First Course on Partial Differential Equations – 1
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Lecture 17
Partial Differential Equations – 1

Hello, everyone.

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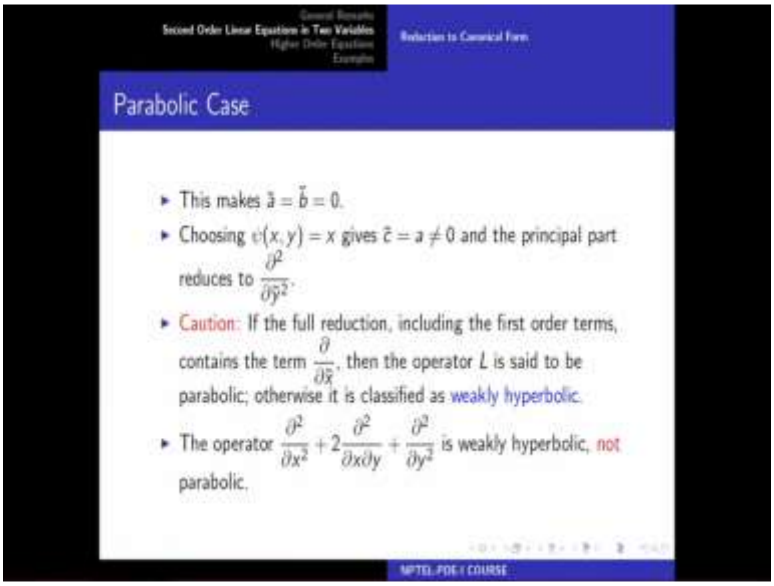

So, in the previous class, we were discussing the reduction of second order Equilinear equations in two variables to a canonical form by using non-singular change of variables and this we have done in the case when the discriminant of the given equation namely ac minus b square is negative and also 0.

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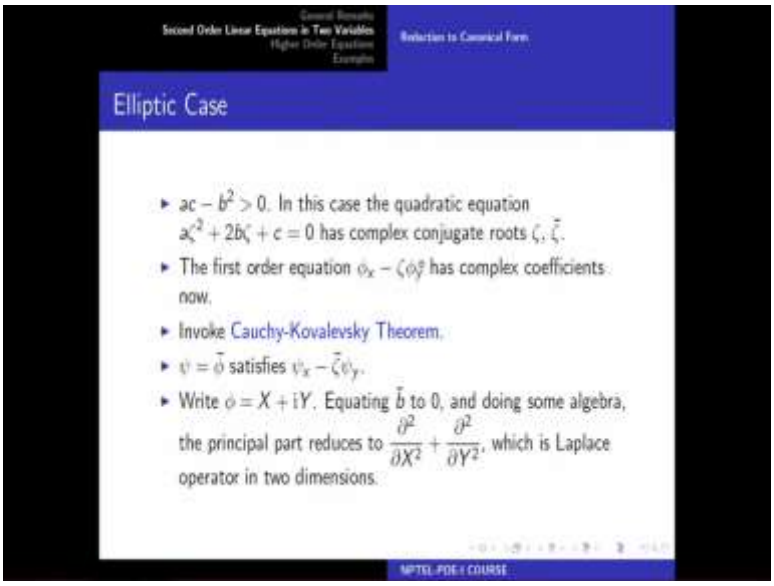

And that leaves us one final case, namely ac minus b square is greater than 0. This is somewhat a tricky case as this quadratic equation now no real roots, it has only complex roots and they come in pair zeta and zeta bar, zeta bar denotes the complex conjugate of zeta. As a, b, c are real functions.

And this leads us to consider the first order equation Phi x minus zeta Phi y equal to 0. This equation now has complex coefficients and we have not covered this part in our course here, because we are all always considered only real coefficients. So, this is the tricky part, now we have complex coefficients so how to deal with them.

And in order to rigorously prove the existence of solution Phi to this equation actually invokes a deep theorem in PDE called Cauchy-Kovalevsky theorem, and that also assumes the coefficients a, b, c are analytic functions. So, we are going a little far from what we want to do here, namely the reduction to canonical form, but in practical problems when you want to do something -- quickly find out a solution of this one.

For example, if zeta is just a constant, we can imitate the, I think method of characteristics to guess a solution of this first order equation, namely Phi sub x minus zeta phi sub y equal to 0. So, in order to rigorously prove that the existence, we have to depend on this deep theorem, Cauchy-Kovalevsky theorem, so if Phi satisfies this equation and then Psi, the complex conjugate of that, satisfies Psi x minus zeta bar Psi y equal to 0, equal to 0 is missing here.

And once we accept that, we have a solution, so that is the tricky part. So, in a given situation, it may be somewhat easier to make a guess and see what that Phi is. So, since zeta is a complex function, so that Phi also has to be a complex function and now you write this complex function Phi as its real part and imaginary part. So, I is square root of minus 1. So, x and y, again they are functions of x and y. So, we got two real functions, namely capital x and capital y.

And again you now go back to that, that is very important. So, you always pay attention to that, a tilde, b tilde, c tilde. And now you equate b tilde to 0. So, you do get an equation and using these coordinates, new change of variable namely x and y. So, this is little tricky algebra and lengthy one, so I am avoiding that.

So, you just do that algebra and see that principal part reduces to del square by del x square plus del square by del y square. So, x and y are our new variables and they are real variables. So, the important thing is here, we should get this Phi satisfying this first order equation, namely Phi x minus zeta Phi y is 0 and that is the tricky part, that is what I was referring too.

So, in this elliptic case when ac minus b square is 0, so it is always possible to reduce the principal part to this the del square by del x square plus del square y by del y square, which is the Laplace operator in two dimensions. So, these are our typical hyperbolic, parabolic, and Laplace equations. So, namely del square u by del t square minus del square q by del x square equal to 0 that is hyperbolic, ac minus b square is less than 0 there.

And then you have heat equation del u by del t equal to a square del square u by del x square. So, x and t both come there, so that is parabolic. And del square u by del x square plus del square u by del y square equal to 0, that is Laplacian, Laplace equation. So, we have typical examples of hyperbolic, parabolic, and elliptic equation.

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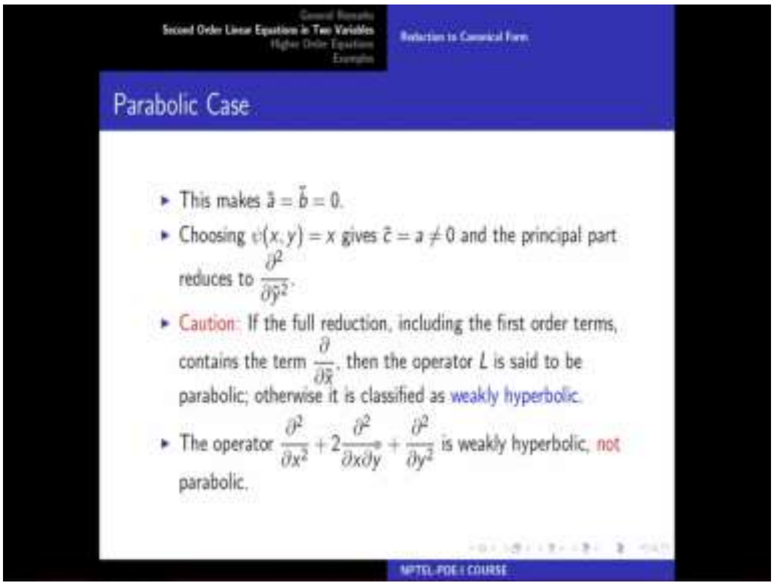

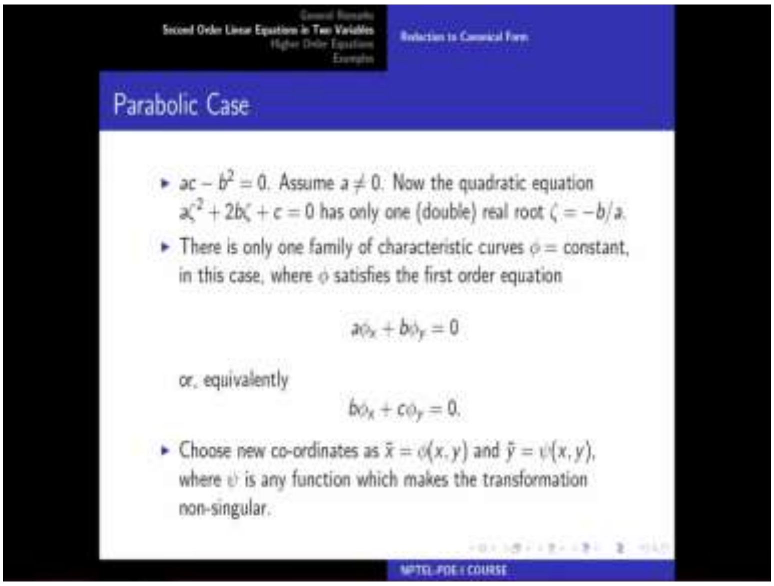

And yet to that, so this usually not mentioned in many textbooks, so that we have, so something very close to hyperbolic as well as parabolic, it is not parabolic. So, it is called weekly hyperbolic. So, it does not have two distinct the ac minus b square is 0 here. So, the quadratic equation does not have to real and distinct roots, there is only one double real root, but this is not parabolic because the full reduction does not produce a first order term in the other variable, that you can easily verify.

Again what we did in the case of parabolic. So, just to go back here. So, you take again -- it falls in that case, so you find the one characteristic variable coming from this characteristic family and you choose any other suitable Psi and you do the full reduction, you get only the second order term and there is no first order term there.

So, we should not classify it as parabolic, but as weakly hyperbolic. So, that is the one perhaps new thing you study in this course. Otherwise, usually, you see in the textbooks, so whenever ac minus b square is 0, you call it parabolic. So, but now, you have some reservation, so you do call it parabolic if some additional thing happens.

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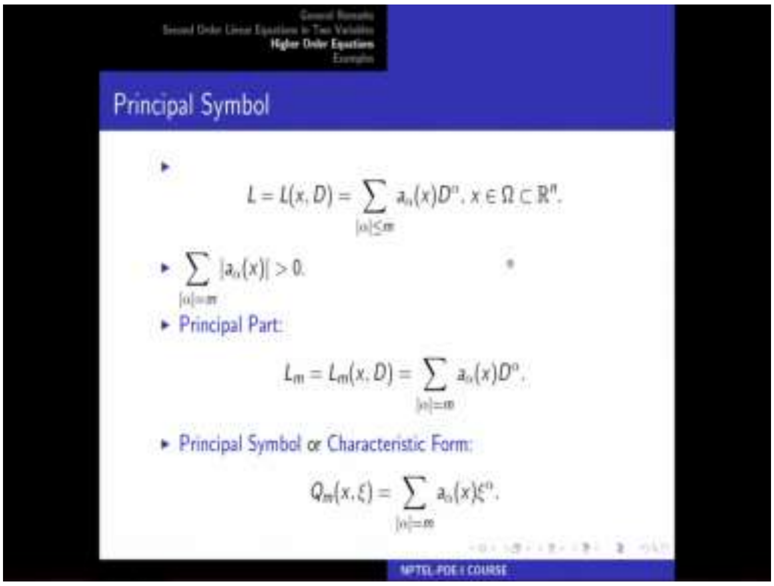

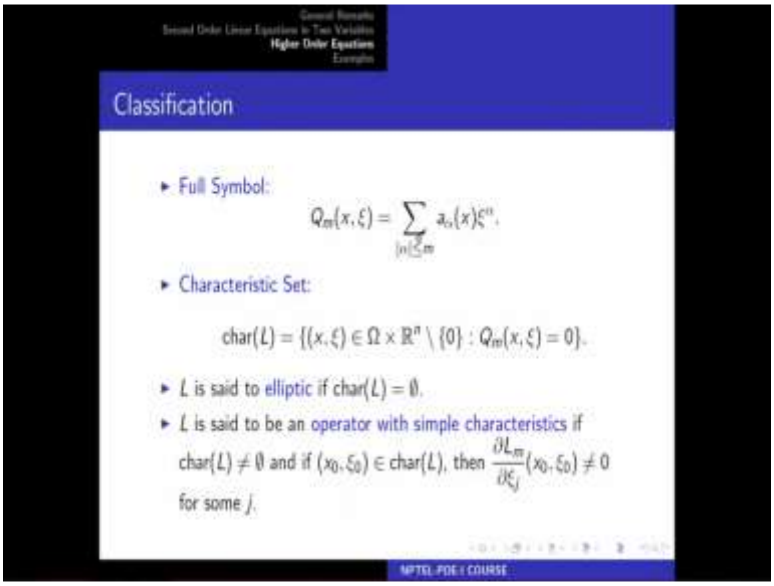

So, we will see now some examples. So, before going to examples, so let us just make some remarks about higher order equations and this certainly even we cannot classify it fully, you will see the reasons soon. So, now you consider and mth order equation in n variables n to Rn. So, this is, so you already seen this notation using d alpha and this multi index notation.

So, this alpha is alpha 1, alpha 2, alpha n, they are non-negative integers. And d alpha is d1, alpha 1, d2 alpha n, dn alpha n. So, these are the coefficients. And like we did in the second order case, so this is genuine mth order equation, so at least one coefficient with model Phi equal to m should be not 0 and that is the condition. So, this is mth order equation.

And so, we collect only the highest order terms and call that as principal part. So, this consists of only highest order derivatives namely mod alpha equal to m. And associated with the principal part is the principal symbol or characteristic form, which is defined by this polynomial. So, formally you replace this operator by this variable in Rn, xi is in Rn.

So, Qm, x xi is a alpha x xi alpha. So, just to replace formerly. So, you get -- so far each x in omega, so this is a polynomial in xi. So, this several variables of who in fact it is a homogeneous polynomials of degree m. So, xi equal to 0 is always a root of this, that is the -- so, just in the second order case, it was the quadratic form. So here, xi is called principal symbol. So, this is general definition, this is characteristic form.

And associated to the operator L, we have the full symbol. So, let me write that also. This m should be removed, so Q of x xi. Now, I am including all alpha less than or equal to mod m. So, now, we define this characteristics set of the operator L. So, that L you remember that is the partial differential operator under steady and that is set of all x xi in omega Rn minus 0. As I say that Qm is a homogeneous polynomial in xi. So, xi equal to 0 is always 0.

So, you remove that trivial root and consider this characteristics set where that polynomial equal to 0. So, you can also define this for each x and then you take union of x or varying or omega. So, important thing is this xi should not be 0, because xi equal 0 is always intercept. So, you take only non-zero xi.

So, for example, in case of second order equation in two variables. So, this just amounts to that quadratic equation being 0, so you are considering the roots of the quadratic equation. And we know that that quadratic equation does not have a real root if ac minus b square is 0 and we called it elliptic. And so, that is the only thing we can immediately define.

So, L is said to be elliptic. So, this is the definition, if this characteristic xi is empty. So, it does not have any real zeros. There we denoted by zeta, just compare it. Take m equal to 2 and n equal to 2, m equal to 2, and you see that this is just quadratic and what it says is the elliptic case is the characteristic set is empty that means it does not have real zeros. So, that is corresponds to ac minus b square bigger than 0.

So, we do not be able to say what type of operate L is, if this characteristic set is non-empty, so that means it has some non-trivial zeros. So, in case of two variables, we could do that hyperbolic and parabolic. So, here it is not at all easy to proceed further, but I just want to make one remark here about an operator or class of operators.

So, when that characteristic set is non-empty and it has some further properties those zeros, so L is said to be an operator with simple characteristics. So, this is the definition. If characteristics set is non-empty, this characteristic set of the operator L is non-empty, and if x 0, xi 0 is in the characteristic set then the derivative of -- sorry for this, so this is not Lm but Qm, sorry for that, this the polynomial Qm. The homogeneous polynomial Qm.

So, the Lm is a differential operator, so I cannot differentiate, so this is Qm. So, you take the derivative of Qm with respect to zeta j and again you evaluate at x0, xi 0 that should not be 0 for some j, not demanding for all j, but some j. So, this is similar to the one variable functions. So, we say that F as a simple 0 at x 0, if F x0 is 0 but its derivative is not 0 at 0.

So, similar to that definition. So, that principle symbol it has a 0, but its derivative in the some direction is not 0. And this class of operators is fairly well studied and there is also we understand more or less a complete picture of that. And so when you gain some basics of PDE and wish to study some modern theory of PDE, so this you can look into literature and make further study.

So, this is -- and as far as -- what about other operators, then you call them operators with multiple characteristics and there is still a lot to be done with those operators. So, in essence when this characteristic set L is non-empty, yet we can just say that the operator is non-elliptic, that is a very, very broad classification. But then as I said in the beginning, there are also classifications called hypo elliptic, that includes elliptic operators and non-hyper elliptic operators.

So, the theory becomes complicated and more difficult. So, you have to really see some advanced textbooks or literature for further study. So, this is the only remark I want to make regarding the higher order equations.

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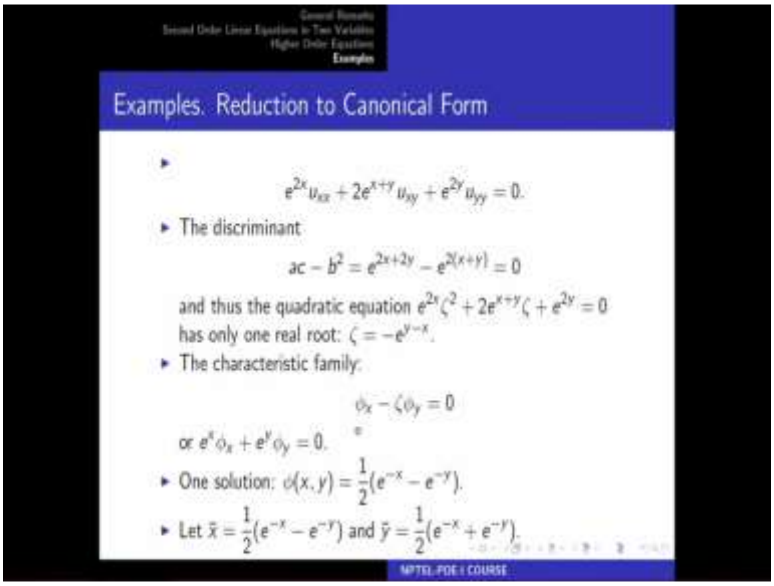

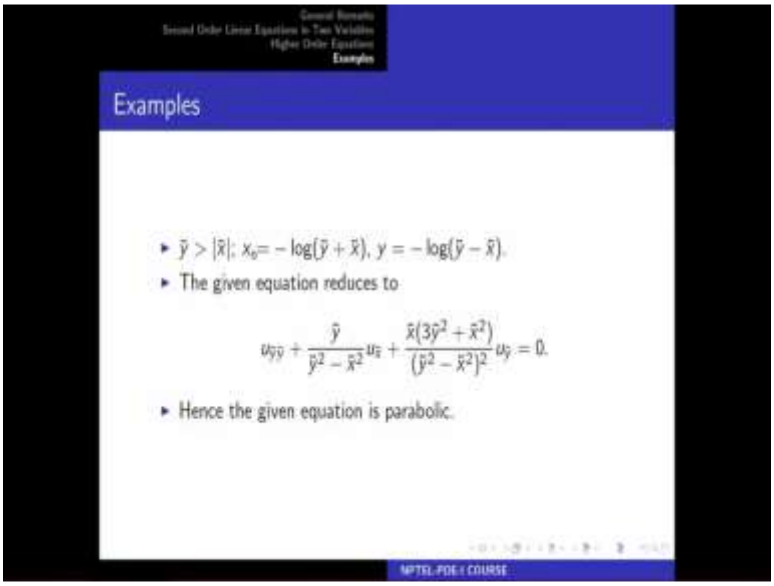

So, with this, now we consider some examples. So, again, this is when it comes to actual reduction of a given equation, it is always a tricky business and lots of computations are involved. So, here I have taken care to not have any mistakes, but still, you should check my calculation and verify yourself whether what all I have done is correct.

So, first consider, so I have just considered some three examples, one of each type in just two variables. So, here the equation has variable coefficients, so e to the 2x, e to the x plus y, e to the 2y. So, since it is exponential, none of them are 0. So, this equation in principle is valid in whole of r2. So there is no problem with that.

And let us calculate the discriminant ac minus b square and you immediately find that is 0. And so, the quadratic equation has only one real root, that is a double real root, namely zeta equal to minus e to the y minus x, just that is b by a -- minus b by a, minus b. So, e to the x plus y divided by e to the 2x one closer, so minus e to the minus x.

So, we have one characteristic family and that is given by the function Phi, which satisfies the first order equation namely Phi x minus zeta Phi y equal to 0, you substitute zeta. So, this is the equation ex, Phi x, e to the Phi y equal to 0.

So, as I said, so this is a first order equation, but we are not given any initial curve or initial data. So, it is just you solve this by the method of characteristics and find one solution. We are not interested even in finding all solutions. So, one solution you can easily check. So, this half is only for the convenience. So, Phi x, y is equal to half e to the minus x minus e to the y. So, you can easily check that, it satisfies this first order equation. So, we got one Phi.

And in this case, we are free to choose the second function Psi in whatever manner we want as long as the non-singularity condition is satisfied. So, this is just one case here. So, x tilde equal to this one and I put y tilde. S, this is coming from the characteristic, that one family of characteristics, so I am taking that one and this is a different choice.

So, this is again for the convenient, so in this form we can easily express x and y in terms of x tilde, y tilde, if you choose any other y tilde, it may not be that easy. So, this is again for convenience and we have to just see the situation and choose this second variable. And just observe, this is important to observe.

So, this x, y, now, we are going to new variables x tilde, y tilde. So, in this setup -- in this new coordinate system, you always see that y tilde is strictly bigger than mod x tilde. And as I said, x and y are easily expressed in terms of x tilde, y tilde. And now comes in several differentiations, so a long calculation.

So, you have to chain these uxx, uxy, uyy, etcetera. The only thing is now only the cross term remains and so you please verify that all these things are correct, this is correct, this is what we get, but then there are lower order terms. The important thing to notice here is, there is a first order derivative with respect to x tilde.

And since this term appears, this given equation qualifies to be called as parabolic. So, I will not call just by looking at this condition this equation is parabolic, but now I have done the full reduction and there is a first order term, so I call it this equation as parabolic. And because of this condition these are all non-zero coefficients.

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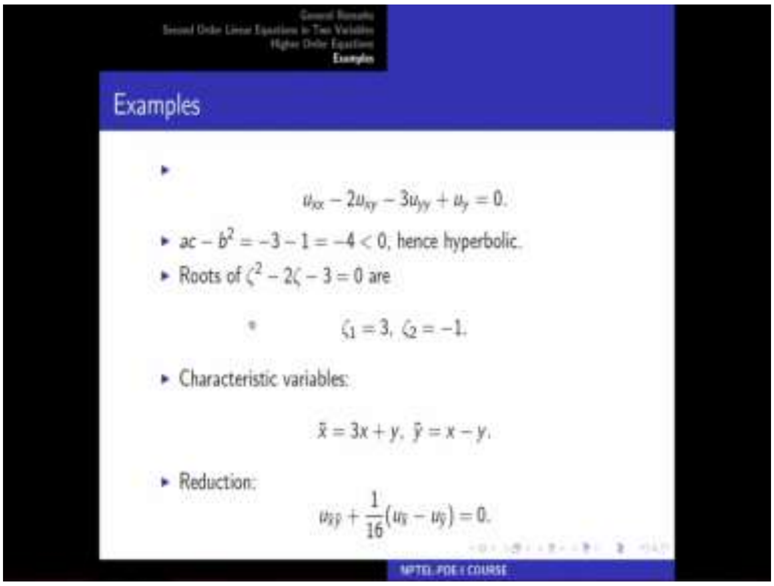

The second example. So, if this is constant coefficient, you can use a linear algebra. But here, it is more straightforward to use the method of characteristics. So, ac minus b square is strictly less than 0 and hence hyperbolic. So, there is no hesitation here, only when it is 0 you have to do the full reduction and decide whether it is parabolic or weekly hyperbolic.

So, in this case the roots are real and distinct they are zeta 1 equal to 3and zeta 2 equal to minus 1. So, you do get two families of characteristic curves, so I am omitting that part. So, again you solve two first order equations and we get these new variables and then the given equation reduces to this second order equation.

And if you want, you do not want the mix derivative, you do one more change of variable, I mentioned that zeta equal to x tilde plus y tilde by two, eta is equal to x tilde minus y tilde by two and you get the typical wave equation. As far as hyperbolic equations are concerned, even this is accepted as canonical form.

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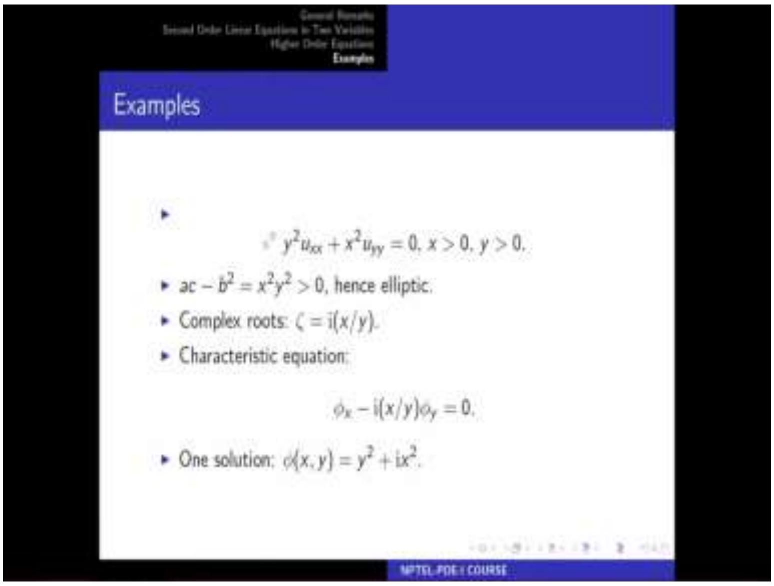

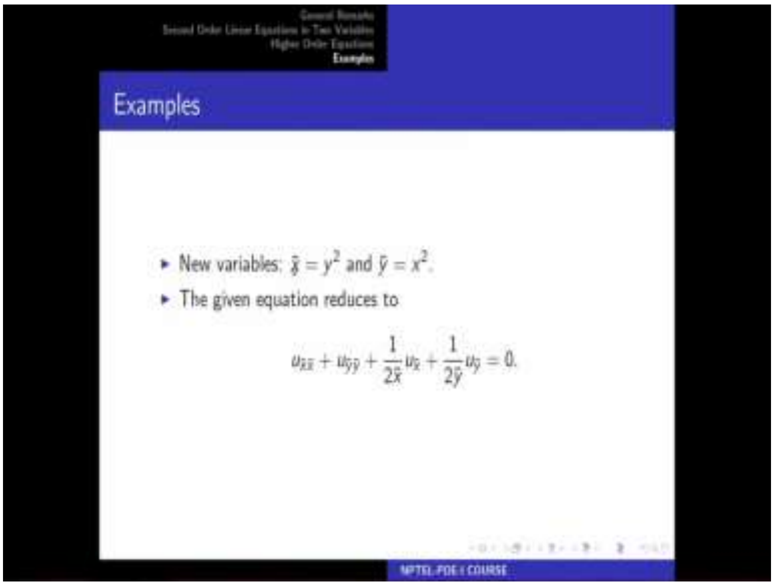

And one, one more example. So, this is y square uxx plus x square uyy equal to 0. So, if either x is 0 or y is 0, we see that one variable is missing. So, these can be 0 only on the axis. So, you avoid both the x axis and y axis. So, you consider this equation in any of the four quadrants, and here I am considering this equation in the first quadrant. So, you can also do it in other -- second, third, fourth quadrant.

So, in this case, we have ac minus b square is equal to x square, y square, and in this domain we see this positive and hence elliptic. So, this is the tricky part we have to deal with. So, the complex roots here, very easy to find out. So, I should write plus or minus i because they appear in complex conjugates. So, this is zeta equal to i by x, x by y, and another one is zeta bar minus ix by y.

So, characteristic equation is Phi x minus i, x by y, Phi y equal to 0. So, as I was remarking, we are not dealt with such equations, equations with complex coefficients. So, but here, since this is very simple we can make a guess for the solution. So, here looking at the equation again, so if you make Phi y a real function and Phi x a purely imaginary function and then there is a possibility of cancellation.

And we see that Phi x, y one solution. So, it is sufficient to find one solution in a given situation and one solution here is y square plus ix square. So, you can easily see that, this satisfies this equation. And now, our new variables will be the real part and imaginary part of this solution. So, a real part is y square and imaginary part is x square, and that is what we use. Now, new variable x tilde equal to y square and y tilde is equal to x square.

And now, if you again do the computation, you have to convert this uxx, uyy into ux tilde, tilde, uy tilde, tilde and you get this reduction. So, in the reduction you see that important thing is this constants are one and one. And because the original equation contains variable coefficients, so it is inevitable that it will produce some lower order equation and these are again variable coefficients.

So, somehow they are shifting, but principal part remains with constant coefficients. So, with that example, we come to an end of this topic of classification of partial differential equations. As you see, we have just touched some simple situations and complicated higher order equations you cannot do such reductions we have to deal directly with them. So, only in case of second order equation and that too in two variables, we are able to do this reduction.

Of course, here the -- it was only considered reduction and not solving the equation. But you should also try wherever possible after the reduction, try to solve the reduced equation and then go back to the original equation. Thank you.