First Course on Partial Defferential Equations – 1

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Lecture: 19

Laplace and Poisson Equations Part 2

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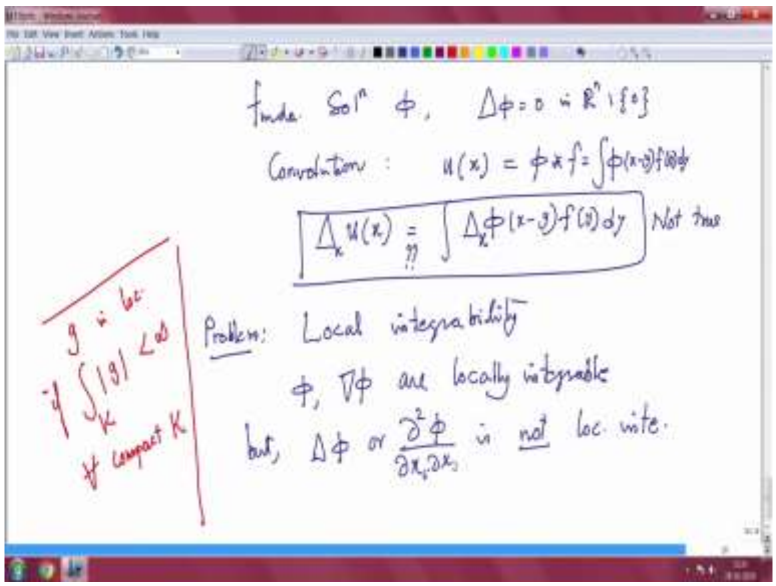

So in the last class we have introduced what is called a Fundamental Solution, fundamental solution Phi. It has a singularity and you have a seen that Laplacian of x, Phi equal to 0 in R n minus 0. Then we also introduced what is called a Convolution. I did not give anything about whether Convolution is meaningful but formally we introduced a function u x equal to Phi star of f.

That is nothing but integral of Phi of x minus y f of y d y. Then we are trying to do something. We are trying to differentiate u with respect to x two times and sum it. To do this one, you need d u by d x i and again do it. The question is that, the question we have asked even if the convolution is meaningful can you do the differentiation Laplacian of x Phi of x minus y f of y d y. But this is not correct. Not true.

The problem is to do such things you need what is called a local integrability. So, I am going to discuss something, local integrability. So Phi, I will, this we will give it as an exercise but we will explain to you something. Phi, Grad Phi are locally integrable but the Laplacian of Phi or d square Phi by d x, the second derivative is not locally integrable. So what is the meaning of local integrability?

Well let me, all of you know it. f is, or g, let me put g is locally integrable if integral of mod g is finite for all compact K. So we want to discuss something now, interesting.

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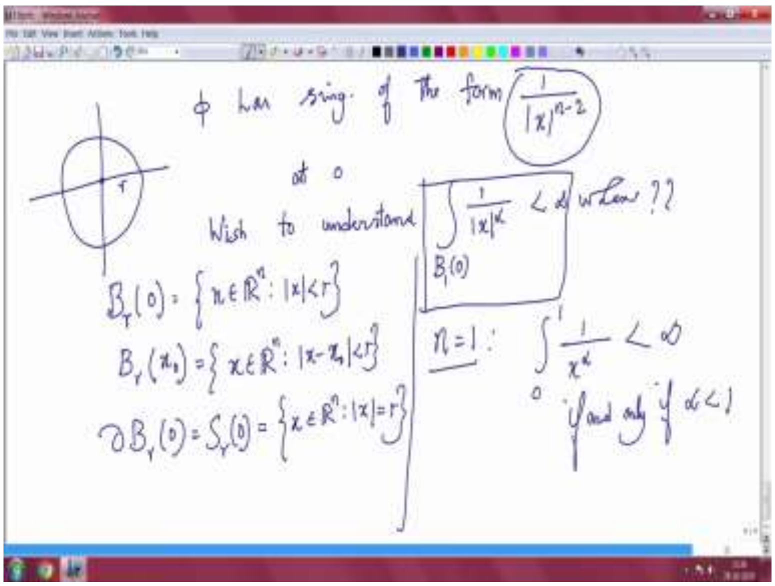

So I want to, if you look at it, Phi, Phi has singularity, of the form has singularity of the form except n equal to 2, you have singularity of the form 1 by mod x power n minus 2. So this is the singularity type. So the singularity is at the origin, n at the origin 0. So you have to see the local integrability in a neighborhood of 0.

So basically you want to understand, so you wish to understand this one, wish to understand integral over a ball of radius, in a neighborhood of 0, any neighborhood, so let me take a ball of radius 0, B1 o is 1 by mod x power Alpha when, is finite, when? This is the question. Where B1 0 is the ball of radius, which you know it, is an open ball of radius x in R n, more generally B r of 0 x in R n such as at mod x less than R.

This is the ball of radius R. This is the ball centered at the origin. So similarly, you can define what is B r of any point x not, if the ball of radius x in R n, such that mod x minus x not less than R. And its boundary, you can have it, boundary of ball of radius 0, this is also S r of 0 if you want, the surface. The surface is, which you already, set of all x in R n and such that mod x equal to R.

So you, since the singularity of 1 by mod x power is at the origin, it is enough to check near the origin. In other point, there is not need to check because the function is smooth till infinity. You do not have to worry about it. So you only have to see when this is. So, what about when n equal to 1? This, you know it. n equal to 1, it is 0 to 1, 1 by x power Alpha.

You know that, this is familiar to you. This is infinity if and only if, when Alpha less than 1. Of course, when Alpha is negative, no problem because then if there is no singularity, it goes up. Only when Alpha equal to 1 also there is no, 0, there is no singularity. Singularity comes only when Alpha greater than 0. So you have to check only the singularity at that point.

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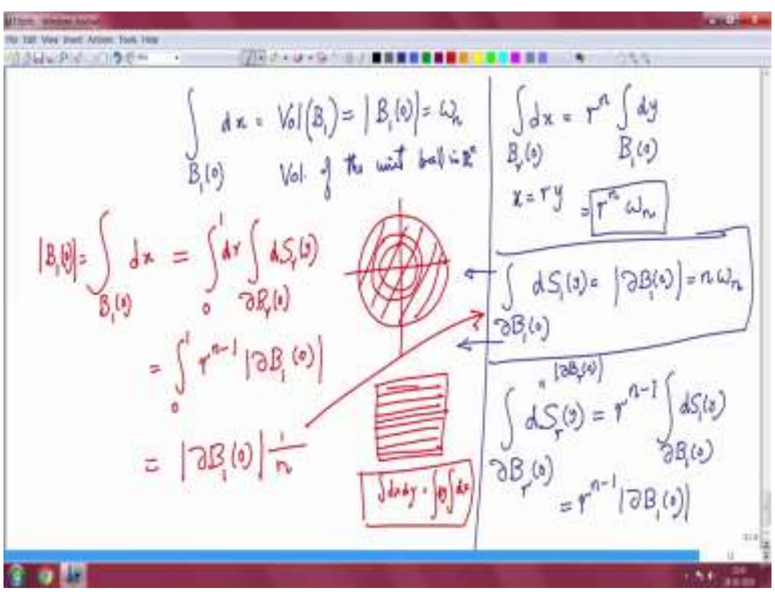

So I want to also tell you few more things because we will be using it frequently. So if you integrate without that one by mod x power Alpha, on a ball of radius 0, d x, this is nothing but it is volume of the ball of radius, volume of B1. So that is, there are different notations. So you write it as Volume, this, you have a precise value. You can refer some books. This is also a notation. We call it, it is the volume of the unit ball. Volume of the unit ball in R n.

You can actually, what I want to tell you here, if you think, how do you make some change of variable. If you have a ball of radius R 0 with d x, you can make a change of variable x is equal to, what do you want to put it, so you want to write it x is equal to, you make a change of variable, x is equal to, make a change of variable, x is equal to r y.

So when x varies from 0 to r, y varies from 0 to y but then d x will become r power n. So this is r power n with B, ball of radius 0. So all this, you should be familiar. B1, that is nothing but r power n Omega n. That ball of radius. It is better to know all this. Similarly, you have a surface area. If you have a surface, you are integrating, you have learned now, the surface measure.

So if you have d of B1 of 0, so you have your, differentiating, integrating, not differentiating, with respect to the surface measure around the unit, so I just mention d S1, this is with respect to y, this is nothing but the surface area of the surface, d 1 of 0. And you can show that, this is nothing but actually, you can prove it.

I will come to that. Maybe I will give a proof of that. You can see that one. And when you do, these are all formulae you have to be familiar, d B r of 0, how do you make these things? of d S, this is a surface measure on the surface of radius r of y, d y, not d y, there is no d y, we already got d S, so we already have d S of y.

In this case, you got r n Omega n, here you will get r power n minus 1, so you, because you are only differentiating, integrating with respect to the surface and surface has the, n minus 1 dimensional object, so you will get only, you can see that the surface on d S, ball of radius r is r power n minus 1 of surface measure of n of y of d B1 of 0.

So this is nothing but r power n minus 1, the surface measure. So the surface measure of ball of radius r is this r power n minus 1 into surface measure of ball of 0. And this we are already telling that how do you do that part. So this is, this formula, this is nothing but the surface area of the ball of radius. By shifting, it will not change. So let me give you, how do you do this part.

So let us compute, so the proof of this, let me give, quickly, because this, we will be using it again and again. So let me use it in a different color. So you have your B1 of 0 d x, this is your volume of the unit ball. That is what I am saying, the surface area is n into, the surface area of the ball of radius 1 is n into the volume of that one. So this, we use Fubini.

Here, I want to comment, this is nothing but 0 to 1 d r integral the ball of radius r of 0 d S r of y. That is natural. So what you do is that if you have a surface and you want to understand the whole volume, you want to understand the whole volume, so you understand the surface area, all these surface area and integrate. This is typically the Fubini Theorem.

That is what you do it. If you want to integrate something like d x d y, what do you integrate? You integrate with respect to all these lines, right? That is what, you, Fubini's Theorem tells you. You write this one as, something like integral of with respect to d x and then you integrate with respect to d y. So you know the condition.

The same, I am doing in so you, surface area, you do that one. So this is equal to integral 0 to 1. You already computed that r power n minus 1. And this is the surface area of ball of radius r equal to 0. B1, so that is, this is not on, this is B1. So now, you integrate, this is surface area B1 of 0, into r power n by n, that is equal to 1 over n. That is same as this formula, you see? You have d B1 of 0 equal to n into that. That is why.

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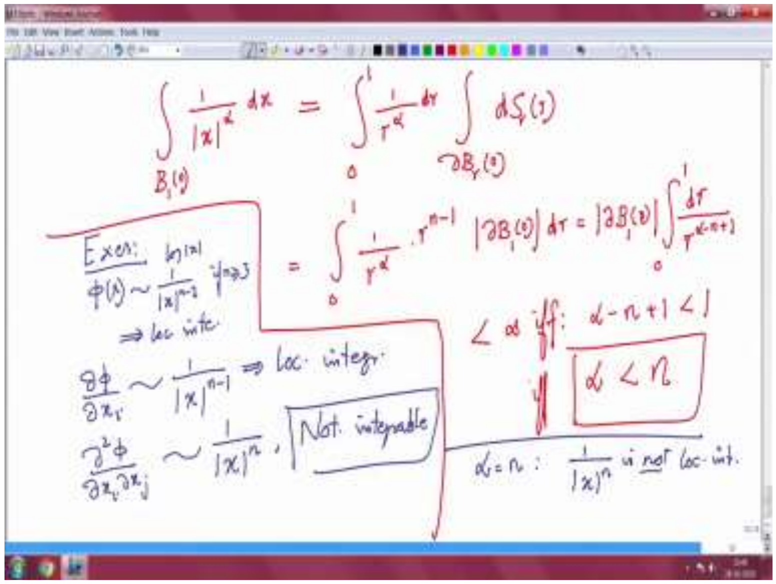

Now, let us go to our case. I want to understand this one. Integral of 1 by mod x power Alpha over B1 of 0 d x. I use that. This is equal to integral of, I use the same thing, this is a radial function so this is nothing but 1 over r power Alpha d r integral of d B r of 0 d S r of y. So you should know how to do this one. But that is the volume.

That is a surface area. So you will have 0 to 1, 1 by r power Alpha and this is nothing but r power, this is nothing but r power n minus 1 plus volume 1, d B1 of 0. This is independent of r. That, we have already computed into d r. So this is equal to modulus of d B1 0 integral, so we reduce it to a one dimensional integration. That is what we have done. Since it is a radial function, alpha minus n plus 1 and this is one dimensional.

And this is finite if and only if Alpha minus n plus 1 is less than 1 or if an only if Alpha less than 1. So this type of singularity, integral of 1 by mod x power Alpha d x is finite Alpha less than n. Now, look at here. So this is an exercise, I am giving you. So you have to do these exercises. Phi x you already know it. So you look at Phi x.

Phi x is, behaves like 1 by mod x power n minus 2, if n greater than 3. n equal to 2, it is log mod x. This is locally integrable, by this reason. For n equal to 2, you can do it separately. Now, you compute d Phi by d x i, you can see that this behaves like 1 by mod x power n minus 1. Again implies, locally integrable. These are some exercises.

But you try to compute d square Phi by d x i x j. You can see that this behaves like 1 by, you just miss it, when Alpha equal to n, it is not locally integrable. You have to see that Alpha equal to n, 1 by mod x power n is not locally integrable. So this is not locally integrable. So you cannot do what we were doing in the integral.

So this is the difficulty, you will see the affect of, so at just missing the local integrability. If something like mod, 1 by mod x power n minus Epsilon, then it will be locally integrable, so you just Laplacian and you take second derivative, you miss by any small number. And you will see the effect of this when you study the Newtonian Potential, perhaps not now, maybe in the second course and leading to the introduction of Sample Spaces.

So this is a small singularity which creates trouble when you take the setting, at the same time, it is stores some information and that prevents you from taking integral under the integral sign and it contributes something. So we are trying to understand it is contribution, what this singularity contributes. So we will do this. So please do these exercises. It is important that second derivative just misses local integrability.

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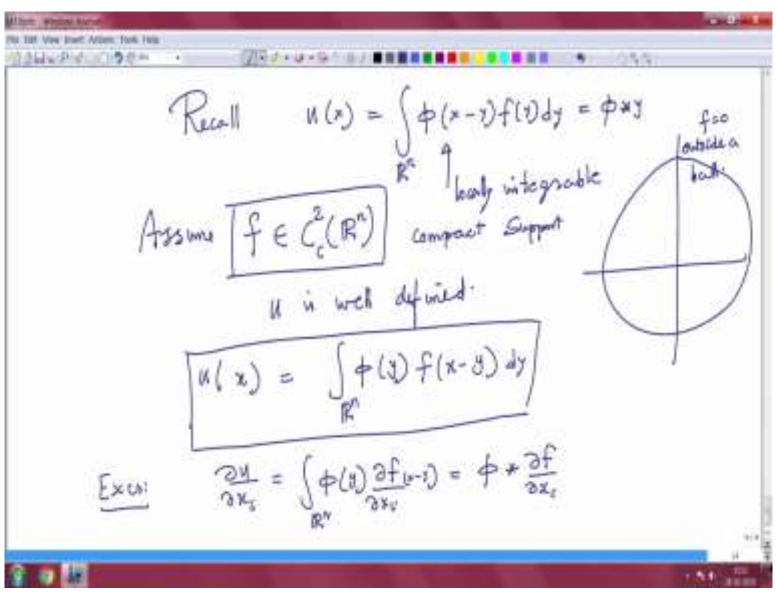

So let us not try to do some results. Being studied this much, let us try to do some things, some interesting things. So we have your, so recall now. So this is the trouble, recall u x equal to integral Phi of x minus y f of y d y. This is not that bad. Only thing, you are not able to take the derivative, two times derivative inside. This is locally integrable, now you know that. So you are doing this R n.

So I want to give you the meaning, R n locally integrable. I do not want any trouble from f. So assume f is a continuous function. So that has no singularity. And also, there is, the trouble from come from infinity so for which, I will use C c, that is, compact support. Compact support - Compact support means, you have something, f is 0 outside a ball or a compact set, whatever it is.

So f is 0 outside a compact set, so at infinity, there is not trouble. And Phi is not outside, throughout smooth, so Phi is entirely smooth, it has only singularity. So the only trouble is local and we already seen that f is locally integrable. So this is u is well defined. That is what I want to say. With this assumption of f is C c or R n and by our proof that Phi is locally integrable,

The f has no singularity, so Phi of x – y f of y d y, f of y, has no singularity. So in any finite compact domain, there is no issue and then outside f is 0 so then u is well defined. So the only problem is, because of this singularity, you may not be able to take the derivative. Again, there will not be a problem to take one derivative.

The issue is that when you want to take two derivatives, because then d square Phi by d x i, d x j will come into picture which is not locally integrable and it will not be meaningful. So we want to get rid of that. So we will assume if Phi is in C2, so we are going to do something or C infinity, you are assuming.

So u x, by a change of variable, u x can also write it as, you put x minus y equal to t, so you can also write it as Phi of y f of x minus y. This is a very nice thing about convolution. Convolution has very interesting property. So you have, we have already seen that this is equal to Phi star of f. This is also Phi star of f.

So you can take, because of the translation invariance of the measure, you can x minus y equal to T n, arrive at that one. Now, you can see, why did I use this expression. But look at now. Now, if I take a derivative of u, that derivative goes on f, not on Phi. So I can immediately do some thing, I can immediately, so this small exercises, you can do that one. Because now, I can take, these are all, you can justify using theorems from Leibnitz or whatever it is.

Because once you take a derivative, you have to justify. So you understand. Now, you are taking derivative with respect to x and Phi x is not coming so do not differentiate in now Phi because it is with respect to y so I differentiate with respect to x. You can prove this actually. I can do as much as possible as long as f has my derivative. That means, I can write this as Phi star of, so I have to differentiate, this is again a special property or my thing, you see.

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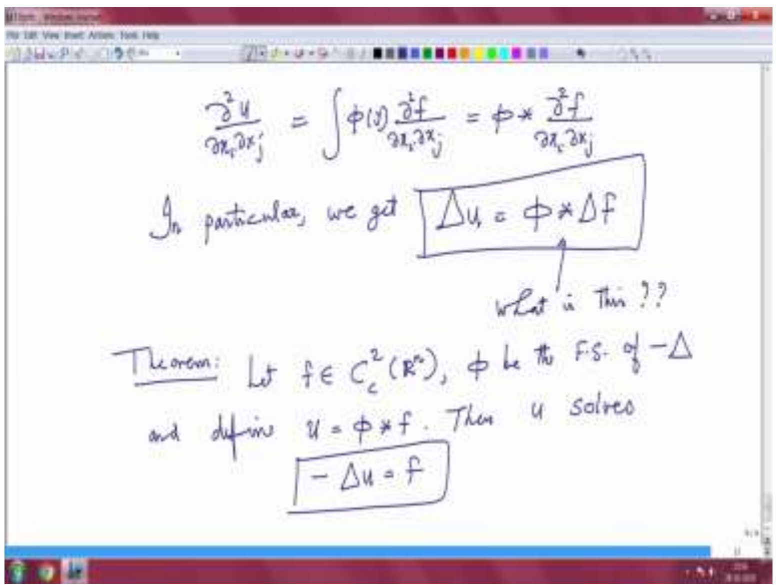

In fact, I can do the next, I can do a second derivative d square u by d x i x j, is equal to integral of Phi of y d square f by d x i d x j. So you see, I am not touching my Phi which has a singularity. So as long has my f has differentiability, I can go on doing this. So this is nothing but Phi star of d square f by d x i x j.

Now, we have assumed only f is C2 twice differentiability and compact support. In particular, if I take i equal to j and sum it, in particular, we get Laplacian of u equal to Phi star of Laplacian, I could write this way, this is a nice formula. Earlier, we are trying to write Laplacian of Phi star f. That was wrong because Phi you cannot make assumptions.

Phi is given as a fundamental solution. So I started with an f which is nice, that means, it is twice differentiable and which has a compact support. So my convolution is well defined and I could put my derivates in the other thing. And this is a general thing, when you have a convolution and convolution operators are used to many purposes, even to smooth non-smooth function by taking convolvment.

So probably when you study more partial differential equations or other subjects, you will come across something, how do you smoothen? Because many times you can do analysis only if your function is smooth. So what you do is that, try to approximate by smooth functions. And convolution is very effectively used together what we call it modifier which we, I do not think we will be defining here, if necessary, we will define as and when it is required, so smoothen this thing.

And you see this here. So one of the functions Phi is, need not be smooth and singular but then you can still do the convolution by taking a derivative. Now, we want to understand what is this? That is all we are trying to do. What is, can you recognize this? Can you do something more? What is this? So what we are going to prove, a very interesting theorem. So let me write a theorem now.

Let f belongs to, so this is not a boundary value problem, it is a problem in Phi R n. Let C c 2 of R n and Phi be the fundamental solution of minus Laplacian. You have to use the same thing. if you are changing the sign and some books may use fundamental solution of Laplacian, there will be a sign change. And define u, u equal to Phi star of f, then u satisfies, u solves minus Laplacian of u equal to f. You will see that is why we have used that. So you see, it is not Laplacian of u equal to 0. So it got back your f via this thing.

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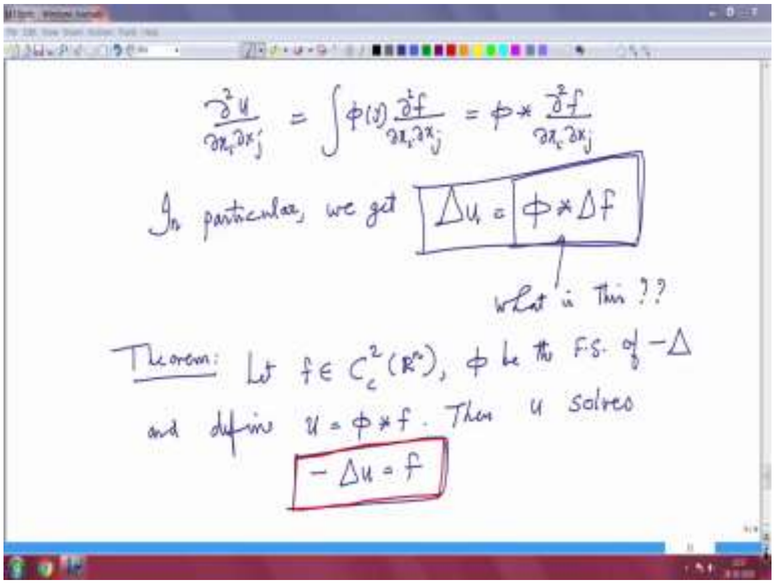

So we are going to prove this theorem. If, maybe in the next class, we will try to prove this one but before that, I want to make some, I will prove this. Try to prove in the next class, I hope I will be able to complete the proof. So you see, so it recovers minus Laplacian of u equal to f is actually minus f. This is what we are proving minus Laplacian of u equal to f.

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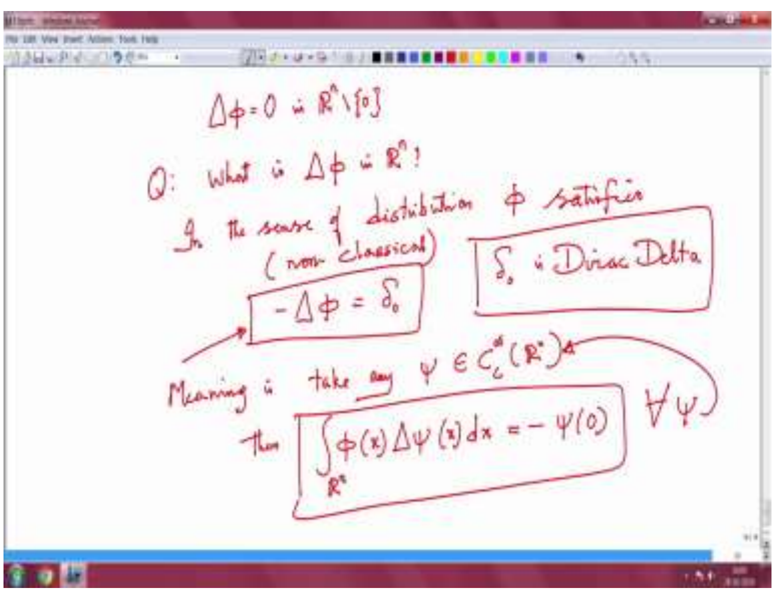

So let me make a remark before thing. So we have solved Laplacian of Phi equal to 0 in R n minus, sorry, 0 in R n minus 0. So the question is that, but what is Laplacian of Phi in R n? What is Laplacian of Phi in R n? You cannot do point wise. So this is a kind of modern or advanced thing, in the sense of distribution which we may not do, maybe we will at the end the course, may not be in this course, maybe in the next course.

In the sense of distribution, this is not classical. That is a modern or advanced p d. Phi satisfies Laplacian of Phi minus Laplacian of Phi, so that is why all we call it a Dirac Delta distribution. What is the meaning of this, what is this? It is nothing like a point wise.

So you take, the meaning is, take any Psi belongs to C c of R n, C c infinity of R n then, integral, what we want to show here integral of Phi of x Laplacian, I can take Laplacian Psi here, Laplacian of Psi of x, this is meaningful because Phi is locally integrable. This is not. You cannot take integral of Laplacian of Phi because Laplacian of Phi is not locally integral but I will do, this is some sort of an integration, is equal to minus Psi 0. This is true for all Psi in this thing.

So that is how you understand. You see, this is the actually, what we are doing is that, we are trying to understand this in the proof of theorems, since you do not know what this distributions, and Phi has information stored at the singularity, something like this, what I explained here, in the sense of distribution, you are trying to understand the action of Phi on test functions.

That is what integral of Phi of x, Laplacian of Phi, it is action and that gives you, and Delta not is called the Dirac Delta. So what in the next class, I am going to prove, to give you, the proof of this theorem. So will stop for the time being before starting the proof because the proof may require some amount of time and hence, I will move to the next class. Thank you.