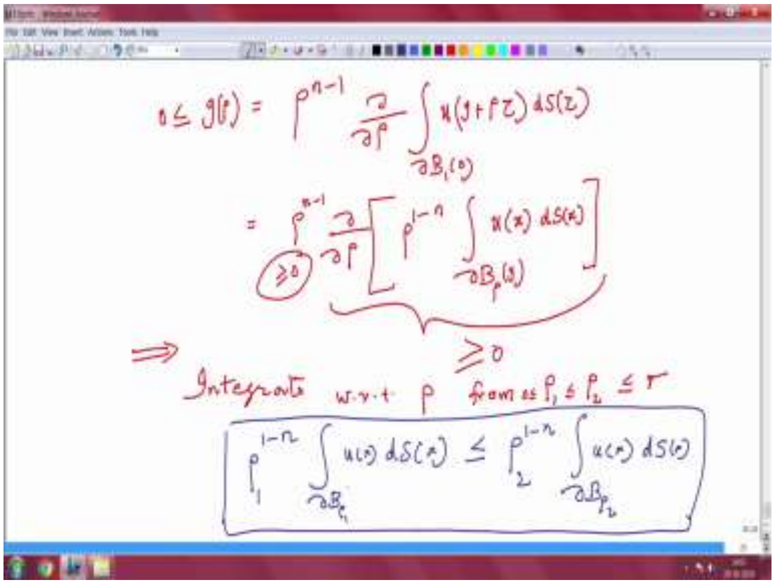

So that will become, so you will have Rho n minus 1 and now I can take it d by d Rho outside and integral d b 1 Rho into u of y plus Rho Tau ds of Tau, you see. Now I have achieved my thing, what I want, of course this is 0 and this equal to this. You have that condition. Now look at here so I will go back by the same procedure after taking this derivative outside so now it is outside.

D by d Rho, I can go back, when I go back instead of, while coming back, translating and dilating in this fashion I got Rho n minus 1 so if I go back to the same ball, I get Rho power 1 minus n but that is within the derivative so I can go back in the same way I get b Rho of y, u of x, ds of x. So that is a kind of trick you keep on using it and this is positive. Rho is positive so this implies, this derivative this is positive because this is already positive, right.

So this implies that this quantity is positive so what do you do, so that is a nice thing. So now integrate. Integrate with respect to Rho from Rho 1 to Rho 2, any 0 less than or equal to Rho 1, less than or equal to Rho 2, less than or equal to r, the ball r is fixed. You can choose any Rho 1 because this is valid, this formula is valid for any Rho 1 and with Rho 2. So if you integrate this is you can integrate so you will get so the integral you get exactly that term so integrate you will get. So let me use a different color.

So you will get Rho 1 power 1 minus n integral over d b Rho 1 of y, u of x, ds of x, this is integral Rho 1 to Rho 2 is less than or equal to, so you will get Rho 2 power 1 minus n integral d b Rho 2. I am not doing anything special, just integrating with respect to and applying the limits, you see. So you have this inequality. This is true for any Rho 1 and Rho 2 such that Rho 1 is less than or equal to Rho 2 and with this it is between Rn.

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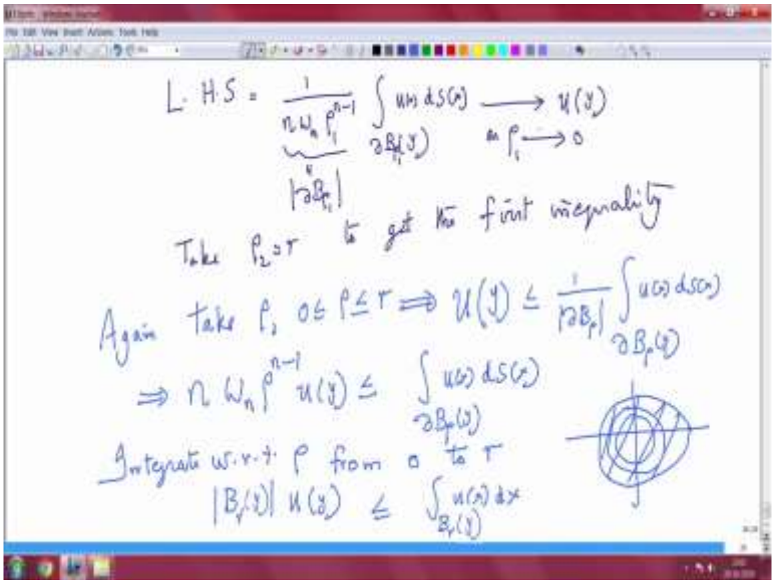

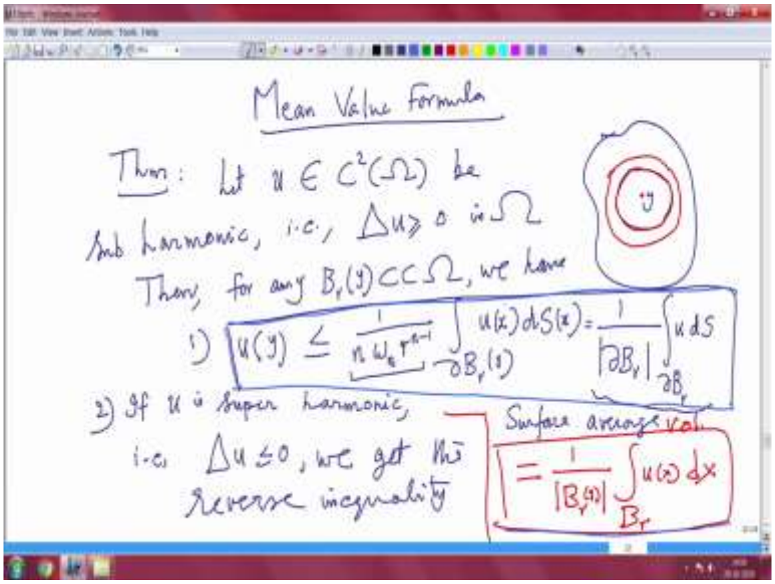

Now I can do two things so what do I do is that, so you look at the left hand side. So look at the left hand side. I want to make it a volume, it is equal to 1 by, I multiplied it by n omega n on both sides. If I multiply by n omega n, Rho 1 power minus n minus 1 and d b Rho 1 and this is nothing but the average of u of ds of x and then you have seen this average yesterday. Whenever you have some average around a neighborhood and if that neighborhood shrinks and if u is continuous it will go to its boundary value.

So this will go to u of y because the ball of radius y, you see, as Rho 1 tends to 0. So you got your, you look at here, so you got your left hand side. On the left hand side here this converges to you of y and you take Rho 2 equal to r to get the result. So you take Rho 2 equal to r to get the first inequality. So let me go back to the inequality, where is the inequality, yeah this is the first inequality, you see. This is your first inequality.

Now I want to prove the second inequality. We want to prove the second inequality. Only I have to prove result 1. The rest of Rho result is immediate so we will go to the first inequality. Again take Rho such that 0 less than or equal to Rho less than equal to r that implies this inequality u y it is true for any r, in particular for any r, so Rho 2 you took r but you can take Rho 2 equal to Rho so you get u of y is less than or equal to 1 by d b r, d b Rho, integral of d b Rho of y, u of x, ds of x.

So I know how to evaluate this. Yesterday we again in the previous lecture if you want to discuss an integral, if you want to get a volume integral what do you do is that you take integral surface, surface integral about all balls. This is again we have and then integrate and that is what we are going to do. So you take this to the left hand side d b Rho.

So once you take it to the left side what you will get is that n, this implies you take this d b Rho so you get n, omega n, Rho power n minus 1, u of y is less than or equal to integral over d b Rho of y, u of x, ds of x. Now integrate from, now integrate with respect to Rho from 0 to r, the right hand side will become nothing but if you integrate that is nothing but your b r of y that is how you get it.

If you want to integrate in the volume integral, you integrate over all the surfaces and integrate with respect to that radius of it, dx. On the right hand side what you get is, you will get n omega n and you integrate this with respect to u of y, you exactly get Rho power n by n and that is r power n that is nothing but the volume so the left hand side will be Rho power n by n, n and n cancels.

You get omega n into Rho power n, r power n because you are integrating Rho from 0 to r so exactly get your b r of y and u of y. So if you, this completes the proof, so if you take that modulus of b r of x on the right side, you get your inequality, volume inequality. So that is a proof of it.

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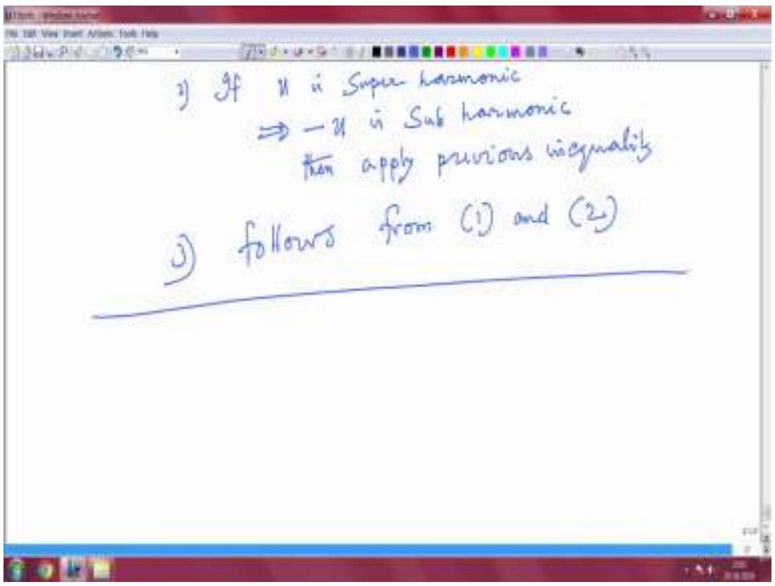

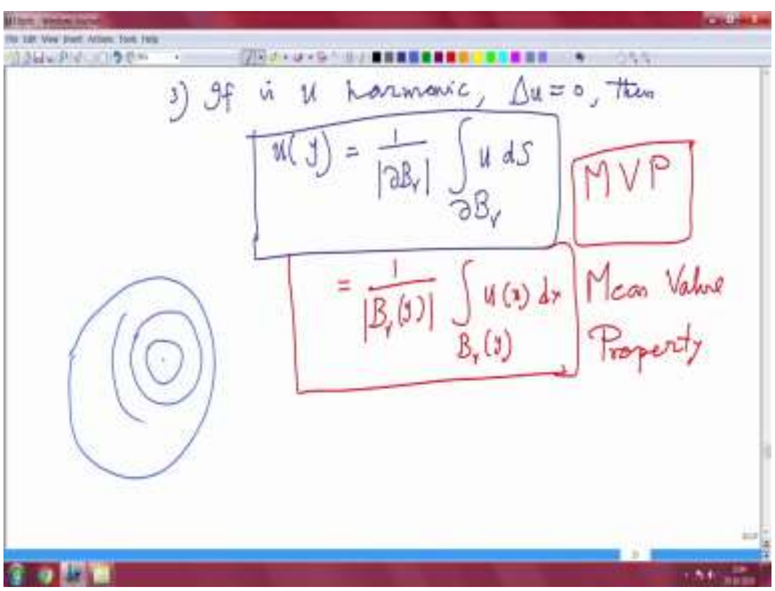

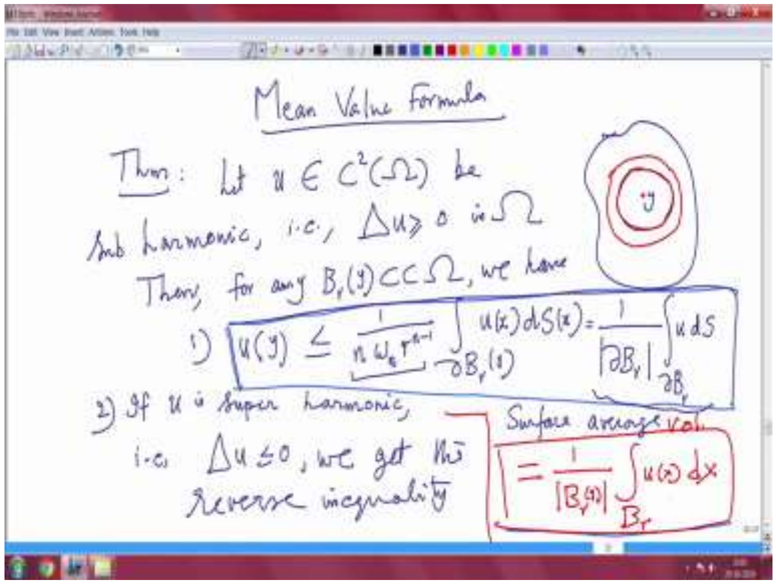

And the second thing, if u is super harmonic that implies minus u is sub harmonic okay, and then apply the previous inequality. To get 3 follows from 1 and 2. So that is the thing we want to, so you have the result. So let me recall once again for your thing. So whenever u is sub harmonic y is less than or equal to either the surface average or the volume average and if it is harmonic then it is the surface, equal to the surface average or the volume average.

And that ball it can be anything. So that is what you have to see. It does not matter. Any ball you can average it out. The more interesting result is the converse to it, converse to mean value property. So let me state the mean value property here. So let me make some remark, maybe remark after stating the theorem and then I will continue it in the next class.

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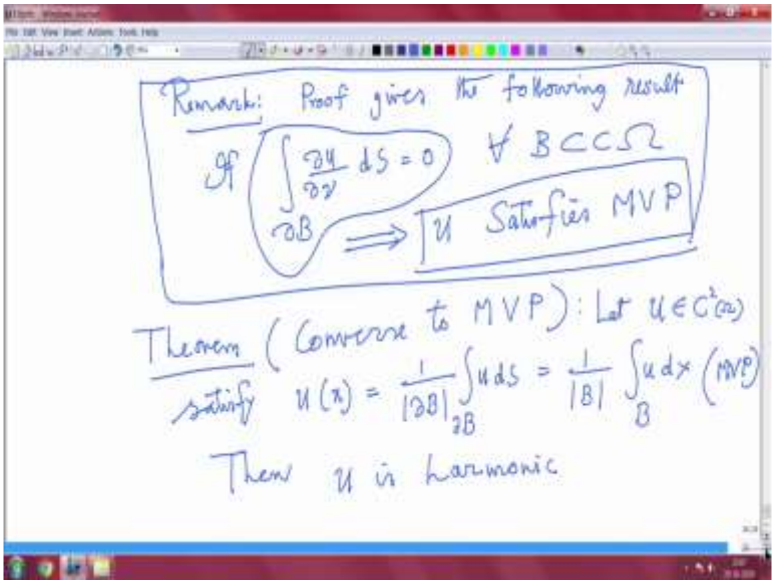

So let me make a remark first, one remark. There are other remarks which I will make today. If you follow, the proof implies that, proof gives the following result. Just think about it, gives the following result. Nothing to do, you just look at the proof, you will see this fact. If integral of d u by d nu, d s equal to 0 over the boundary, equal to 0, for all b contained in omega, all balls contained in omega that implies u is harmonic or u satisfies mean value property.

So you can check when, so if you check your this averages, not even average, this quantity for all balls then you can check that u satisfies the mean value property. That is all we have used it. We have used this is positive, derive the inequality. If it is negative, you will derive the other inequality and you follow that up. So let me state the theorem and which we will prove in the next class.

Theorem, converse to mean value property and then we will continue in the next class. Let u belongs to C2 of omega satisfy u x equal to, this is called the mean value property, 1 by the boundary of integral over b, u of ds, boundary, same as, this is the mean value property, b integral over b, u dx, this is the mean value property that means for a function twice differentiable function satisfies the mean value property then u is harmonic.

This is the converse. Earlier I have proved that if u is harmonic then it satisfies the mean value property.

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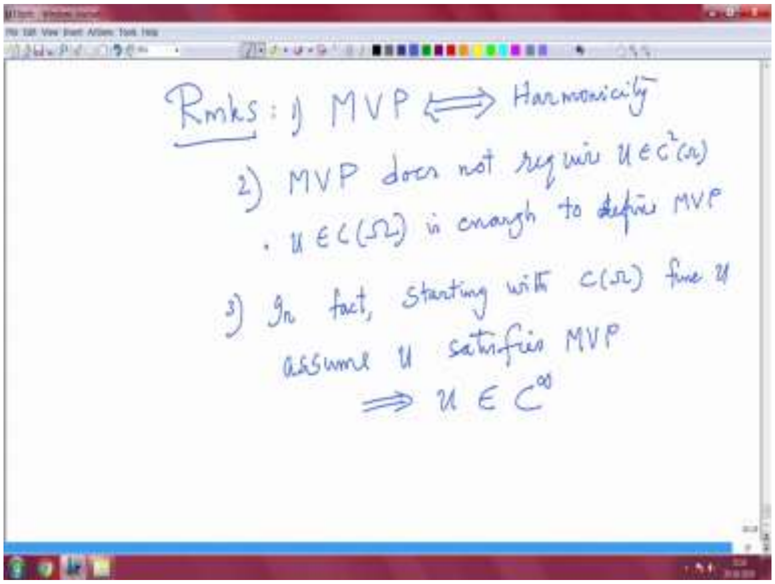

So let me make a comment now which we may not prove here. Remarks, so this converse implies in some sense mean value property is equivalent to harmonicity. That is a crucial thing. If it is harmonic it is mean value property satisfied, conversely if u is twice differentiable and mean value property is satisfied then Laplacian of that u is equal to 0. In other words, u is harmonic.

The second remark is another interesting thing. If you observe the mean value property, mean value property does not require that, though in the theorem I have added that, does not require u is in C2, u is continuous is enough to define mean value, is continuous is enough to define mean value property. So you see the mean value property can be defined even without having its smoothness.

So the question is that what can you say about if the mean value property has satisfied this thing. In fact this is a very interesting thing. If we get time we may prove this. In fact, so starting in fact, starting with a C omega function, starting with continuous function u define, assume, this is a more general theorem, assume u satisfies mean value property so you are not given the smoothness.

Of course in the theorem we have given the smoothness but what I am saying is that even without the smoothness of C2 smoothness, continuity you can define the mean value property. What this stronger theorem tells you that then u actually is the infinity. So you do not need to start with the smooth function to define the mean value property.

You can just start with the continuous function, then that continuous function is the mean value property satisfy that gives you u is infinity in fact, in particular it is C2 and hence it satisfies the harmonicity. So to prove this without this result, you, maybe we will indicate some other result later but then you can also prove use in the convolutions and mollifiers and all that which I will now do that one.

It is given in the reference of our book. So those who are interested should see this proof based on the mollifiers in convolutions. So I will stop here, so we will continue the proof of this converse to mean value theorem. Thank you.