First Course on Partial Differential Equations – 1

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Lecture 16

Partial Differential Equations – 1

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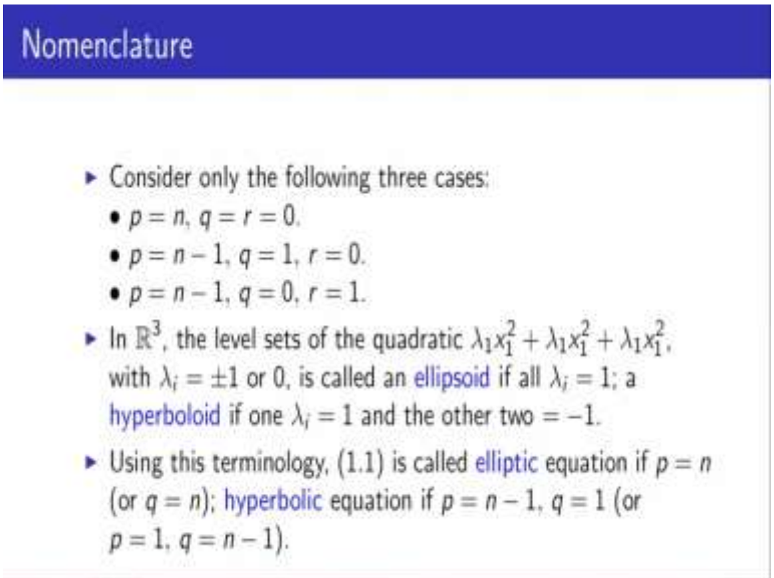

So, in the previous class considering the second order PDE with the constant coefficients, we classified them as elliptic or hyperbolic depending on the number of positive Eigenvalues of the coefficient matrix A. So, if the coefficient matrix has all the positive Eigenvalues or equivalently on all the negative Eigenvalues that just a nice change of sign. So, we call it a elliptic equation.

And when the coefficient matrix A has n minus 1 positive Eigenvalues and 1 negative Eigenvalues or equivalently 1 positive Eigenvalue and n minus 1 negative eigenvalues, we call it an hyperbolic equation. So, the other third case now we will discuss in detail, namely n minus 1 positive Eigenvalues and one 0 Eigenvalue, equivalently n minus 1 negative Eigenvalues and one 0 Eigenvalues. So, there is one 0 eigenvalue, so now the matrix is singular there is a 0 Eigenvalue.

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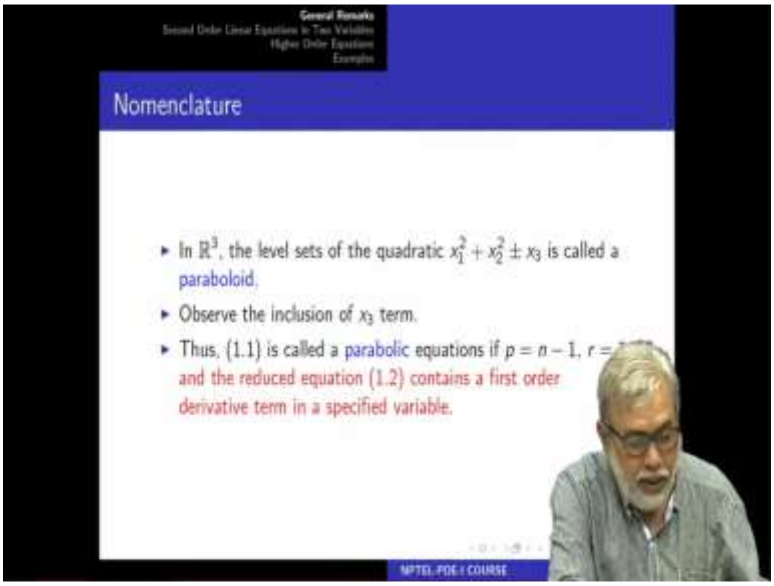

So, how do we classify this? Again, we will go back to three dimensional geometry or even two dimensional geometry. So, since we have done for elliptic and hyperbolic three dimensional geometry, so let us go to again three dimensional geometry. The level sets of the quadratic, the level sets are nothing but you take this quadratic and equate it to some constant and see what curve or surface it describes in that R3 space.

So, here you can see that this quadratic x1 square plus x2 square plus or minus x3, so x3 is only of -- so, in fact, I should not call it a quadratic because I am adding x3 there. So, it is not a quadratic it is just function. So, this is called a paraboloid in three dimensions. So, you just cost -- you observe the inclusion of the term, x3, so we are adding that. So, it is not a quadratic, but we are also including a lower order term and that too in the third variable.

See the quadratic terms are x1, x2 and the first degree term is in the x3, not in x1 or x2. So, that is also we have to observe then only you get a paraboloid. For example, if it is x2 or x1, again you can absorb that in quadratic and you will not get a paraboloid. So, that inclusion of x3 term to get a paraboloid is very important and same thing holds for the PDE.

So, we call the equation 1.1 a parabolic equation if p equal to n minus 1, r equal to 1. So, I have highlighted that. The reduced equation 1.2 contains a first order derivative term in the specified variable, that is important. So, just like for the paraboloid, the linear term should be in x3 and not x1, x2.

So, same thing should happen for the reduced equation, then only we call it parabolic. Otherwise, the some other classification needs to be done. So, we will see in some examples later, why this additional remark and stress is there on this. So, this is very, very important. So, in some sense the classification into parabolic class is somewhat difficult. It is not that easy.

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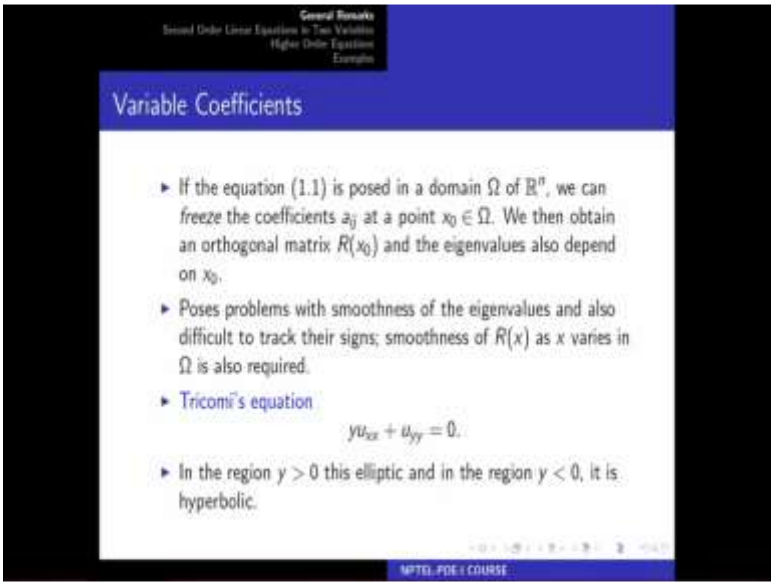

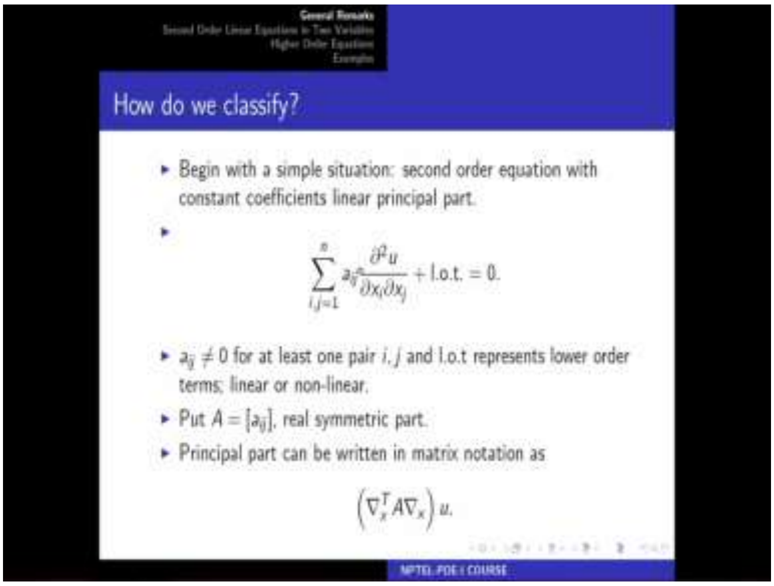

So, now we move on to the variable coefficients. So, that is again, so what we can do. So, again let me just so go back there. So, this equation and now we are considering variable coefficients. So, this aij they depend on a variable x in some open set in Rn. So, how do we classify now? So, if you want to imitate what we did for the constant coefficient case, what we can do is you freeze the coefficients at a point x0 in that open set and then you get the constant matrix namely aij x0.

So, you evaluate all aij's at this point, so we get some numbers and so you will get a symmetric matrix and then we obtain an orthogonal matrix. Again that depends on x0 and Eigenvalues also depend on x0. And as we vary x0 in omega, we get matrix valued functions are x0 and Eigenvalues are also functions of x.

And that makes the life complicated, and since we are dealing with partial differential equations, it is not enough that we get this matrix valued functions R x, x not and the Eigenvalues also depending on the variable x, but we also need some smoothness of the eigenvalues and also on the rotation matrix Rx.

And these are difficult questions and not dealt with in a typical course on linear algebra. So, in the usual linear algebra, we never consider matrix functions. And what might happen is the Eigenvalues at some x, they may all be positive, but in some other point, some of them may be positive and some of them may be negative. So, hence the type changes, the costs changes and it is very difficult to keep track of all those developments.

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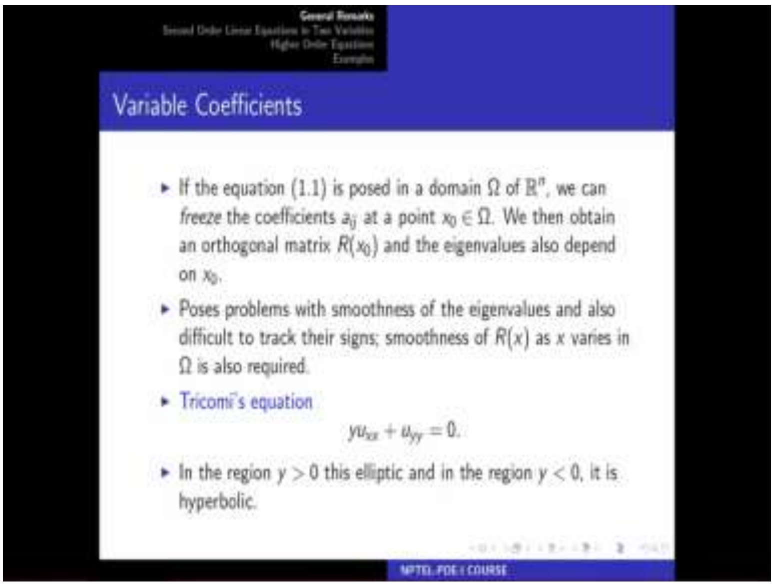

So, here is a simple example, Tricomi equation. So, the simple second order equation in two variables yuxx plus uyy equal to 0. And in the region y greater than 0, it is elliptic, and in the region y less than 0, it is hyperbolic. So, we will see when we do classification of the second order equation in two variables, you see why this is elliptic and why positive and hyperbolic and why negative.

So, the type changes and these are more difficult problems to deal with. So, we cannot possibly approach this classification problem of variable equations through this linear algebra. So, that is the message I want to drive at. So, using the usual linear algebra is not possible to classify equations with variable coefficients. Then what we will do?

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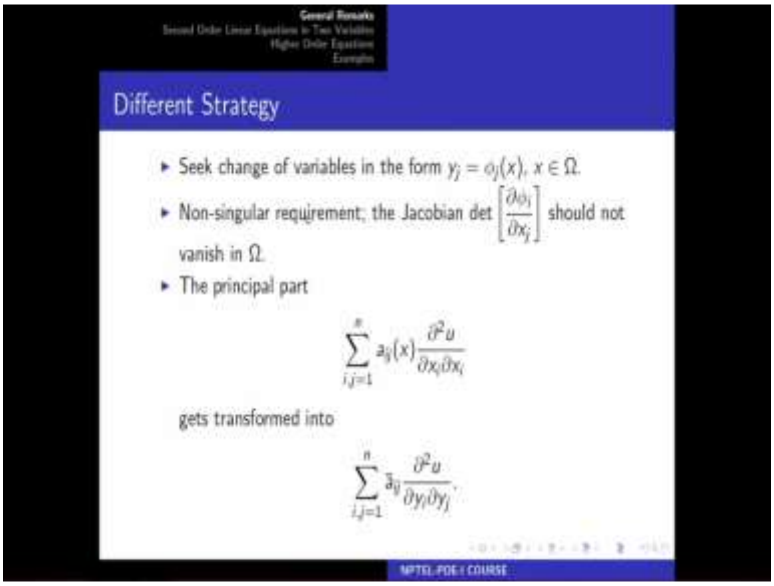

So, we will seek for another strategy, say a different strategy. So, do not look for linear change of variables, but now you look for some function. So, the change of variable from the x variable to y variable, so yj equal to Phi j of x. So, we seek some functions Phi j, the first requirement that this Phi j should process with that this non-singular requirement in case of constant coefficients these r was an orthogonal matrix and so that is non-singular.

So, here the non-singular requirement means the Jacobian of this transformation namely determinant of del Phi --- so, Phi j is Phi j is a function of x1, x2, xn. So, we are in n variable situation. So, you consider this matrix of the first dot Jacobian matrix of this transformation namely del Phi i by del xj, and then you take the determinant of that. That should not vanish in omega, that opens at omega. So, this is the requirement. Why this requirement?

Then the from one variable from x to y, we can go in from y to x, suppose in the variables y if it is easy to solve the given equation then we can come back to the original variables. So, it is like a system of algebraic equations you want to solve, so usually you bring it to either diagonal form or at least upper diagonal form so that you can easily solve the given set of algebraic equations.

So, as similar strategy we want to add up for the PDE. So, now our principle part, so I am ignoring the lower order terms, so lower order terms are important for the full reduction but as far as the reduction is concerned, it is only the highest order terms that is the principal part. So, using this change of variable you transform this equation into this. So, while doing so, you do get some lower order terms here I am not considering them.

So, there are lower returns here and even this reduction going from this x variable to y variable also we will produce some lower order terms because the coefficients are variables now, but you club all the first order terms here, first order terms here and you will get first order -- but we are concentrating only on the second order terms.

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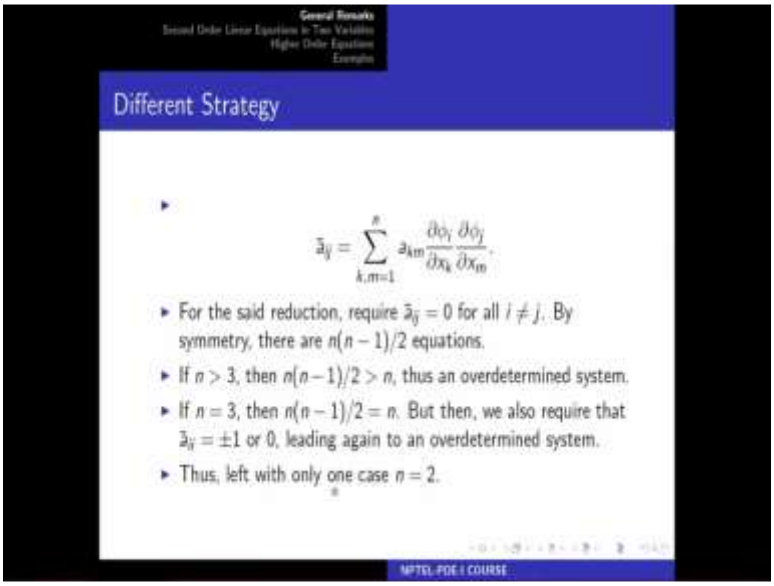

So, what are these aij tilde? So, they are easy to compute, some tedious calculation but simple differentiation, so you compute them. So, here I am writing, so aij tilde is given in terms of akm and the transformation Phi i. So, it is a little complicated but straightforward to compute, so you should do that. So, it is just a chain rule and implicit differentiation, so you can do that. So, now we want this aij tilde to be 0 for all i not equal to j, just see that.

So, just if I make all the mix derivative 0, then this is reduced to a diagonal form and that is what we want to begin with. So, at least the first requirement is that if possible, we want this aij tilde for all i not equal to j. And since these are all symmetric aij tilde is aji tilde, there are

-- just to count them, there are n into n minus one by two equations. And these come in the – this are the equations for the Phi i's we want to find out.

So, there are n into n minus 1 equations. How many unknowns are there? Unknowns are only Phi 1, Phi 2, Phi n. So, there are n unknowns and we are getting n into n minus 1 by 2 equations. So, a simple observation, so if n is bigger than 3, then this n minus 1, n into n minus 1 by 2, by 2 is typically bigger than n. So, that means, there are more equations and less number of unknowns and this is an over determined system. So, very difficult to solve.

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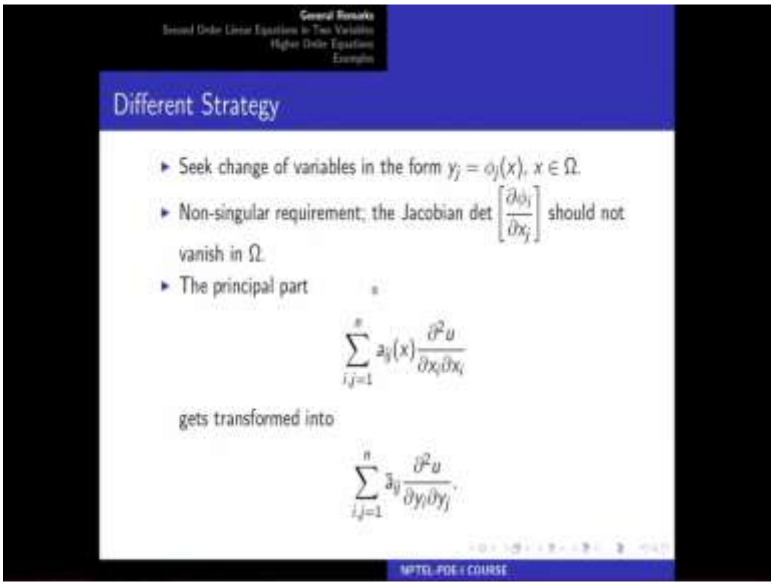

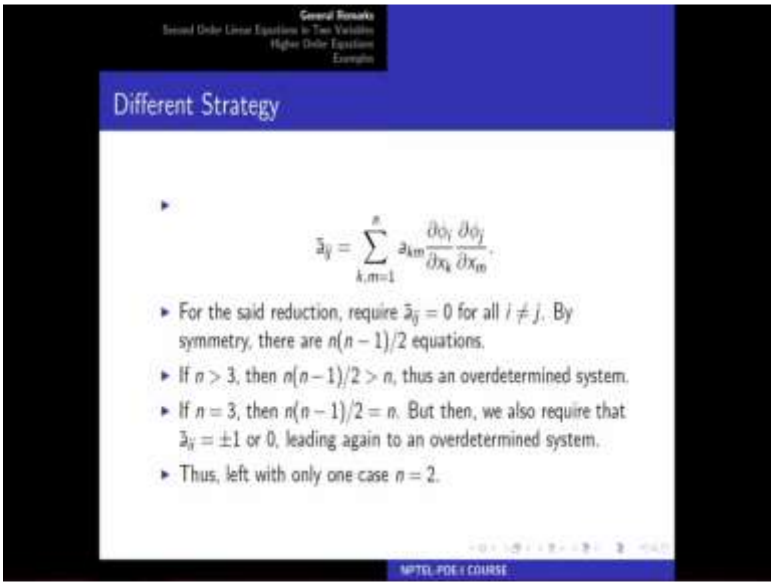

So, for n bigger than 3, we cannot hope to get such a transformation that transformed the given equation is principal part into a diagonal part, so this is not possible. So, what about n equal to three? n equal to three, so there are a number of unknowns and number of equations are the same, so there is some hope.

But then we also want the diagonal terms, namely, aii tilde to be plus or minus 1 or 0, because that is what we did in the constant coefficient that is the canonical form. But then these equations are added to that, so again we are getting an over determined system. So, even for n equal to 3 there is no hope of obtaining such a transformation, and that leads -- thus, we are left with only one case namely n equal to 2.

So, such a transformation. So, again let me repeat that finding such a transformation to reduce the principal part of the given differential equation into a diagonal principle part with diagonal terms plus r minus one is not possible if n is bigger than or equal to three, and we are left with only case of n equal to 2 and now, we proceed to do that.

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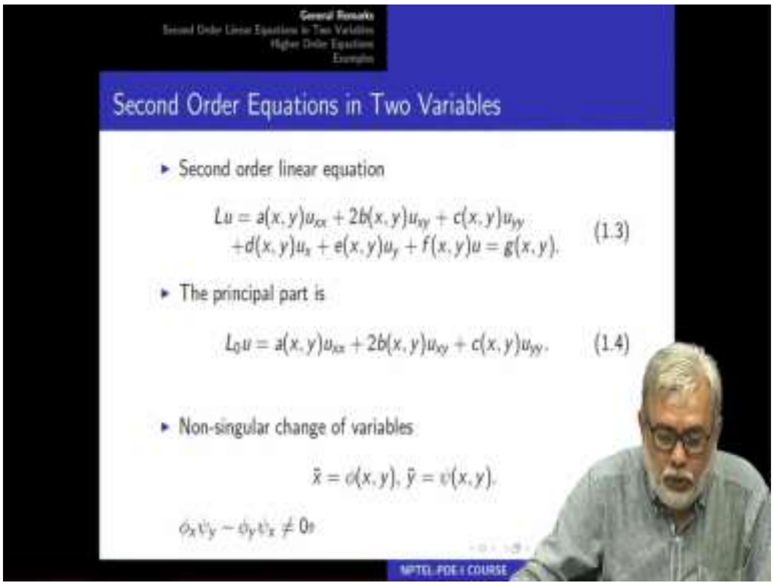

So, thus this is the only case we need to consider and whether there is a possibility of reducing the principal part of this equation, this is the principal part only consisting of only the second order derivatives to a diagonal form. So, somehow in the transformed equation, we do not want to see this beta. So, that means this mix derivative should vanish. And let us see whether that is possible and we will show that that is always possible.

And before proceeding, so the one important simplification we have in case of two variables. See the quadratic form associated with this principal part is a xy square plus 2b xy eta plus c eta square. So, let me write is a xy and eta variables. And this whether it is definite or indefinite depends only on one quantity and that is -- that we seek.

And this is the requirement that this transformation is non-singular. So, that determinant, I have written it explicitly here because it is a 2 by 2 determinant. So, just you compute that, so this should not be 0 in the domain we consider.

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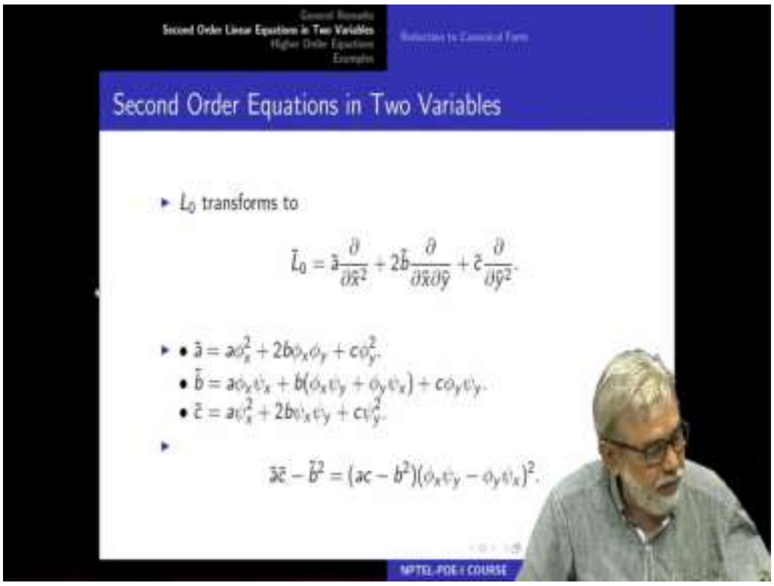

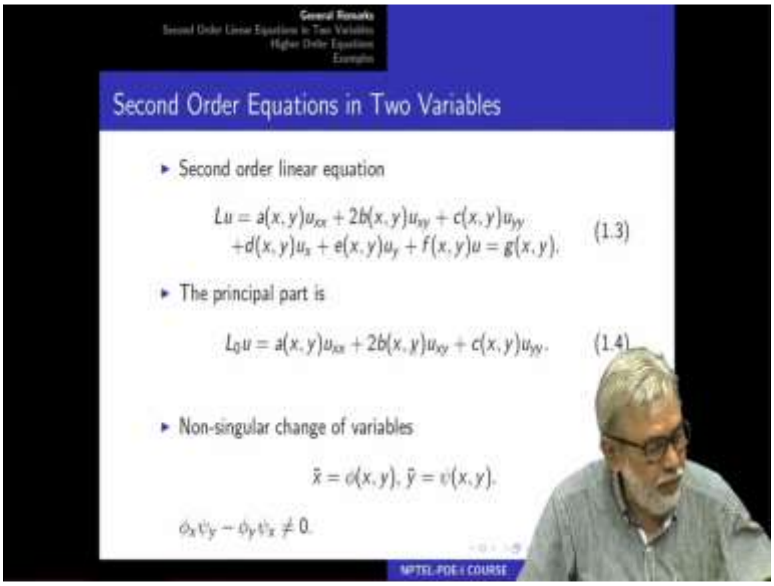

So, let me again -- so, in case of general second order equation in n variables we wrote this transformation and now we again write explicitly here. So, just you observe these three quantities, so here, so there are only three quantities a tilde, b tilde, c tilde and they are given in terms of the coordinate functions. So, that we seek namely Phi and Psi. So, we observe these things, because this is what gives us the transformation Phi and Psi, this is the important thing.

So, I was telling the nature of the quadratic form whether it is definite or indefinite depends on only one quantity, namely the discriminant of the quadratic ac minus b square. So, if ac minus b square is positive, then the quadratic form is positive. And if ac minus b square is negative, then the quadratic form is indefinite.

And then ac minus b square is 0 it is semi definite, meaning it is always either greater than or equal to 0 or less than or equal to 0 that all these things just follow from considering the quadratic. So, that is why it is easier here in two dimensional case just with two variables. So, what it shows is, so again this computation it -- this require some computation algebra here, as what it says is the discriminant had the same sign whatever the L had.

So, ac minus b square whatever and L 0 tilde also has. So, if it is definite in xy variables, it is also definite in x tilde, y tilde variables. So, that is an important observation. So, the time does not change under non-singular change of variables, non-singularity is used here. So, this is non zero, so you are multiplying only by a positive quantity.

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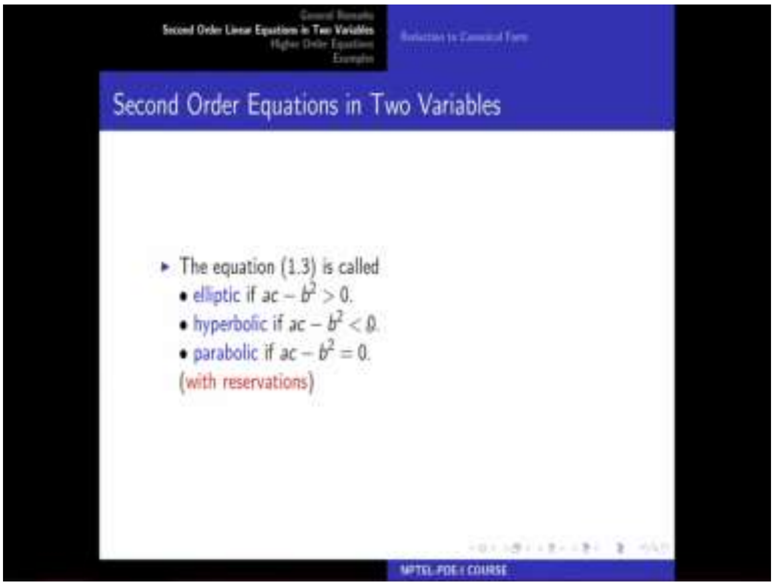

So, the type does not change, that is very important. And so you immediately conclude that the equation 1.3 so a name, so the second order equation, so only the first three terms matter. So, we call this elliptic if ac minus b square is positive, hyperbolic if ac minus b square is negative and parabolic if ac minus b square equal to 0 and for the last one that with reservations. So, all the time we will not be able to call it parabolic as we see some examples.

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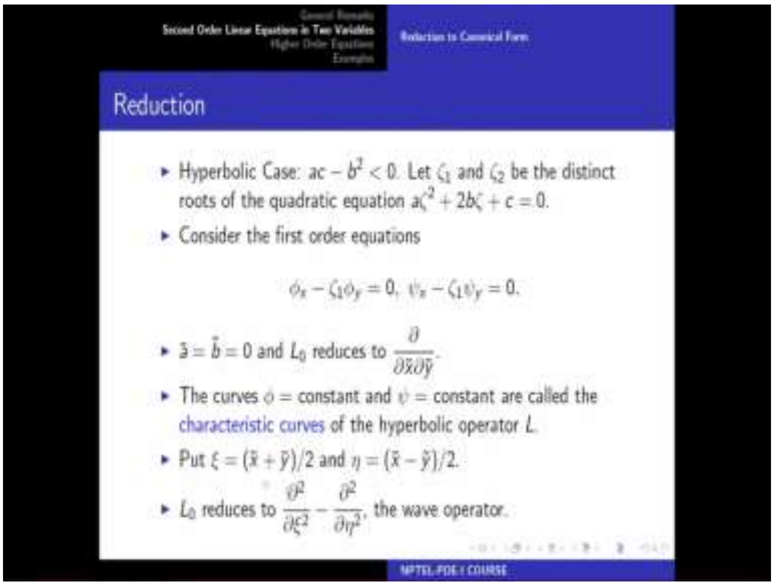

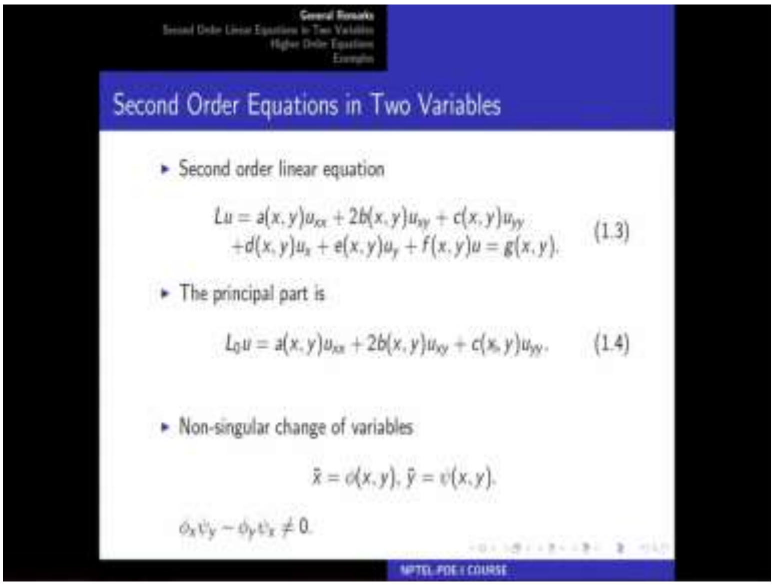

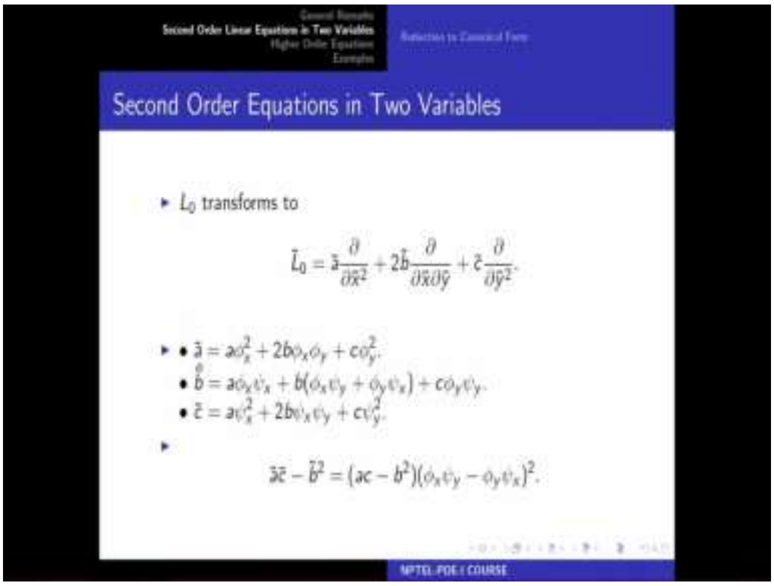

So, now, we are proceeding to find the transformations. So, that is our goal now. How do we find this Phi and Psi. So, the first case, we consider is the case, ac minus b square is negative, and that should happen not just at one point, but in that open set, that is important. And in that case, this quadratic equation, an ac minus b square is negative, this quadratic equation has two real and distinct roots and that is what we learn in high school.

And since a and a, b, c are functions of x and y, this xy1 and xy2 are also functions of x and y. And this quadratic equation plays an important role why is that? So, let us just again go back and observe these three quantities a tilde, b tilde, c tilde. So, for example, if Phi y is not 0, I can divide by Phi y square and then I get a Phi x by Phi y square; 2b Phi x by Phi y, plus c, and that is exactly this quadratic, if I replace zeta by Phi x by Phi y.

And Phi x and Phi y it cannot be equal to 0 simultaneously because of the non-singularity condition. So, you have to pay attention to all these things. So, Phi x and Phi y cannot be 0, simultaneously. And similarly Psi x and Psi y cannot be 0. And similarly, so since Psi x and Psi y are not simultaneously zero, so that Psi x by Psi y also satisfied that quadratic equation.

So, this zeta we can replace it Phi x by Phi y, or Psi x by Psi y as the case may be. And since xy1 and xy2 real and distinct, so we choose that Phi x by Phi y by the by, so for example, what happens if a equal to 0, now you take zeta square here, so that is always possible.

If a and c are 0, then we are already in good case. Let me just mention that, we will see that. If a and c are 0 there is only one mix derivative we will see how that can be converted into fewer derivatives. So, that is the easiest case. So, you begin in this case that one of them is not 0, simple things, but you have to pay attention to these things.

So, as I said now, so we can choose this Phi x by Phi y equal to zeta 1 and Psi x by Psi y equal to zeta 2, sorry for this, this should be zeta 2. So, this Phi x and Phi and Psi satisfy this first order PDE. So, xy1 and xy2 are variable because a, b, c are variables. So, xy1 and xy2 are functions of x and y. So, this we obtain Phi and Psi by solving this first order equations.

One thing you should remember here, so though they are first order equation, so we have already learned the method of characteristics to solve first order equation, and here they are linear in fact, this xy1 and xy2 are only functions of x and y, so there is absolutely no problem, but there is no initial line or initial condition.

So, in practice when you want to really obtain these Phi and Psi, some simple functions will do. So, we need not go to the full strength of the method of characteristics. So, even by observation you can write down the some solution, we are not interested in all the solutions. We are interested only in finding some simple solutions. Our only concern is non-singularity, that is important.

So we have to find Phi and Psi, so that we get no-singularity. And in this case, hyperbolic case, it is very easy to verify this Phi and Psi are non-singular in the sense that Phi x, Psi y and Phi y, Psi x is not 0, because xy1 and xy2 are different, some simple calculation you can easily verify that.

So, again go back to the expressions a tilde, b tilde, c tilde. So, with our choice of Phi x by Phi y equal to zeta 1, and Psi x by Psi y is zeta 2, where zeta 1, zeta 2 are solutions of that quadratic equation, you immediately see that a tilde and c tilde vanish. That is the -- just by looking at the those expressions of a tilde, b tilde, and that zeta 1, zeta 2 are solutions of this quadratic equation.

And so, L 0 reduces to -- so, only the middle term, the mix derivative. So, a tilde is 0, c tilde is 0, so you get only this del squared -- sorry for that, again del square, del square, del square. So, just del square, second order derivative and just this one more step to get rid of that mix derivative, so this is a very simple transformation.

So, the curves Phi equal to constant and Psi equal to constant are called characteristic curves of the hyperbolic operator L in this case, so just like the first order equation. So, here there are second order equations. So, there are two characteristics, two families of characteristics curves. And this mixed second order derivative we can convert into two pure second order derivatives by the simple transformation.

You put zeta equal to x tilde plus y tilde by 2 and eta equal to x tilde minus y tilde by 2, and then this becomes a constant multiple of this operates, I have not written that constant but that constant you can non-zero constant. So, that you can absorb in the first order terms. So, in this hyperbolic case, we are able to find Phi and Psi explicitly and finally getting the equation reduced to a pure second order form. So, this is a typical wave operator, so del square by del xy square minus del square by del eta square.

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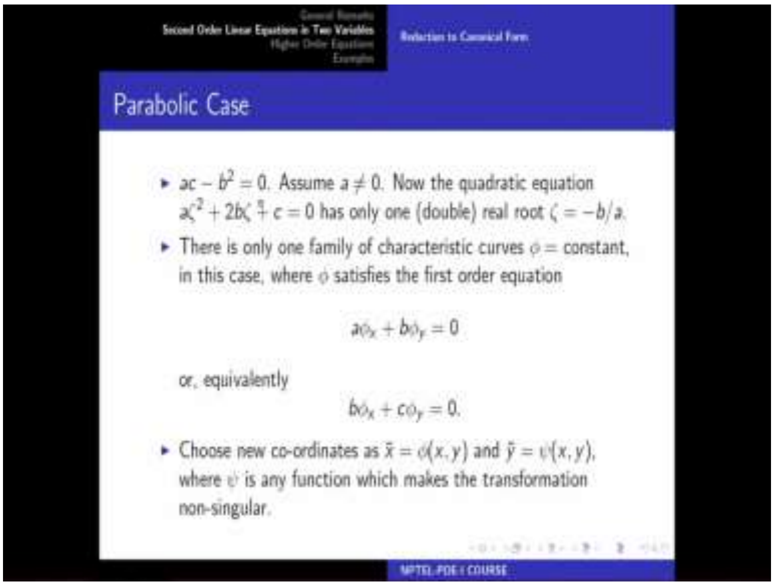

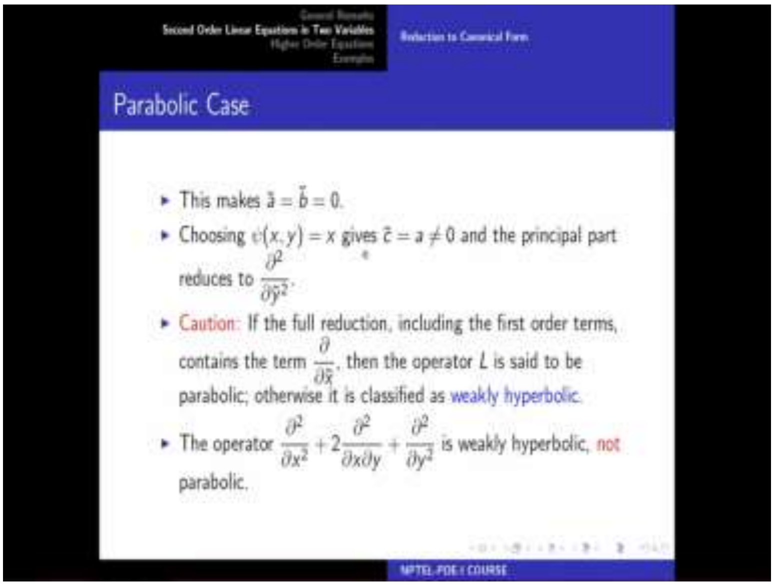

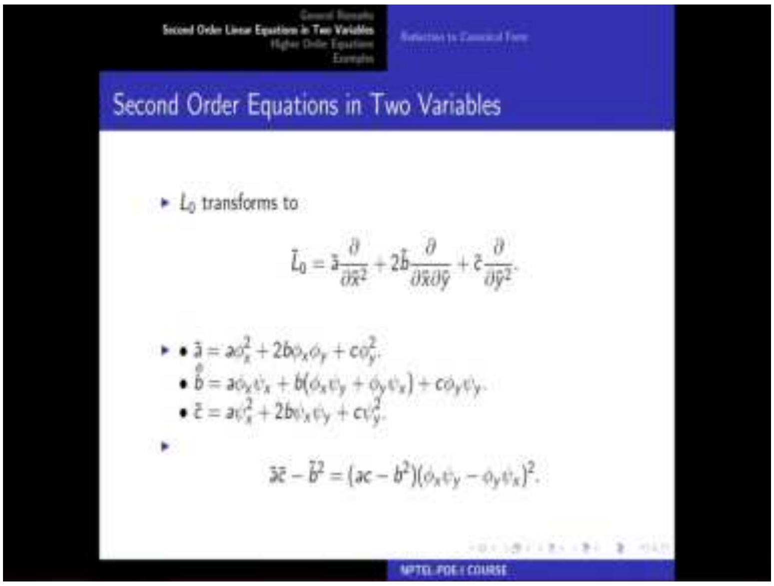

So, now we go to the next case, namely parabolic case, ac minus b square equal to 0. So, assume a not equal to 0. So, if a equal to 0, what happens if a equals 0? Then this term is gone, so b is also 0 and then again there is no second order equation at all. So, if a, b, c are 0 all second order terms are 0, so we are getting only first order question. Because we are considering second order equation, all three cannot be 0 simultaneously.

So, either a should not be 0 and see rc should not be 0. So, for definiteness, we are assuming a not equal to 0. And similarly, c is not equal 0. And in this case again that quadratic equation it has only one double real root namely zeta equal to minus b by a. And in this case, the given PDE has only one family of character stickers Phi equal to constant and that Phi satisfies now the first order equation again.

And because ac minus b square, this is also equivalent to b Phi x plus c Phi y equal to 0. So, we could now only find one function, but now we can choose any other suitable function that makes this transformation non-singular. And now you observe these two equations and again go back and see the, this is important again, a tilde, b tilde, this time we are not Psi is arbitrary for us, for the time being, we have only Phi.

And so, choosing the Phi, we get a tilde 0 and again here if you group it, so you get a Phi x plus b Phi y, Psi x and b Phi x plus c Phi y Psi y, so you plug in these things and observe that in this case, a tilde and b tilde are 0. So, there is only one-term remaining. So, this you get del square by – So, for example, here I can simply choose Psi x, y equal to x, just for the argument sake. So, this is we can freely choose, we will see in some examples.

And that gives important thing is the coefficient of c tilde is not zero and that gives me del square y -- del square divided by del y tilde square. So, again -- so this a caution, so we have the full equation here again go back in this case. We have the second order equation. And in that case ac equal to b square you reduce this equation in the new variables x tilde and y tilde, and the leading term, the second order term is definitely del square by del y tilde square, but unless there is a lower order term in del by del x tilde.

So, this is similar to the comment I made on the parabolic. So, you have to have a lower order term in the other variable, not y tilde but x tilde, then only you call it parabolic. And here is a classic example when that is not possible for a given differential equation, so that means the full reduction will not contain this del by del x tilde term then you call that as weakly hyperbolic.

And here is a simple example, you take this operator in second order, and so there is only 1 -- here ac minus b square is 0. So, a is 1 b is 1 and c is 1 . So, this ac minus b square is 0. So, there is only one real characteristic, but after the full reduction you will not get this lower order term, and this is classified as weakly hyperbolic and not parabolic.

So, in this case for two variables we could do full analysis and the analysis just depends on one quantity, namely the discriminant, ac minus b square and we could do reduction in case of hyperbolic and parabolic, and with this caution, so either parabolic or weakly hyperbolic we can do that. And now the only case that we left is ac minus b square is greater than 0 and that is bit tricky, that will take-up in the next lecture. Thank you.