Partial Differential Equation - 1
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Lecture - 02
Introduction - 02

Welcome back to the course. In the last lecture we were given a brief introduction about our course in the first lecture and then we wrote the general kth order PDE and we were trying to classify the PDE.

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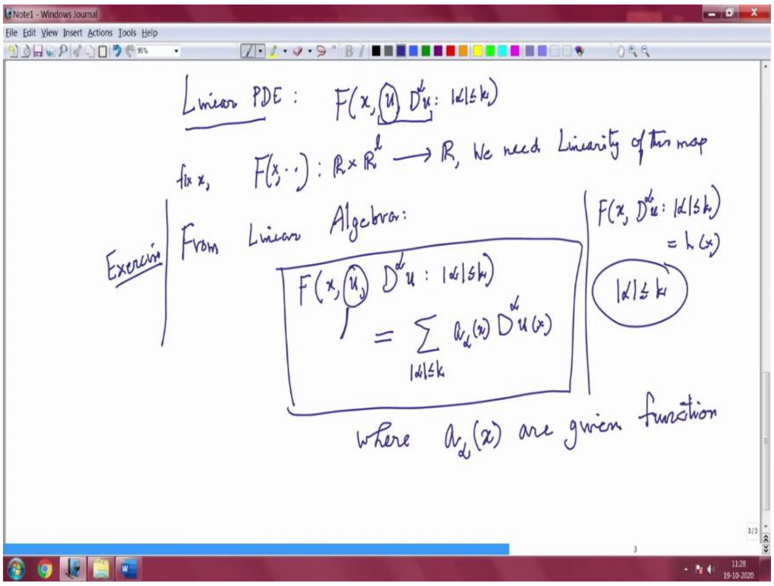

So, we introduced a linear PDE let me recall what is mentioned there for a linear PDE you the this is last slide of the last class. Any linear equation can be written in this form where so maybe I will use the same color, where a alpha of x are given functions you have to understand, are given functions, u is your unknown this is given. So, let me go to write a first order PDE in that case.

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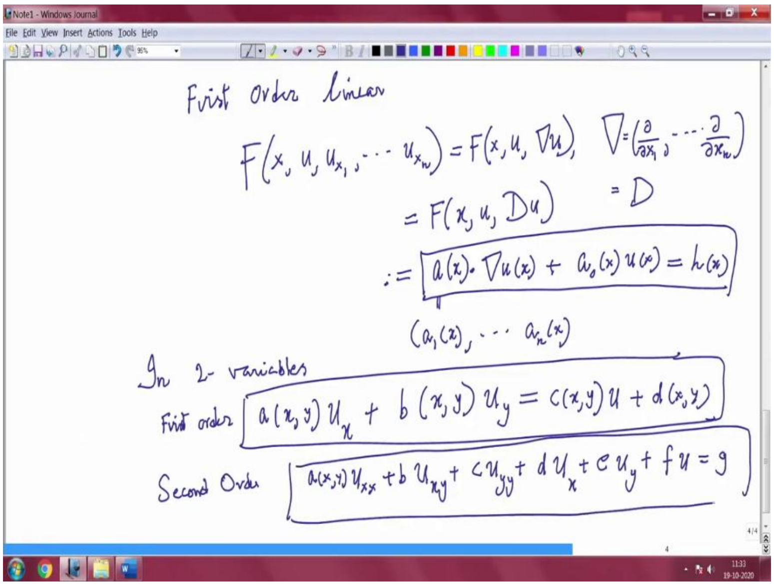

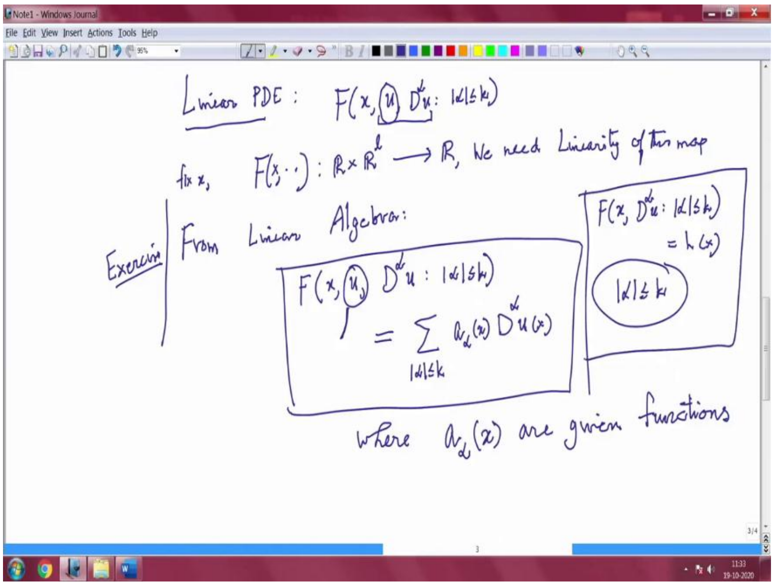

So, let me write a first order linear because this you will be using first order linear. In this case your F you will have x you will have your u you will have your u x 1 if you want etcetera u x n. So, many books you can write it whatever way you like it I can write x here I can write u here I can write grade u here.

So, sometimes we use for grade is nothing but this vector this we are going to use it so let me have the introduction as and when necessary we will introduce more notation this we will sometimes also use it at D u. So, I am giving you various forms which we may use different type. So, u here D.

So, the first order equation will be of this form you will have a vector of function say x this is a vector function a 1 of x etcetera, a n of x and then you are taking its dot product with grade u. So, that covers you will and then you have your first order term so this formula format we will be using it a naught of x into u of x. So, your first order this is your F by definition and your first order PDE will be some unknown given nonhomogeneous term so you see you have your first order term.

So, when you write it as in two variables, so let me do that also in two variables, this is much more easy variables, we are going to use this so later in the next lecture. So, it will be of the form a of x y normally when it is two variable instead of x1 x2 in most of the books they use it as x1 comma x2.

So, or you can also use it x comma y a of x y into u x plus b of x y into u y so you can write it for convenience which I am we are going to use it so I will write it this form does not matter where you write it because c can take the side what you want u plus D of x y. So, this is your first order equation two variable first order.

What about second order, this is first order, how does second order in two variable, second order? Is easy, similarly, a, a will be a function of x and y we may not write all the time x y, so a of x y u x x so you will have b into u x y you will have c where a b c are not constants need not be constant c into u y y plus D u x plus e u y plus f u equal to g so you see you can write here.

Some of the terms may be absent not that every term this is the most general form of first and second order equations, the last two are the most general form of first and second order linear equations in two variables. And the first order equation this equation, this is the most general form of first order equations in n variable. That way you can write second order, third order but then when you go it, it will be more and more difficult, that is how the notation is so comfortable you see here, so you see that is why this notation of nth order equation is very, very comfortable. This is the thing, so that says.

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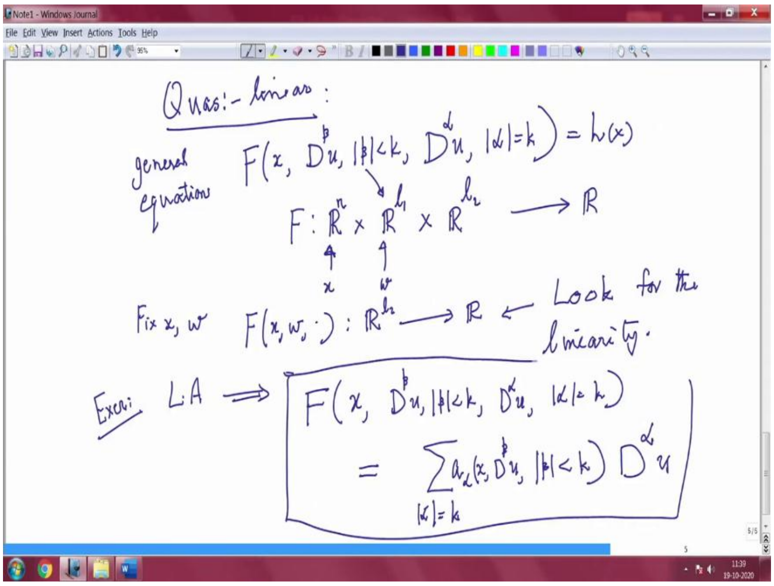

Now, go to the quasi linear equations, we will do quasi linear quasi linear. Basically, we are looking for see in the earlier case you look for the collective linearity with respect to all the unknown and its derivatives u the ux, ui, u xx and all variables and thing you look for a collective linearity.

What in quasi linearity you look for? Is the linearity with respect to the highest derivatives. So, we I can write my F of x a general equation, general equation I can do it in a different style general equation, you have x, I will collect all my derivatives D beta of u with the mod beta strictly less than k, I can write. So, this D beta of u consists of all derivatives less than the order k, not the highest order k and then I will write you can also write D alpha of u with mod alpha equal to k you see equal to h x.

So, you have this equation, I am same equation, I am writing I am separating the terms involving mod alpha equal to k and separating the words thing. So, in this case you can view again the same map you may be able to view from some Rn and then this corresponds to my x variable, x belongs to that, that is for x variable or some domain omega and this may be some, this is not the previous l it may be some l 1.

Where this belongs to and this belongs to maybe some R power l2 to R. And now fix these two things so you can view that F is a mapping, fix the x, fix some vector here it may be a mapping the unknown it you consider that as a mapping because when you have u x D u by D x for each x it is a value so you can view it from here so I may put some number say w so I will fix x w, fix x w and then view it as a mapping from R l2 to R.

And I look for the linearity of this map look for the linearity. So, again from linear algebra the linear algebra if you, do now again prove this exercise. So, from linear algebra I can write my F will be x D bar beta of u mod beta less than k D power alpha of u mod alpha equal to k will become because I need a linearity with respect to D power alpha of u part only.

So, only that D power alpha of u with the mod alpha equal to k gets separated. So, this will be of the form summation you will get only D power alpha of u with the mod alpha equal to k. And then my coefficients which is given will depend on not only x it will also depend on my D power beta of u. So, I need some space so let me write this D power b so mod beta less than k into D power alpha of u. So, you see my quasi linear equation will look like this. So, let me write down the first order so what about the first order now.

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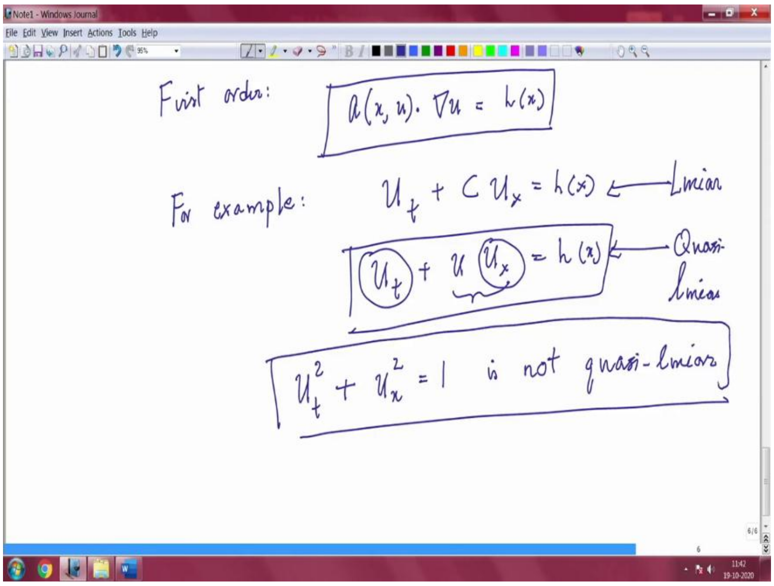

I need linearity, I do not need linearity with respect to u I need a linearity only with respect to grade u. So, your first order equation and a earlier if you looked at it, a depend only on x not on u so, if we wrote in the first order linear equation a naught of x u plus a of x a grade dot u. But you need only with respect to grade u. So, this is how it will look like you see so you can write down the first order second order any order so you experiment with it if you wish to learn it and write down.

For example, this is a very classical example anyway you will be studying and again and again use it, u t plus which we are going to next class c u x equal to say h x this is a linear equation I wrote it because we want to study linear. But then this is the conservation law u, u x equal to h of x this is quasi linear, quasi linear. You see you look at this term because of the product u and u x this is not linear but then in the courser linearity you would only look at these terms, which is got separated but if you mix it u t and u x then it will not be quasi linear.

So, it will not, say for example if I write it u t square plus u x square equal to 1 is not quasi linear. Keep this in mind. So, now you have a very immediately the differential equation is given without even verification look at the differential equation stare at it, you can immediately write down whether it is a linear equation or quasi linear equation or general nonlinear equations. Anyway we are going to study these things more and more. See in the next 15 minutes or so next to 15 minutes or so I want to make some remarks.

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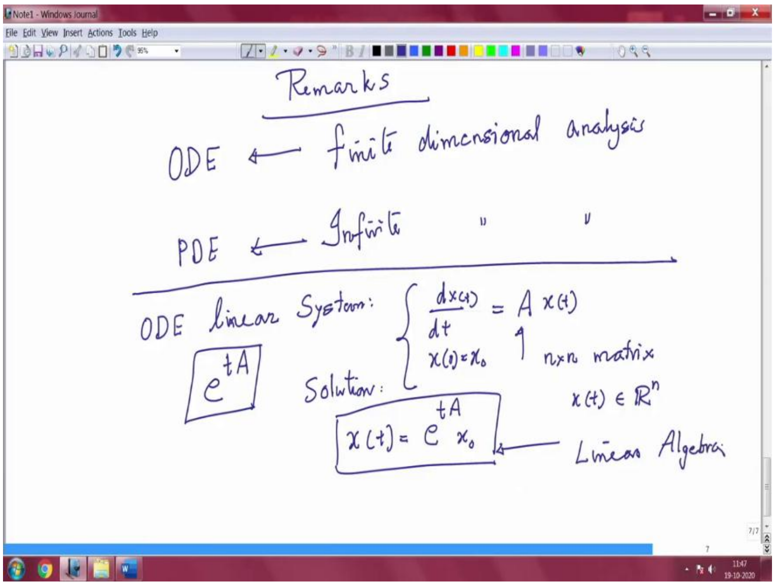

This is a general talk remark, remarks. The study of PDEs is not that easy compared to the study of ODEs. You require more sophisticated rules quite often there is a kind of feeling that PDE consists of two independent variables, ODE only one variable so it is a reduction from two variable analysis to one variable analysis. But that is not the way to look at it, so if you look at it ODE and PDE, you will eventually get to get the idea, ODE is basically a finite dimensional analysis that is what you have to see.

Even though one independent variable ODE you do the analysis in Rn because you will have a system. So, ODE is a finite dimensional analysis, whereas, PDE is an infinite dimensional analysis, infinite dimensional analysis. That makes the things whole analysis different. You know that in finite dimension you have the analysis, there are many things which you can do it, there are many results like the one of the major issues are the, their compactness and you have nice compactness theorems, which you can recognize by close bounded sets.

On the other hand, in the infinite dimensions, things are not that easy. So it is not about ODE is a one variable differential equation, PDE is a two variable differential equation, two independent variable equation. The analysis is based on where you look for solutions so here you look for basically solutions in the finite dimension, the spaces which you soon come to see the spaces which you are working for PDE is come are infinite dimensional spaces.

You will get to see very little in this course that is what you needed more sophisticated tools for to study PDE. That is a one remark, I immediately want to tell. The second most important remark which I want to tell about the study of even linear equation, so look at ODE system. ODE linear system, how do you write an ODE linear system dx by dt x is a function of one variable one independent variable that is equal to some a into x t.

Let us look at the autonomous system, that means a system with constant coefficients basically. So, A is an n by n matrix and x t is a vector is in Rn that is what I said, x t is in Rn for each t. Then you have a concept of exponentiation, so given a matrix A you can define what is e power t A, can define and you can write your solution, at least you can write in principle, you can understand that solution is a different issue and you can write your solution x t is equal to e power t A into x naught, where x naught is your initial value.

So, you see, so this is an initial value problem basically and you can write our solution, so any linear system, n by n linear system with constant coefficient you have a representation of solution, you can immediately write down the solution you can, now you can to understand the solution you need linear algebra that is true. But still a good linear algebra course you can understand a reasonably well about your ODE system. But, the life is not that easy with non-linear systems.

Even if you choose linear system two variable linear system just two independent variable linear equations in, in second order forget about the highest order, nevertheless there is a good theory about the first order equation, first order PDE there is a good theory and that is what we will be covering in four lectures.

We will see after this but then the moment you go to second order linear, so you are making the, your life easy two variable, two independent variable linear and then in equations and then you can see that you do not have a general theory. So, I just want to make that remarks, some comments to distinguish that, so let me write down your three operators.

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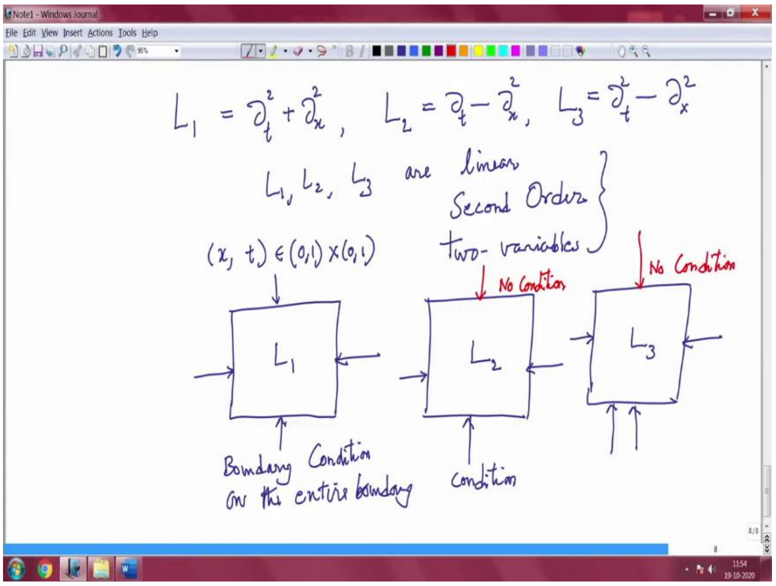

L1 is equal to dt square dx square or dt square whatever it does not matter. So, write down dt square plus dx square one operator this is you see t and x are independent variable, only two variable in the second order equation linear because we separated it out and L2 dt minus dx square and L3 dt square, so let me use the same notation dt square minus dx square.

See all these equations L1, L2, L3 are linear you have all easy generally in general linear equations are much more easy to handle than non-linear equations. Second order equations are better because there are second order, then first order fine but and then only two variables you see not too many variables, linear second order, so I am putting everything easy thing and only two independent variables.

So, making the life the most thing. I want to know variables. So, I am trying to consider an equation I you will see this soon in our this is what these three equations are what you are going to see it because eventually you will see a classification, how do you really classify different equation? So at least in that class you have a general theory and that is what you want to obey.

You cannot give a very general theory even for linear equations a common general theory like in ODE what you have seen but can we classify into different things, can we find a family of equation, a class of equations for which it behaves similarly. And in eventually you will see that these are the representatives of different class of equation that is important but how do you just given a the second order three, these three equations how do you distinguish because you see only a change of some signs.

So, let us say just I want to comment something you will see later so let us look at this equation where x t is in the unit square 0 1 cross 0 1. You what you will be seeing in this course not all these things here I want to see how only to convince you that imposing conditions, initial conditions and boundary conditions are not that trivial what do you put conditions, what kind of conditions do you put there, which problem it looks like a thing and probably this is because of the diverse nature of physics scale problems.

Differently problem physical problems even though it may lead to nice linear equations it will not be handled in the same way so the physics and physical phenomena may come into picture. You have to not understand that just this classification is not enough. For example, if I look an L1 operator here, this is the L1 operator here, you need to different conditions here, one condition each in all these places. You have to put boundary conditions everywhere, boundary conditions, on the entire boundary.

But on the other hand, if I take the same equation, on the other hand if I take the same domain and I consider the same and if I consider the L2 operator here I need a condition here, I need a condition here, I need a condition here, condition to be imposed to get a good theory. On the other hand, you should not put a condition here, you have to be careful, no condition. You may ask how it is done that is what the difficulty. On the other hand, if you look at the third equation L3 you need a condition here, you need a condition here, but you need two conditions here, but no condition.

So, you see very simple equations the conditions there we, I am not looking at its physics so this forces you need to understand these equations well you do understand how do you put conditions. So later as we go along in this course you will see that these equations are coming from different physics. For example, the L1, L1 even though it is a two variable equation it is actually a two variable equation in the equilibrium state of a two dimensional phenomena, so you have an two dimensional phenomena which is dynamics and time is coming into it.

So, two (dimensional) independent variable where t and x are both a special variable with a time variable third variable and it is equilibrium state of that, so it is a two dimensional problem. On the other hand, the L2 and L3 that t is a time already in the first one t is not at time it is a spatial variable the second and thing physically t is a time variable and x is the spatial variable.

So, it is a phenomenon from physics of the one dimensional phenomena and third one is also like that. So, it looks like an another way of distinction differential operator, whether it is an equilibrium state or it is a time dynamical system of equation. So, L2 and L3 you feel that maybe it is a similar type of equations yeah you cannot judge immediately. With one more last thing last comment maybe I will stop here.

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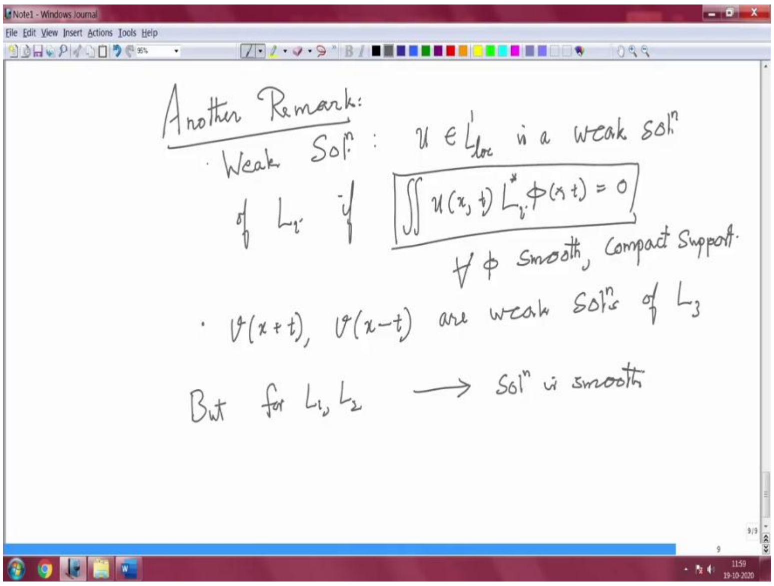

So, this is one observation which you do it so we want to have a another observation of the same another. This is not easy to obtain another observation only I am making comments which you will see in the course. I do not call it observation because we are not observing it we are only remarking it another remark.

Another remark is regarding a kind of weak solution you can define a, another solution. So, you ask a integrable, locally integrable function I am not telling anything, locally integrable means it is integrable in some compact set, it is finite in a compact set, you will see. So, it is not you I am not assuming u is a differentiable function is a weak, weak means you are is not directly verify it.

So if you do not understand right now it is okay, but I want to make this solution of Li if double integral wherever you are having omega cross or whatever it is u of x t into some Li star some opera (integra), this is basically some integration of paths f x t equal to 0 for all phi smooth some smooth enough c infinity if you want with combat support that is more important.

Because otherwise that terms will not be there, compact support. So, you have an another weak concept of a weak solution, so these concepts you may not learn much but we will try to indicate here and there and that will be part of a third another PDE course as I said you need more sophisticated tools.

But what is the remark I want to make it here, you can immediately see that you take v of x plus t locally integrable function or you take v of x minus t are weak solutions of L3 you can get. That means you can immediately get non smooth solutions in this sense in the sense of this definition you can get non smooth solutions but for L2, L1 and L2 nodes the it will automatically give the solution is smooth.

This is just to show you L1 and L2 has something common now earlier you have seen that dynamical way where L2 and L3 are somewhat common because you said I told you that you have to view it as a dynamical system and you see that one boundary is removed in putting the conditions. So, you view that t is a kind of ODE operator, x is the spatial variable but here you see the L1 operator and L2 operator in some sense is a smoothing operator.

So, even if you are looking for a non-smooth solutions in a different sense you the such solutions turned out to be smooth. So, you will see that a drastic difference when you study the wave equation, so the L3 is the wave equation basically it is a hyperbolic equation L2 is a modeling of a heat phenomenon, is a heat equation which is a parabolic equation and L1 is an elliptic equation it is a Laplace equation.

And so we want to, so the course is to understand these three equations and first order equations in as in a minimal way I am telling you, so we will only trying to touch the ocean of PDE is just a question of touching and trying to find a small hole on the surface so you can see that PDE, the vast amount of PDE to be studied.

I think the so as I said you need to have more sophisticated tools to study and one of the major sophisticated tools if you want to study a little further even not very advanced, even if you want to have a really not this PDE 1 and PDE 2, told you PDE 1 and PDE 2 is a combined course of a one semester, but if you want to have a real one another second semester course you need more tools from functional analysis, Fourier analysis and then some good amount of differential geometry because if you know some good amount of differential geometry you get a better feeling of your PDE.

So, with this we stop this lecture and next probably two hours or maybe one hour, maybe two hours we will tell you some preliminaries. I told you we cannot do elaborate preliminary discussion here because the preliminaries which I will be discussing is a some of the courses you would be studying in your masters if not please go and through that cover your basics understand it in a better way. So meet you in the next class, thank you.