First Course on Partial Defferential Equations – 1
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Lecture: 20
Laplace and Poisson Equations Part 3

Welcome back. And last class, we have trying to solve a problem. So let me recall the problem once again and then we will prove it.

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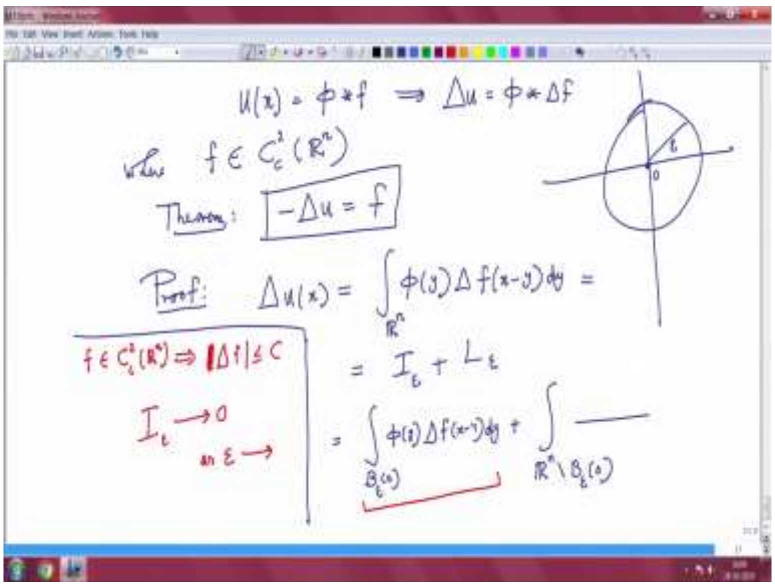

So what we have done is that we have introduced a function u x. This is a convolution of Phi and f. And then we have seen that this implies minus Laplacian of u equal to Phi star of Laplacian of f where f belongs to Cc 2 of R n. This is smooth, twice differentiable and compact support. So the theorem I already stated last time. So the theorem is minus Laplacian of u equal to f, where Phi is the fundamental solution.

So we are going to prove this today, now, in this lecture. So this is a kind of I am, going to give a full proof here with some exercises which I leave it here and there but then it is important that you complete the steps in between. This also tells you how to handle singularities. When you have a singularity, you have a difficulty in handling it.

So the proof is giving, proof is, not that it is that difficult, but it teaches you, makes you understand how you can handle your singularity. So let us recall this thing. So you have your Laplacian of you, so Laplacian of u of x. So you have to understand and take care of the variables correctly. So this is equal to Phi of y Laplacian f x minus y d y.

This is in R n. I may not write all the time. So this, I am going to write it in two terms. You have a singularity, right? I want to, you have a, I want to take care of the singularity, separately. That is the problem. So I will take these, suppose I have a ball here. And then this is the origin. So I consider ball of radius Epsilon.

So you have to understand the normal and all that. So I will write this as I Epsilon plus L Epsilon. And what is my I Epsilon? That is integral over B Epsilon of 0, that is where the singularity issue. So I want to get rid of the singularity slowly. Phi of y Laplacian of f x minus y d y, the same term but it is outside R n minus B Epsilon of 0. This is my L Epsilon. Same thing.

But now, look at the first term. So that I do not have to carry it to the, so let me change the color here. So look at this first term. f is Cc 2, f is in Cc 2 of R n. That implies, my Laplacian of f is known, if you calculate, this is bounded, whatever it is, Norm or f or modulus, this is some constant, modulus of that constant, independent.

So you can take that constant outside and then it is a local thing and you can see that a function is integrated on a set of small measure. That you know it, whenever it is a function f which is integrable in B Epsilon, so Phi is integrable in B Epsilon of 0 and a function is integrable, then on a set of integral value on a set of small measure will be small.

In other words, integral of f Phi, so you can estimate integral of I Epsilon, it will be a small quantity. So when take, Laplacian of x outside because it is a constant, you are integrating Phi over a set of small measure when Epsilon is small, B Epsilon of 0. So when Epsilon tends to 0, you will be integrating on a, even though there is a singularity, that is a meaning of local integrability.

Even though there is a singularity, when Epsilon becomes smaller and smaller, the integral value becomes smaller and smaller, if it is not locally integrable, this will not happen. So what I am trying to say that, I Epsilon will go to 0. So you do not have to worry as Epsilon, I Epsilon. In fact, I leave it as an exercise.

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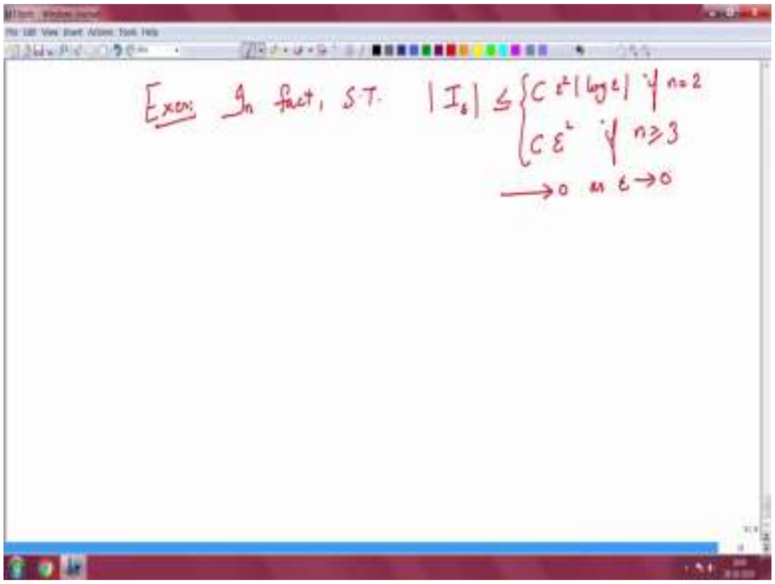

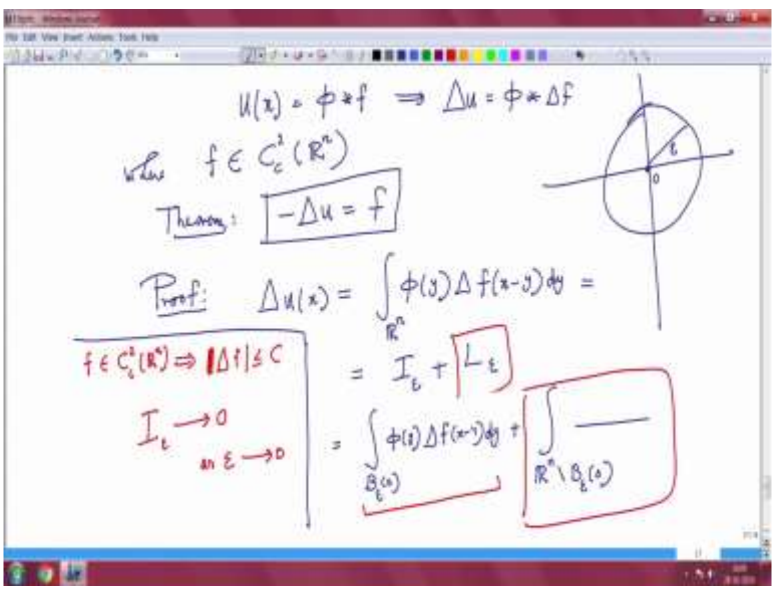

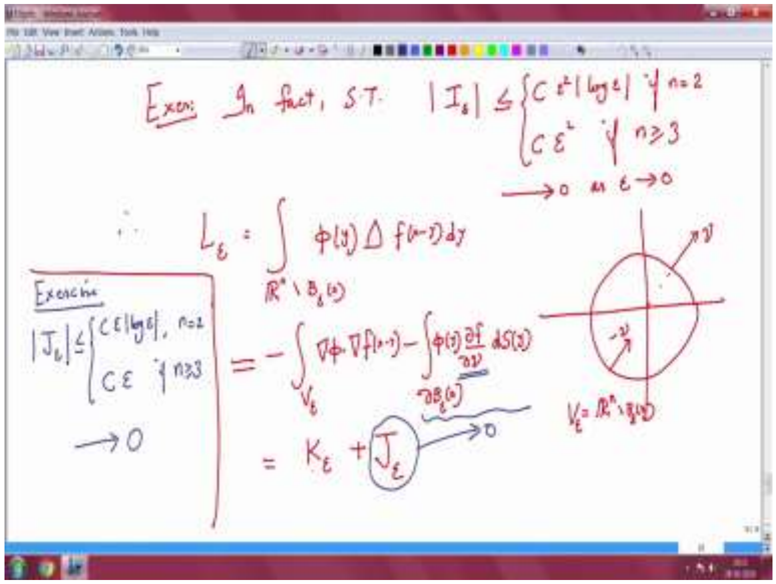

So this is your exercise. In fact, you can precisely prove it. Please prove this. In fact, you can show that modulus of I Epsilon is less than equal to. You prove this. Though you do not need this precise estimate to conclude I Epsilon go to 0, you can actually prove very precisely, some constant into Epsilon square into modulus of Log Epsilon if n equal to 2 and constant into Epsilon square if n greater than equal to 3.

Of course, in the first term, you get log Epsilon tends to infinity, mod log Epsilon. But then there is an Epsilon square here and that will make you go to 0, so this goes to 0. So you have a precise estimate as Epsilon goes to 0. So now, recall. You have only, these L Epsilon to be considered. So that is your L Epsilon.

So first time, therefore, I have to consider now my L Epsilon. So what is L Epsilon? L Epsilon is integral of R n minus B Epsilon of 0 Phi Epsilon, there is no singularity but now when Epsilon tends 0, it will go near 0, so you have to be careful while dealing with it. L f x minus y d y.

Now, do an integration by parts by Divergence Theorem, Green's Theorem et cetera. You have to be careful with the normal you are taking. So let me fix my normal. So you have your, here. So, I always consider, this is my Nu. By now, my R n minus B Epsilon 0 is outside. So if I take these are my normal, for this, the external normal is minus Nu.

So you have to be careful. External normal is minus Nu. So I fix always normal outside the B Epsilon ball but my integration is of R n minus B outside that. So it is external normal outward, unit normal. It is going inside the ball. So I will do my integration by parts. This, we have done it in the preliminaries, already.

So if I do it, I will, so let me write, this is equal to Phi Epsilon. I do not to write it all the time. So it is a V Epsilon, I can take my d here, so I will get my Grad Phi, Grad, this is dot product, Grad d, with respect to f and then I will get actually plus here but since the normal is minus, I will get again minus here.

Phi of y d f by d Nu, d by d Nu will be minus d by d Nu, that is why this minus here, and that is on the ball of radius Epsilon. That is a boundary ball. So you will have a ball of radius, this is about your d S y, surface integral. This, I call it K Epsilon plus L Epsilon. Now, not L Epsilon, so let me do it J Epsilon. I already used L Epsilon so let me use J Epsilon. So I use J Epsilon.

So careful, this is on a big domain, you may think that immediately this will, what will happen to this one. So what again, I will leave it as an exercise for you, so let me leave it as an exercise, all this, you have to work it out, unless, you do exercise, you will not learn the subject. Doing exercise is the only thing. So exercise, prove that my J Epsilon, you can compute all this. Again, there is no worry of d f by d Nu, this is bounded.

The problem in Phi, you know it exactly. So you compute that one and do it. Log Epsilon, J Epsilon will be less than equal to, so you see the power is reduced, here I Epsilon you get Epsilon square mod log Epsilon but you get only Epsilon mod log Epsilon because there some differentiation is coming, the amount is reduced n equal to 2 and other one, you get only C Epsilon if n greater than equal to 3.

Of course, this tends to infinity, mod log Epsilon tends to infinity, this tends to 0 but you know that this product goes to 0 so again this goes to 0. So you see, in the thing, so this basically or this goes to 0, this J Epsilon component, so only remaining with my K Epsilon. So you have split the whole integral.

If you recall once again, you have your Laplacian L Epsilon and L Epsilon, I Epsilon we have seen that it goes to 0 so we do not bother about it and then we have seen that L Epsilon is split into 2, in that case, J Epsilon goes to 0. The only remaining thing is K Epsilon. So you have to work out K Epsilon. That is what we are going to do now.

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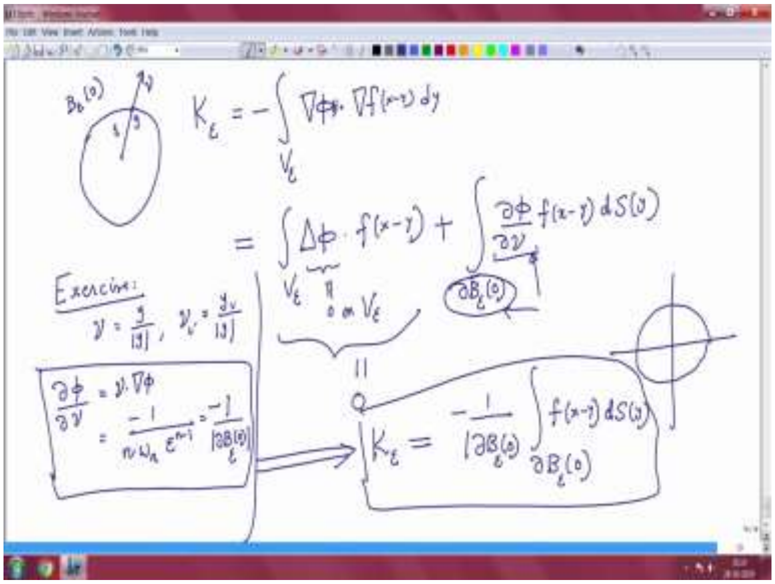

So you look at the K Epsilon. K Epsilon, now, you have to be a little more careful. Integral, you write it as V Epsilon, so let me recall again, from here. It is Grad Phi Grad f. So you see Grad Phi dot Grad f x minus y. This is y d y. Now, again, I will do an integration by parts. I can do that because V Epsilon has no singularity. So the singularity, I am fixing Epsilon.

So when I am doing it I am, so K Epsilon is with minus sign here, let me look at it. Yeah, there is a minus sign. So I have to put this minus sign here. So I can do this integration by parts because V Epsilon has no singularity and your Grad Phi and Laplacian Phi are all smooth in V Epsilon, there is no problem.

So I can do that one because singularity comes only when Epsilon goes to 0 now. That is why I removed that one. So I can do this integration by parts, I can get V Epsilon, this will become Laplacian and f of x minus y, again, the sign has to be taken care, in this case, it will be d Phi by d Nu and this will be one the ball of Epsilon 0 into f of x minus y d S of y, you see.

Now, in this case, this is not the one to go to 0. The one go to Laplacian of Phi equal to 0, this is equal to 0. So you see, this is equal to 0 on V Epsilon because it is 0 in R n minus, hence this term is equal to 0. So now, this integration is only on Epsilon but it is on the surface. So you have to be careful it will go to 0 or not.

So already you have seen that in one step the order of Epsilon reduced. But now you have to understand this contributes, how much is it is singularity. And there will be a singularity here in d Phi by d Nu, there will be a smallness here so it is a kind of trade off between the singularity and this thing.

In the earlier case, the smallness was bigger than the singularity, extent of singularity. In the second case also the smallness but then now a trade of the singularity coming from here is similar to the smallness here and the singularity and smallness will exactly get cancelled to produce something now real. That is what we are going to now, for a computation, want to do.

So here again, I am going to give you an exercise. So this is the thing. Now you can precisely calculate your exercise because Phi is a very given function. So you can actually. So you have a ball of radius Epsilon, right? This is for B Epsilon of 0. And this is, so if you have a ball of radius 1, a ball of radius 1, Nu is the thing, your normal, so this is only Epsilon, and you can see that your normal Nu, this is your normal Nu.

External normal, that is what you want to compute it. Your Nu will become, y is not a unit vector anymore. If this is y, this is not y, so it is y by mod over, or Nu y will be y I by mod V. And you can compute your d V by d Nu. Your d Phi by d Nu is, in this case, is Nu dot Grad Phi.

So compute, this Grad Phi, I already told you to complete in the earlier lectures. If you do this one, this will be exactly minus 1 by n Omega n Epsilon power n minus 1. This, you already seen that this is nothing but the, these are all, we have computed in the previous lectures. This is nothing but the area of a radius Epsilon and centered at t. So you can exactly get that. So do not forget that one.

So you can exactly compute your d Phi by d Nu, that is nothing but minus 1 by, so if you write down your K Epsilon here, so this will become, so this implies d Phi by d Nu, that is independent of that one, that comes out d B Epsilon of 0 integral over d B Epsilon of 0 into f of x minus y, you differentiate with respect to y. So you see, this is what your K Epsilon is. Now, so I want to compute the limit of that. I can, this is a very familiar thing, it is basically an average.

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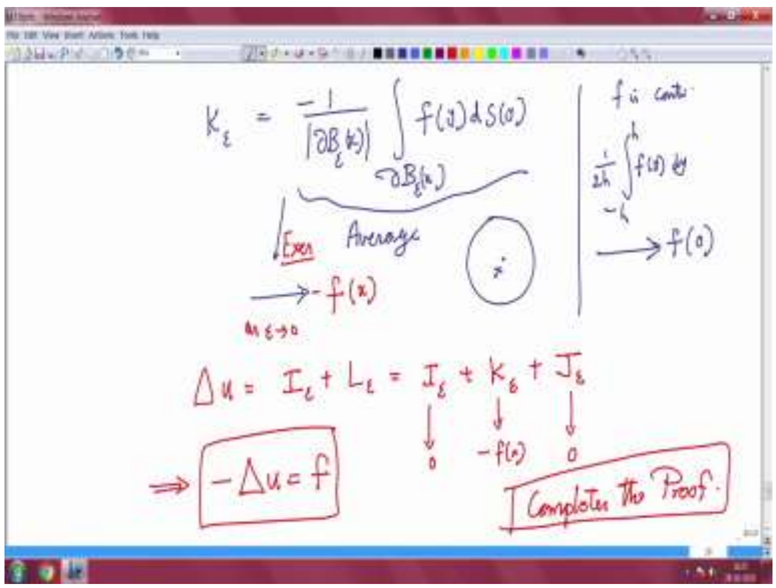

So that K Epsilon becomes minus 1 by, I can make a translation. So I will get, when you do that translation, I can translate to the origin, B Epsilon of 0, integral of d Epsilon of 0 f of y, no sorry, that is what, so I can make a translation, oh everything has gone. So this is not B Epsilon of 0, I can translate that to the origin, I can write it as x here, I can make a translation. I can make a translation of x f of y d S of y.

And look at this one, this is nothing but the average. And you know that for any continuous function, that average goes to it is value. This is a well known thing from an analysis. For example f is continuous and integral of 1 over 2h, this is the average. You try to prove this, if you have not seen it. f of y d y goes to f of 0. This, at any point, you can do that as long as f is, so you are taking basically any point x and taking average along the surface of the function so as Epsilon tends to 0.

So if you have not seen that, you see this as an exercise. You have a lot of exercises to do it, this goes to f of minus, there is a minus sign, f of x as Epsilon tends to 0. So this proves the result. So you have written, so what you have written is, Laplacian of u equal to I Epsilon, so recall, I Epsilon plus L Epsilon. That is what we have started with it and then that is written as I Epsilon plus K Epsilon plus J Epsilon.

And you have seen that this goes to 0, this goes to 0 and this goes to minus f of x. So that implies, minus Laplacian of u equal to f. You got the solution, completes the proof. So you have already seen one theorem in this direction but then more than the theorem understanding, you have to understand the small, small proof proofs which you are doing and how these singular points contribute to it. So you learn all these things, in the formula.

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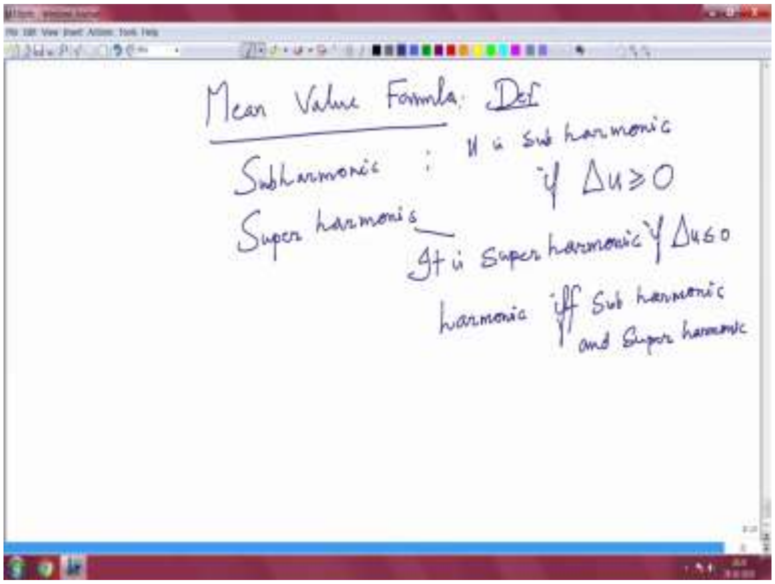

Now, in the, just let me set up for the next class. So the proof is complete for one case. When you have an f is a Cc 2 function and Phi is a fundamental solution to your minus Laplacian then it is convolution with f solves this. Of course, you have an issue where you have assumed that f is Cc 2 of R n. And it is a solution to, in the Poisson equation. So what you have done is prove a solution to the Poisson equation in R n, in a strong assumption when f is Cc 2 of R n.

Whatever aim is to do a boundary problems, when Laplacian of u equal to f in Omega and then u equal to g in d Omega. That makes that life is much more complicated. So what I am going to prove in the next few lectures is first, I will introduce what is called a Mean Value Formula. In this setup, you will understand what is a Sub harmonic function. So you will understand the meaning of Sub harmonic et cetera as we go along.

And we will call it super harmonic. So sub harmonic means u is Sub harmonic, this is definition, u is Sub harmonic if Laplacian of u is greater than equal to 0, it is Super harmonic, Super harmonic if Laplacian of u is not equal to 0, and harmonic, of course harmonic, if and only if Sub harmonic and Super harmonic, and Super harmonic.

So what I am going to do is derive some average values, if captures that, some nice formulae of harmonic functions and later you will see that harmonicity is equivalent to mean value property. That means the averages will determine the value at the center, any point. So it can get back your values of u by averaging the value of u for a harmonic function. So it restricts, you cannot have some arbitrary functions as harmonic.

So I think I will stop here instead of starting something new, and which I will do it in the next class. Thank you.