First Course on Partial Defferential Equations – 1

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Lecture: 18

Laplace and Poisson Equations Part 1

Good Morning. So, today we will start a new topic known as Laplace and Poisson equations. We will try to understand these two equations, Laplace equation and Poisson equation and what we have completed so far are the first order equations. In the first order equations we had given a general theory but then we also introduced what is called classifications and now you know that that the partial differential equations can be classified into elliptic, parabolic and hyperbolic.

All these three categories have different behavior. So studying the equations in generally is a little more difficult. So what we are going to do is study a representative equation. So what we are planning in this five to six hours of lectures, even then we will not be able to complete much, we will do only a part of it. The remaining part of this will be seen in the next course, not even in this course. So let me write down second order elliptic equations, Laplace and Poisson equations.

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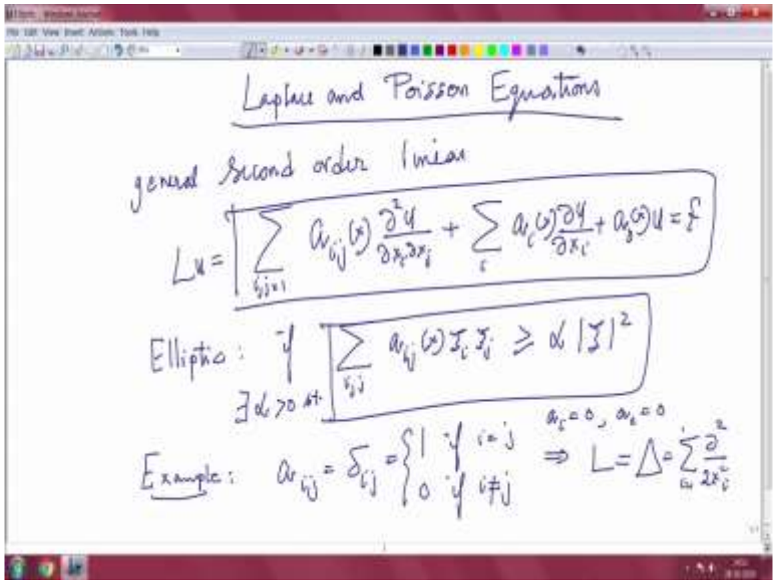

Laplace and Poisson equations. So how is a general second order linear? General second order linear. The general form is Sigma a i j or x d square u by d x i d x j, I am just recalling what you already know. Sigma a i of x d u by d x i plus a not of x u equal to f. This is the general form of equations. Here i j varies from 1 to n, here it is i. That is it.

This is elliptic, so I am just recalling, elliptic with summation, we do not write summation all the time if there is a repeated indices is used, Psi i, Psi j is greater than equal to alpha into mod Psi square. If there exists alpha positive, such that this is true then it is called an elliptic operator. An important elliptic operator is Laplacian.

So if you take a i j equal to delta i j, that is equal to 1, if i equal to j equal to 0, i equal to 1, equal to 0 if i not equal to j and a i equal to 0, a not equal to 0. So this is, let me write down the elliptic equation. L of u equal to, that is the operator L of u. That implies, my L is equal to the operator Laplacian. Laplacian means summation d square by d x i square.

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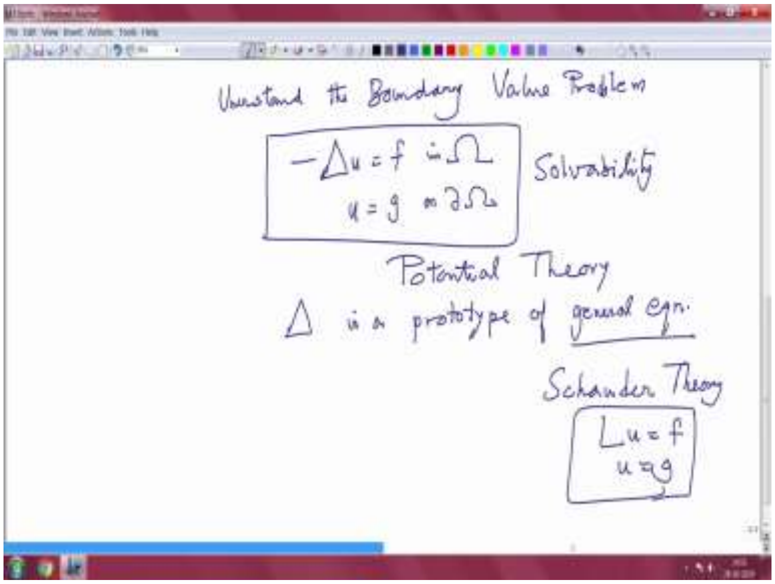

So, I want to study, so the purpose of this course is to understand the equation, understand the boundary value problem, boundary value problem minus Laplace n of u, minus, we use it for some reason but you can also use it without minus, Omega, because minus Laplace n gives you the positive definite operator in some sense. So let us not bother. And u equal to G and Omega.

So our aim is to understand this problem, understand the boundary value problem or rather you want to understand the solvability. It is not an easy thing. So studying the Laplacian problem, instead of a general theory, two fold, there are two reasons, there are many, but two important reasons why we are studying first Laplacian, this Laplacian operator appears in large number of applications.

So it is not that this operator is just an example, it is a part and parcel of any differential operator theory. And in any physics, lot of physical and engineering applications, this Laplacian operator appears frequently. And this study of this is called the Potential Theory. The second reason for studying, it is very particular example, it is not just particular example, it is also a prototype.

Laplacian is a prototype of general second order, prototype of general equation. What do I mean by general equation? Prototype means this Laplacian, even though it looks like a very simple operator, it has many features of general operator. So the study of general equations is called the Schauder Theory and it is, so if you want to understand the general second order operator and it is boundary value problem, instead of Laplacian, you want to study L u equal to f with u equal to g. But then even to understand this equation, understanding Laplacian equation is important.

And through the lectures you will see that even to get various properties and solvability of Laplacian itself is a challenging problem. It is not easy. That is, even in the five, six hours I can only state certain properties and maybe solvability in very special cases. So the general solvability, we will only discuss it in the next course, as and when that happens. So that is the reason why we study Laplacian operator.

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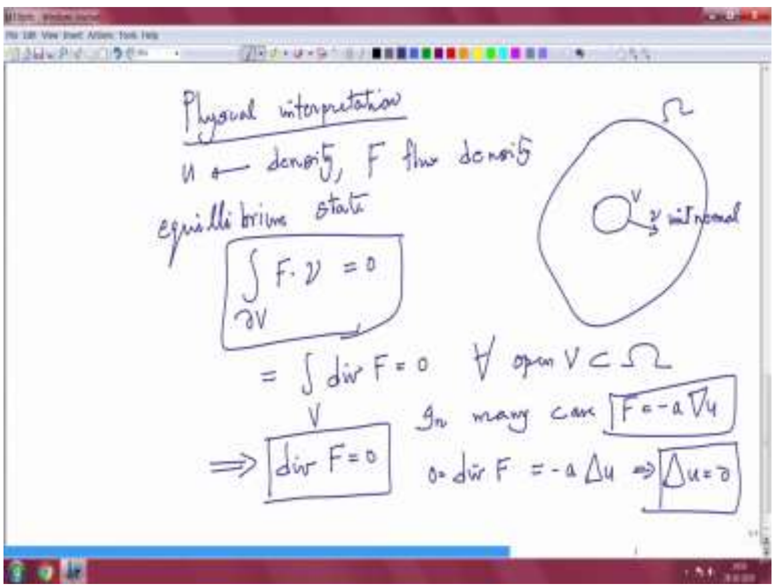

So let us go to the case now. It appears in various ways. Yeah, one more thing before I go further. Just like Laplacian is a prototype of an elliptic equation and we are spending time on understanding this equation, you will also study the Heat operator which is a representative of the parabolic class and you study Wave Equation which is a representative of the hyperbolic equations.

So basically in the rest of the course, we will be studying these three equations, its solvability questions, its properties and various things related to these three equations. So let us go to some physical interpretation. Let me give you very general physical interpretation. Physical Interpretation - This is very general interpretation; you can solve it as I said, Suppose u is a density, some density of or some quantity.

So that is what I say it is a very general physical density and f is the flux density. And you have your equilibrium state. Equilibrium state - On equilibrium, so you have your domain Omega. And you take any V. Since it is in equilibrium, it satisfies the flux through any small region, so you take a small region, look at it is boundary and look at it is flux across the boundary. This has to be zero, where nu is normal, unit normal.

Now, you know that in our preliminaries, we have introduced you what is that one. f dot of nu equal to 0. Now, you apply the Divergence Theorem. This is nothing but Integral over V over Divergence of f equal to 0. And this is true for all open V contained in V. These are all some simple analysis. So when all open V contained in n then this will imply your divergence of f equal to 0. This is the equation, basically. Flux density.

But in many cases f will be proportional to the Gradient. It will be something like a Gradient. If that is the case, you get Divergence 0 is equal to Divergence of f is nothing but, you get Laplacian of u. That implies Laplacian of u equal to 0. So Laplacian of u equal to 0 is called the Poisson equation and if you have some external forces, you get Laplacian u equal to f. So that is how the general thing comes into picture.

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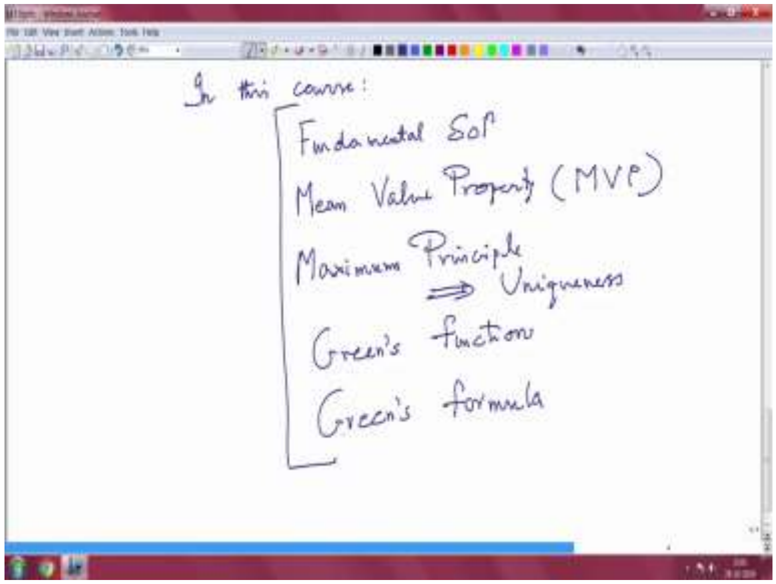

You can give other, so many other physical interpretations but let me not get into that one. So what am I aiming in this course? So we are, want to derive certain, what I want to introduce in this course, what is called the concept of Fundamental Solution, Mean Value Theorem or Mean Value Property, we call it MVP and you derive some Maximum Principle and this leads to what I call the Uniqueness Result.

And then we will introduce Green's Function and then Green's Formula. This is what basically, I want to do that. As I said, I am not going to study the existence in general but on the way, en route, we will see some existences. And we see also the construction of Green's Function in some cases. Because obtaining Green's Function itself is a difficult job. So let me do one by one. So first we want to go to what is called a fundamental solution.

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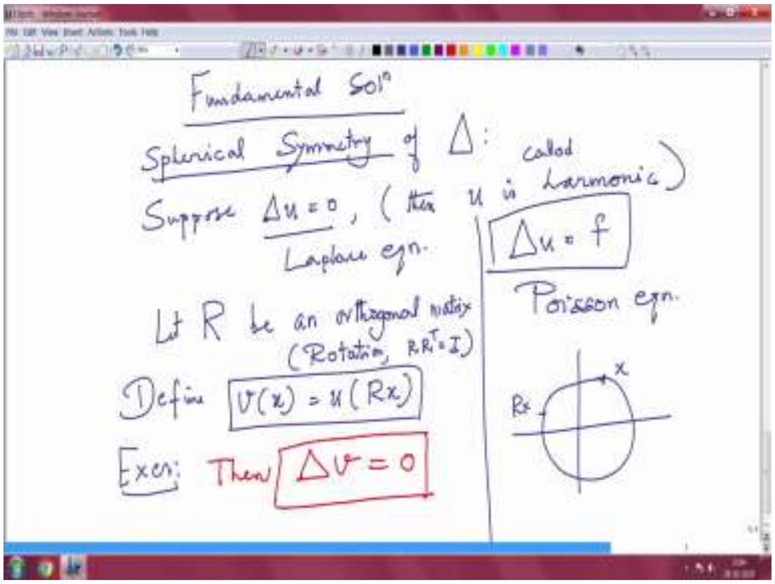

Fundamental Solution - First we try to understand a property. First we want to understand the spherical symmetry of Laplacian. So let us understand one by one. Spherical symmetry of Laplacian: What does this represent? Suppose Laplacian of u equal to 0. That means, so u is satisfies Laplacian equal to 0 then u is called Harmonic. Them u is called Harmonic. And this is called the Laplace equation.

So let me set the names here. Laplacian Equation and Laplacian of u equal to f is called the Poisson Equation. So we have Laplacian Equation. So what is this spherical symmetry? Suppose u is Harmonic function that Laplacian of u equal to 0 and let R be an orthogonal matrix. R be an orthogonal matrix. Orthogonal means, orthogonal matrix, that is, rotation. R R transpose equal to Identity. Define v of x is equal to u of R x.

So that means R is a rotation, it rotates the vectors, you know it, right? If you have a vector here, it rotates, rotates along that. So this is x, R x maybe somewhere here. So you have rotation, you are defining the value of the function by rotating. Then, the exercise which I want to give you, this is your exercise. So let me give it in color so you can prove it. Then, v is also Harmonic. So this is what your spherical symmetry. So, if you have a function u and if you rotate and that rotated function is also a Harmonic function.

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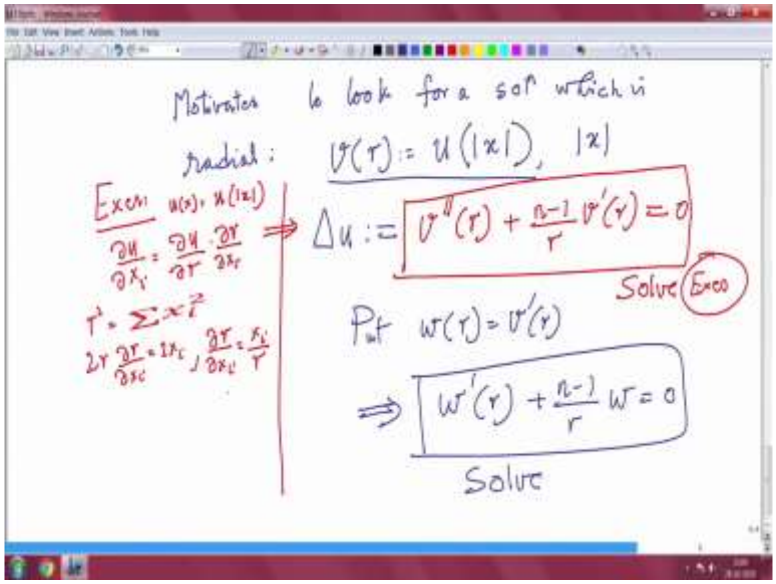

So this motivates us, so this is the motivation. It motivates us to look for a solution also symmetric. Motivates to look for a solution, to look, motivates to look for a solution which is radial. Radial means, which is radial. So we are looking for a solution as a one variable function r is equal to some sort of a, something like of this form, u of mod x. Mod x is equal to r. So that, the value does not change.

Because Laplacian, when it rotates does not change, you expect, you are not sure, the differential operator has a property, maybe it will transfer to the solution but the mean, that is the least we can expect. By rotating your solution, you are getting another solution. So you, right from the beginning, assume that the function u is independent of the rotation. That means, it does not change when it rotates.

It will change only if the radially it is changed. That is what I am saying the function v of r is u of mod x. Then, you want to compute your Laplacian of u. How do you compute your Laplacian of u. So I will leave it as a small exercise here. So you have, if you want to study this topic, for that matter, any topic. You want to compute your d u by d x i.

So you write everything, if it is u is equal to, if your u of x is equal to u of x i, you can write it as u of d r into d r by d x i. This you know. So that is a u of mod x, if you have it.

If it does not depend on that, but how do you calculate dr by d x? R square equal to, so you have 2r into d r by d x i equal to x i, 2 x i.

So that means, d r by d x i is equal to x i by r. Use that to prove this is equal to v double prime of r, this is the exercise, plus n minus 1 by r into v prime of r. So you want to solve this equation. Solve it. So that is your exercise. Solve, I will give you the idea. Solve. But this is an easy second order equation. How do you solve it?

I am giving you the steps, you just complete it. Put w of r is equal to v prime of r. That implies w prime of r plus n minus 1 by r w is equal to 0. This is a first order equation. Solve it. You can solve. So first solve your W and then solve for integrate to get your v. So you get your v prime of r. So you solve it. That is an exercise, you can do it.

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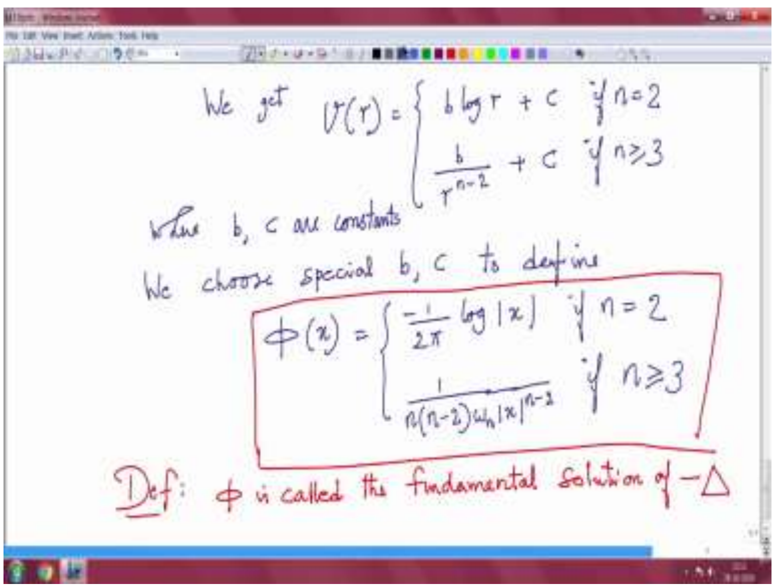

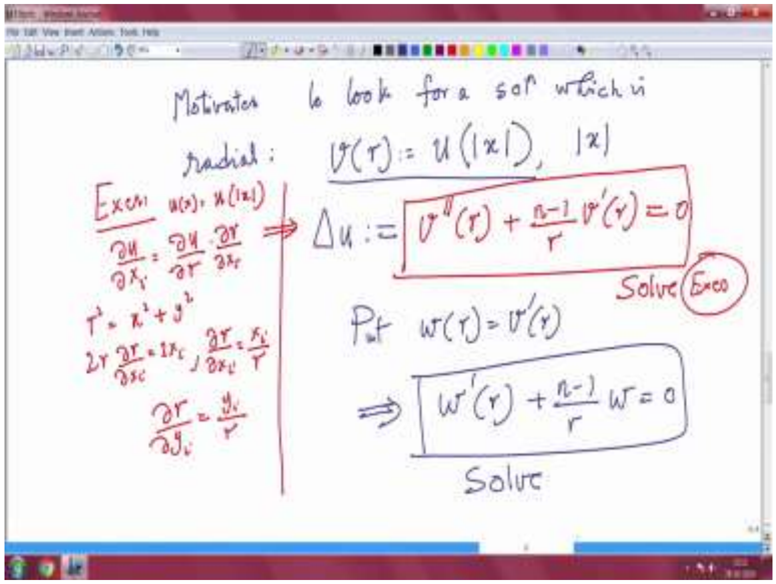

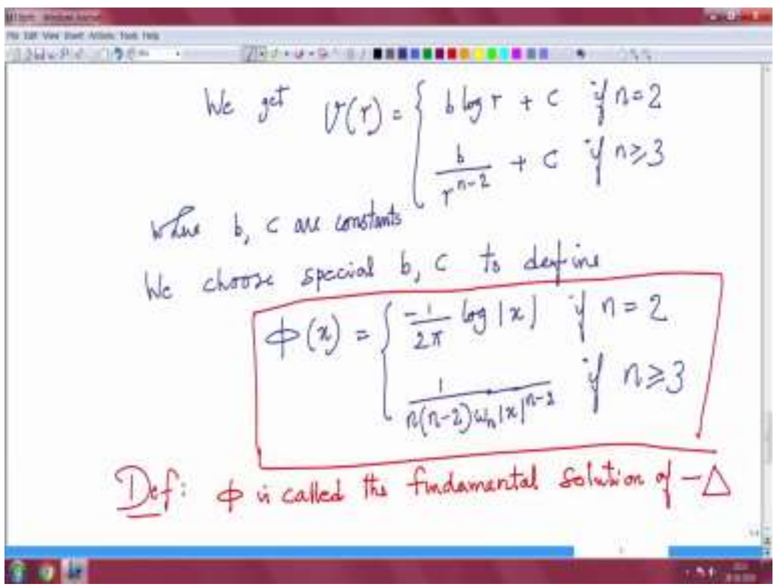

So with that, let me write down the solution, immediately. We get to this, if you want to course, it is better do these kind of small exercises as and when proposed before you are reading further things. You get v of r is equal to v r or r, you have to solve it separately, this is a second order equation so you have two equations. Log mod x, log mod x or log r plus some c if in the two dimensional case.

And this is equal to some b by r power n minus 2 c if n greater than equal to 3 where b, c are constants, c are constants. Now, we choose special b and c. We choose, you will see later as we go along, special b c to define Phi x equal to, the special b and c I have chosen. It will become clear as we proceed further.

You can choose other constants, but then there are reasons why we choose b and c as things. I choose c to be 0 and here, I choose minus 1 over 2 Pi log mod x. And here, this is if n equal to 2 and here, 1 by n into n minus 2, n into n minus 2 into Omega n mod x power, x power n minus 2, if n greater than equal to 3.

So you have this one. It is better to remember this. Then the definition Phi is called the Fundamental Solution of minus Laplacian. Be careful here, some books use Laplacian, in that case, you may have to change the sign and here you have to put a minus. Right now both are positive. When x is small, you get a, that is why the singularity is there.

So you have started with this of u, you go for it, so if you, so, if you, yeah, let me go to previous page, so if you look at it, you started with the solution of, look for a solution. So you want to look for a solution minus Laplacian of u equal to 0. By looking for a solution u, not a general solution, by looking for a solution u, as a special form of a radial solution you arrive at the function Phi x. You got it as a solution. So you have, but look at it. This has a singularity, Phi has a singularity.

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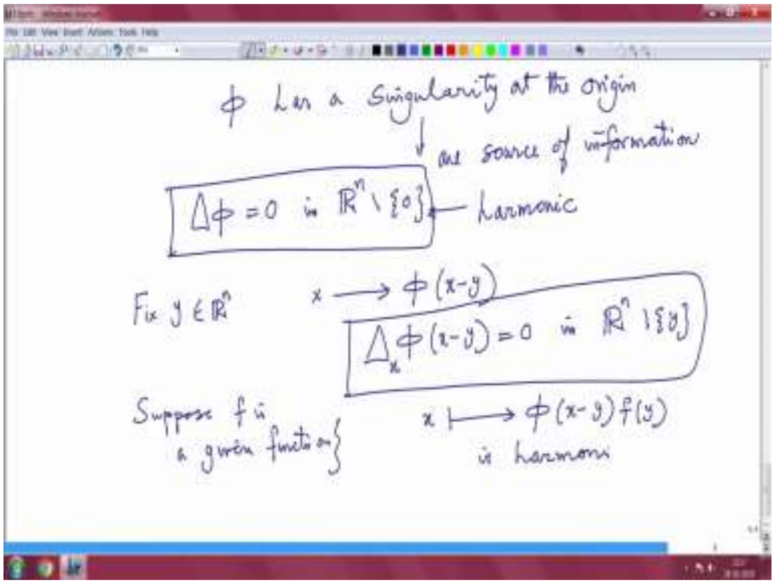

So Phi has a singularity at the origin. You have a singularity at the origin. So, quite often singularity is a source of information. Singularities are source of information. So you do not want to miss it, source of information and importantly Laplacian of Phi, so you are looking for a solution in R n, but then you have a singularity, you think. But except that that singular point, Laplacian of u equal to 0, this is a point wise thing, R n minus 0. That is true. So Phi is Harmonic here.

So Phi is Harmonic in that removed point, 0 removed is Harmonic, not in entire R n, you got a function which is a solution to Laplacian Poisson equation but then the solution is radial, depends on the radial direction which has a singularity and it satisfy the Laplacian equation except at that point.

So that, so strictly speaking this fundamental solution is not a solution but still we call it a fundamental solution now, as we go in this lecture throughout this course or Laplace equations, you will see that Phi is used to construct solutions. Even though Phi is not a solution, it is used to construct, so that means the singularity contains a lot of information, so when we want to extract the information contained in Phi at singularity, at all the singularity.

So you can change this direction now. So you fix y, fix y is R n. I can define x going to Phi of x minus y. Then again, this, then this function, this is, so Phi, Laplacian of with respect to x to represent Phi x minus y equal to 0 in R n minus y. So you are trying to look for singularity, so you want to have, Phi, you have a singularity at the origin.

Now you are shifting this point to another point and looking at the singularity. So basically, you are trying to look at the kind of information carried out on all the points there and we want to see how this function reacts to another function. These are all have more physical motivations, let me not get into that.

So suppose f is another function. Suppose f is a given function. F is a given function, given function. Then, look at this function, x going to, this is a constant now, f of y. Action, Phi acting on that. Not really acting, it is a point. It is a product; we will see the action later. So this is also harmonic, is Harmonic.

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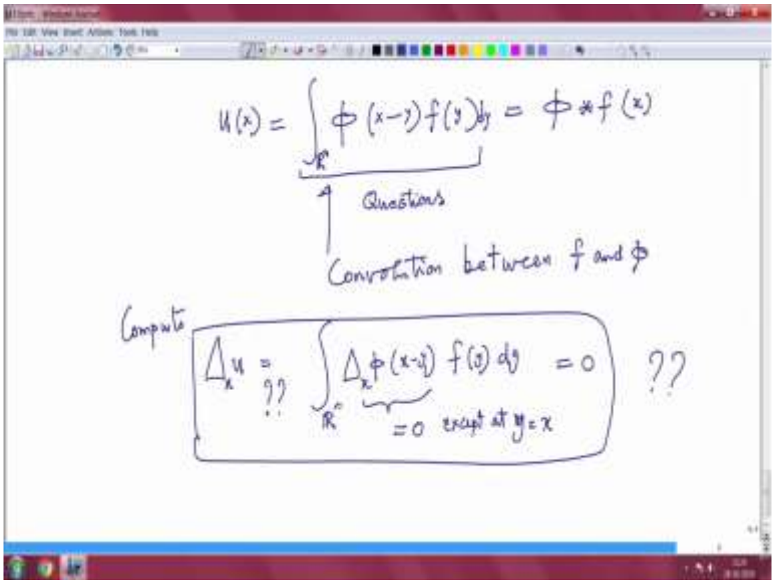

So you want to combine all your singularity actions and that motivates us to define what is called your, summing all the information singularity. So every point, if you shift it, you have a singularity. Phi of x minus y, it is Harmonic except at that point, you take a finitely many, sum it, you still get a harmonic function except at that finitely many points.

So you do this kind of infinite summation and in duty picture and you introduce an integral called Phi of x minus y, so you define integral of Phi of x minus y, summing all the singularity, and I call this to be u x. I do not know, you need to have conditions to whether this is finite or not. So there are questions you can ask.

That we will answer soon. So we will do and ask such questions. But what is this is called, the Convolution of f and Phi, Convolution between f and Phi. And that is denoted by Phi Convolution f. This integration is d y, it is in R n, I am not sure, this has a finite value, this is meaningful, but let us symbolically introduce the Convolution operator.

So let me do something blindly. I want to compute minus Laplacian. Compute Laplacian of u. Suppose if I can inside, this is a question, now, big question, can I take my Laplacian, this is with respect to x, can I take my Laplacian x Phi of x minus y, there is no x here, f of y equal to 0 d y. But then, this is 0 except at one point, right? So when you have a function which is 0 except at one point, except at y equal to x, then integral has to be 0. So looks like that we got a solution.

So that is a question, what we are doing here, is correct that you are trying to differentiate function and trying to differentiate inside. That seems, it gives a solution to your Laplace equation in R n but unfortunately you cannot take that differentiation inside and that is what we will be discussing in the next, when we can take the differentiating under the integral side. You need something. So let me stop it now and we will continue from here. Thank you.