

# First Course on Partial Differential Equations – 1

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Mathematics

Lecture 14

Partial Differential Equations - 1

So, welcome back. So, before taking up the quasilinear case, so let me just show you some slides.

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A hyper-surface  $S$  is called a characteristic surface for  $L$  if at each  $x \in S$ , the unit normal  $n(x)$  to  $S$  at  $x$  is in  $\text{char}_x(L)$ . This is equivalent to saying that the vector field  $a$  is tangent to  $S$  at each  $x$ . A hyper-surface  $S$  is called non-characteristic for  $L$  if it is not characteristic for  $L$  at any point  $x \in S$ .

Thus, the geometric interpretation for  $S$  to be non-characteristic is that the vector  $a$  should make an acute angle with the normal to  $S$  at every point on  $S$ . In terms of the notations used in §1, this implies that

$$\det \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_{n-1}} & a_1(h(x)) \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial h_{n-1}}{\partial x_1} & \cdots & \frac{\partial h_{n-1}}{\partial x_{n-1}} & a_{n-1}(h(x)) \end{bmatrix} \neq 0, \quad (2.3)$$

assuming that the surface  $S$  is parametrized by the functions  $h_1, \dots, h_{n-1}$ .

**Remark 2.2** The transversality condition given in case of two variables is the same as the non-characteristic condition.

With these terminologies, the analysis follows exactly as in the two dimensional linear case. First introduce the characteristic curves  $x(t) = (x_1(t), \dots, x_n(t))$  in  $\Omega$  give by the system of ODEs

$$\frac{dx}{dt} = a(x). \quad (2.4)$$

$\sum_{i=1}^n a_i(x) \frac{\partial u}{\partial x_i}$  and  $a_i, u_0, f \in C^0(\Omega)$ ,  $1 \leq i \leq n$ . The characteristic form of the operator  $L$  is defined by

$$\chi_L(x, \xi) = a(x) \cdot \xi, \xi \in \mathbb{R}^n, x \in \Omega$$

and the characteristic variety of  $L$  is defined as

$$\text{char}_x(L) = \{\xi \neq 0 : a(x) \cdot \xi = 0\}.$$

Thus,  $\text{char}_x(L) \cup \{0\}$  is a hyper-surface, orthogonal to the vector field  $a(x)$ .

**Definition 2.3**

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So, here I collected all the things I shared about the linear case so that non-characteristic condition on the surface, so this characteristic form and then we define this characteristic set or variety and so defined what the characteristic surface and non-characteristic surface. Again, if you go back and see the condition that non-characteristic condition, so this unit normal should not be perpendicular to  $ax$ , that vector  $a$ , and then that you translate into the this parametric functions  $h_1, h_2, h_n$  defining the surface  $s$  and you arrive at this determinant condition.

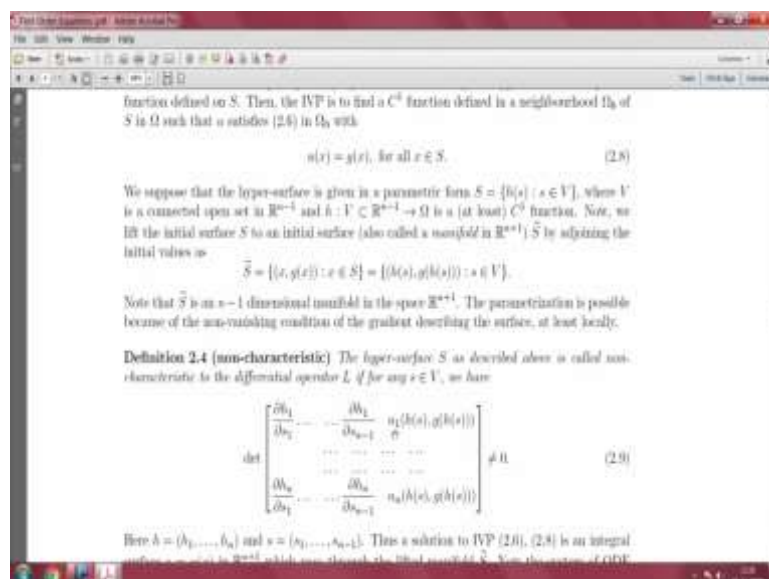
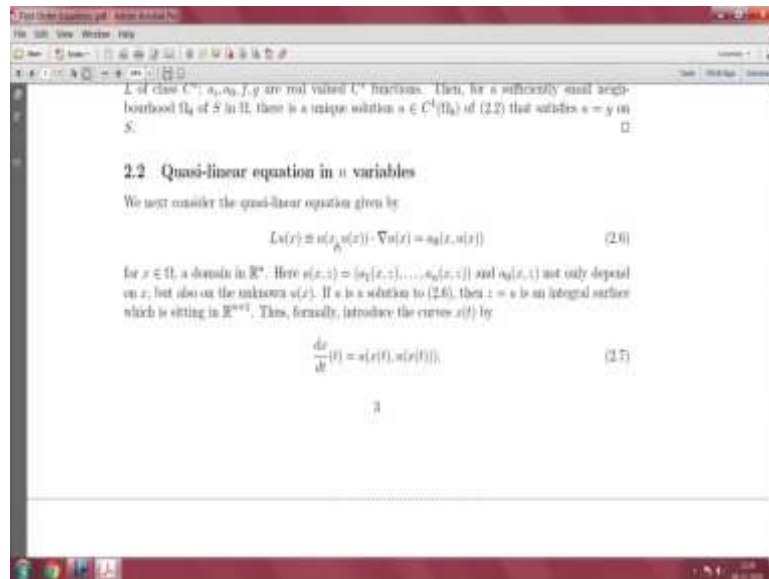
So, this determinant, so you praise the -- so, this is  $n$  by  $n$  minus 1 matrix. So, you just add one column here  $a_1, a_2$ , and again restricting the values to the surface because  $h_s$  is on the surface. So, now you get an  $n$  by  $n$  matrix and this determines should not be 0. So, this is the analytic statement of  $s$  being non-characteristic.

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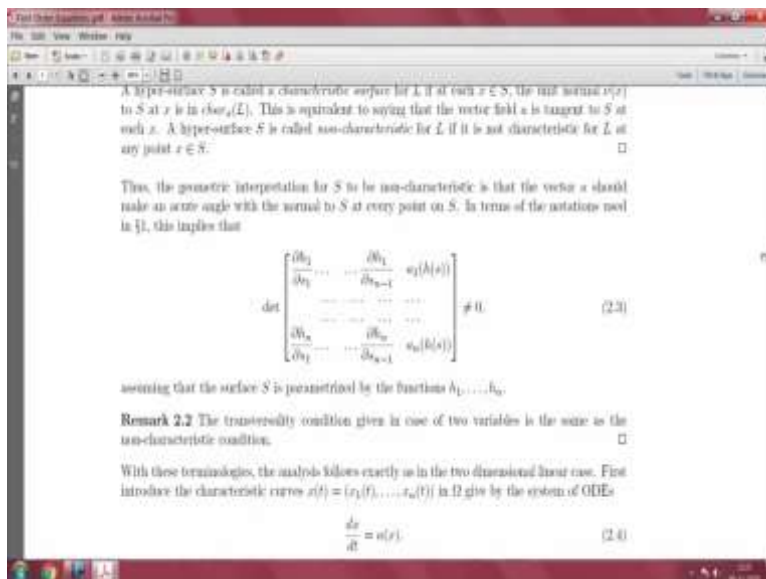
And then you just form these characteristics and then the given differential equation also is converted to an ODE on the characteristic and you derive -- you show that these are all -- you can solve all these ODE's and then again you get back your solution using inverse function theorem, using the non-characteristic condition.

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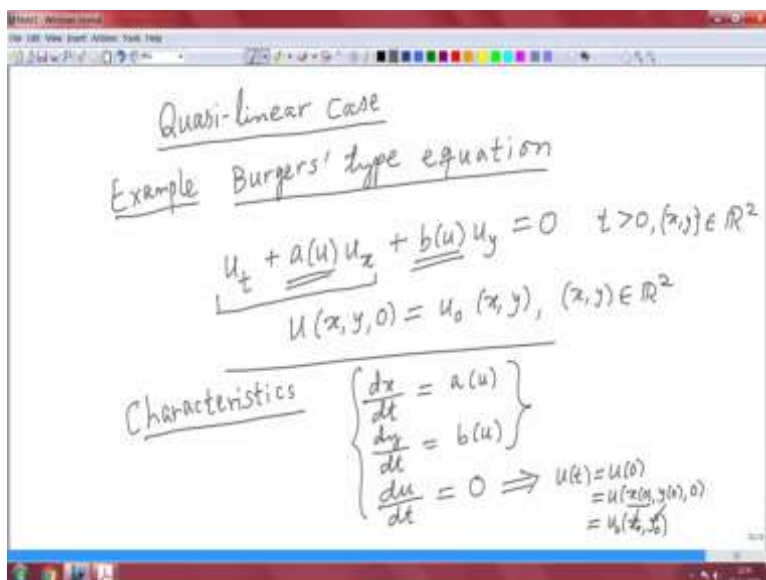
Similar thing for quasilinear equation, it is almost same but now only thing is the coefficients  $a_i$  they may also depend on  $u$ . So, that is how – that is why this system of characteristic equations first order system of ODE's is incomplete because the right hand side also depends on  $u$ . And so, we have to somehow ad-joint that and that procedure is called again given the surface you lift it to the  $n$  plus one dimension and there you define  $s$  tilde, and again the non-characteristic condition becomes.

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So, same thing as you see here. But now, these are also functions of  $u$ , so, that you are added and this condition is on  $s$  tilde. But in any case,  $s$  tilde is obtained by  $s$  by just lifting and you get again this non-characteristic condition and again the same procedure, you solve a system of ODE, and again apply implicit function theorem to get back the solution.

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So, exactly similar to the two variable case. So, let me just explain this procedure quasilinear case using some examples, examples are always better. So, quasilinear case: So, the first example is Burger type equation. So, again let me use the variables as  $t, u, x$  plus  $b, u, y$ . You have already seen this part that means only in two variable and you have seen that.

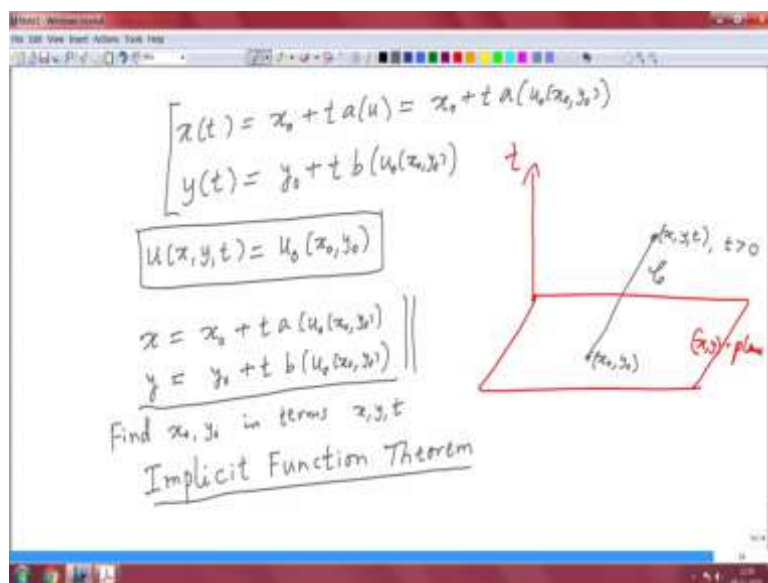
For example, if you take  $a$  equal to  $u$ , so in comparison with linear equation, so however smooth initial condition is, we see that the solution develops singularities at a finite time. So, similar phenomena we observe even in this two dimensional case. So,  $x, y$  belong to  $\mathbb{R}^2$ . So, this is quasilinear equation because  $a$  and  $b$  they depend on the co-efficients, depend on  $u$ .

So, let me again use the variable  $t$  itself as the parameter for the characteristic. So, whenever convenient we should do that instead of going for the  $s$ . Of course, to develop a general theory we have to develop in that way, but in a typical example like this, if some parameter -- some variable if it is possible to use one of the variables itself as a parameter, we should do that.

So, given this thing, so I would like to increase value and try to find solution for all  $t$  positive and let us see, just like the burger equation in one d, whether this solution develops singularities or not. So, here the characteristics again. So,  $dx$  by  $dt$  is equal to  $au$ ,  $dy$  by  $dt$  equal to  $b$ . Of course, since  $u$  itself is unknown, we cannot solve. So, that is how we have join that  $u$  also. In this case, you immediately see that  $du$  by  $dt$  along the characteristic  $0$ .

So, this is very easy to solve. So, this is just  $u$  equal to -- because it is a constant  $u_0$ . And so, that means again just see that we are abuse of notation  $y_0, 0$  and that is given as  $u_0$  let me call this as  $x_0$  and  $y_0$ . So, this is  $x_0$  and that is  $y_0$ . Since  $u$  is constant along the characteristics, we can also integrate these two equations. So, let me write that.

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So,  $x$  at  $t$  equal to 0, it is  $x_0$  plus  $t a$ . So, let me write that because that is a constant and we know that what that constant is a of  $u_0$   $x_0$ ,  $y_0$ . And similarly,  $y$  of  $t$  is equal to  $y_0$  plus  $t b$  of  $u_0$  at  $x_0$ ,  $y_0$ . Our task is -- so, again by this also a straight line. So, let me just draw that. So, this is the  $xy$  plane and this is the  $tx$ . Now, our next task is -- so, this starting anywhere  $x_0$ ,  $y_0$ . So, this is the  $xy$  plane and that is  $t$  equal to 0 there, so this is a straight line.

So this is  $c$ , characteristics. Now, our task is to find the solution. So, given any  $x$ ,  $y$ ,  $t$ ,  $t$  positive, we should be able to draw a characteristic meeting the  $xy$  plane at  $x_0$ ,  $y_0$  and then we should be able to actually figure out what  $x_0$ ,  $y_0$ , and in that case, we have  $x$ ,  $y$ ,  $t$ , because  $u$  is constant along the characteristic. So, given this  $x$ ,  $y$ ,  $t$ , if this  $x_0$ ,  $y_0$  unique then we do get the solution at  $x_0$ ,  $y_0$ , just like the burger's equation in the one dimensional case.

So, our task is, so now consider this equations,  $t a$ ,  $u_0$ ,  $x_0$ ,  $y_0$ , and  $y$ , 0. So, task is to find  $x_0$ ,  $y_0$  in terms of  $x$ ,  $y$ ,  $t$  and these are nonlinear equations. So, we will not be able to find them explicitly. But the implicit function theorem comes to our rescue, whether such a thing is possible or not even that we are not able to decide just by looking at these two equations.

So, we have to apply implicit function theorem in order to find out whether  $x_0$ ,  $y_0$  from this two equations are expressible in terms of  $x$ ,  $y$ ,  $t$ . So, we want to solve these two equations for  $x_0$ ,  $y_0$  in terms of  $x$ ,  $y$ ,  $t$  and that those are given to us. So, given a point  $x$ ,  $y$ ,  $t$ , we want to find the  $x_0$ ,  $y_0$  and then the solution is given by this. Once we find the  $x_0$ ,  $y_0$ , the solution is given by that.

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The image shows a whiteboard with handwritten mathematical work. At the top, a 2x2 matrix is written with its determinant set to not equal zero. The matrix elements are: top-left is  $1 + t a'(u_0, v_0) \frac{\partial u_0}{\partial x_0}$ , top-right is  $t b' \frac{\partial u_0}{\partial x_0}$ , bottom-left is  $t a' \frac{\partial u_0}{\partial y_0}$ , and bottom-right is  $1 + t b' \frac{\partial u_0}{\partial y_0}$ . Below this, an arrow points to a boxed expression:  $1 + t (b' \frac{\partial u_0}{\partial y_0} + a' \frac{\partial u_0}{\partial x_0}) \neq 0$ . Underneath the box, it says "We can expect this to be non-zero for small  $t > 0$ ".

And what does the implicit function theorem tells, so you look at the determinant of the Jacobian. So, this is a 2 by 2t. So, let me just write it. So, 1 plus t, a prime at  $u_0$  x. So, everything we are writing at  $x_0, y_0$ . So, let me just try it once and then I leave it. So,  $\frac{\partial u_0}{\partial x_0}$  by  $\frac{\partial u_0}{\partial x_0}$ , t b prime. So, here the derivative a prime, b prime are with respect to that corresponding variable namely da by du, dv by du and this is  $\frac{\partial u_0}{\partial x_0}$  not by  $\frac{\partial u_0}{\partial x_0}$  not, and similarly, here t, a prime  $\frac{\partial u_0}{\partial y_0}$  by  $\frac{\partial u_0}{\partial y_0}$ , 1 plus t, b prime.

So, that should be b prime  $\frac{\partial u_0}{\partial y_0}$  by  $\frac{\partial u_0}{\partial y_0}$  should not be 0. Then the implicit function theorem tells us that the  $x_0$  and  $y_0$  are expressible in terms of x, y, t and what does this condition -- so, this is just 2 by 2 matrix, so you can easily determine and this is 1 plus t, b prime,  $\frac{\partial u_0}{\partial y_0}$  by  $\frac{\partial u_0}{\partial y_0}$  plus a prime  $x_0$  should not be.

In case of one variable, we have only had one term, now both the terms are combined here. And that t equal to 0 this determinate is 1, so certainly not 0. So, we can expect this to be non 0, this means this left hand side for small t positive. Of course, if this term in the bracket remains non-negative, so similar to the one dimensional burger's equation, this always true.

But when that changes sign, then there is a finite time when this becomes 0 and the solution becomes singular there, namely it is derivative when so, similar to the burger's equation in one dimension a theory can be developed for this also and even higher dimensions same procedure, but the single equation. And so, you just like burger's equation we continue at the

solution for all time as a weak solution all those things, are all that theory can be developed here also.

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Example  $u_t + (x+u)\frac{\partial u}{\partial x} + (y+u)\frac{\partial u}{\partial y} = 0, t > 0, (x,y) \in \mathbb{R}^2$   
 $u(x,y,0) = x+y$

Char  $\begin{cases} \frac{dx}{dt} = x+u \\ \frac{dy}{dt} = y+u \\ \frac{du}{dt} = 0 \end{cases}$

$\frac{du}{dt} = 0 \Rightarrow u(t) = u(x_0, y_0, 0) = x_0 + y_0 = u_0$

$\begin{cases} x = (x_0 + u_0)e^t - u_0 \\ y = (y_0 + u_0)e^t - u_0 \end{cases} \Rightarrow \begin{cases} (x,y,t), t > 0 \\ x_0 + y_0 = \frac{x+y}{3e^t - 2} \end{cases}$

$\Rightarrow u(x,y,t) = \frac{x+y}{3e^t - 2}, t > 0$

Nonlinear Eqn  $F(x, u, p) = 0$

My second example again a quasilinear equation similar to burgers equation, but now we have something different there. So,  $x + u, \frac{du}{dx} + y + u, \frac{du}{dy} = 0$ . So, examples tell us there are many different situations, so we saw in that burgers type equation similarities will develop, but here we see in this particular initial value problem, again  $t = 0$  is a non-characteristic here.

So, we should take different initial conditions and try to see what happens. So, this is also quasilinear, because the coefficients depend on the solution  $u$ . But now, there is an  $x$  there,



there is an  $y$  also. So, the coefficients also depend on  $x$  and  $y$ . So, again the characteristics, so  $dx$  by  $dt$  is equal to  $x$  plus  $u$  and  $dy$  by  $dt$  is equal to  $y$  plus  $u$ . And similar to the previous example, we also have  $du$  by  $dt$  equal to  $0$ .

So, that implies, again like previous thing, so  $x_0, y_0, 0$ . And this is by the initial condition, this is just  $x_0$  plus  $y_0$ . So,  $x_0, y_0$  are the initial conditions for these two equations. So, plugging in this extra information, we got in these two equations and integrate them. So, it is very easy to do that. So, we get  $x$  is equal to  $x_0$ . Let me just directly computation, so just I write it here  $x_0$  plus -- so let me call this as  $u_0$  or simplification.

So,  $u_0, e$  to the minus  $t$ , minus  $u_0$ . So you integrate this -- so, it is no more a constant, but there is an  $x$  term. So, that is how you will get that exponential term. So, and similarly,  $y$  is equal to  $y_0$  plus  $u_0, e$  to the minus  $t$ , minus  $u_0$ . So, again given  $x, y, t$ , a point, with  $t$  positive, we want to see -- now, this is no more a straight line, it is an exponential thing here.

So, it is a characteristic curve and we want to find out the characteristic curve meeting the  $xy$  plane. So, this is  $t; x, y, t$ ,  $t$  positive, wherever it hits. From these two equations, it is easy to solve for  $x_0, y_0$ , but to do not need that. What we need is,  $x_0$  plus  $y_0$  together, that is much easier to obtain. So just little algebra, just leave it.

So this implies  $x_0$  plus  $y_0$  is equal to  $x$  plus  $y$  divided by  $3e$  to  $t$  minus  $2$  and therefore,  $u, x, y, t$ , so this is also quasilinear equation. But we are able to find an explicit representation of the solution, which is valid not only for all  $x, y$ , but all  $t$  bigger than equal to  $0$ . So, just because a equation is quasilinear, we should not say that it will develop singularities at finite time. So, here you do not see any singularities, very smooth. So, obviously, some effect is coming from addition of this two.

So, now, let me again from this move on to the fully non-linear case. So, non-linear equation, of course, this even in two dimensions you have seen is quite complicated. So, it required the introduction of more geometrical objects like Monge cone, characteristics strips, and so many other things. So, you can expect same difficulty to continue you want in higher dimensions. So, this  $x, u$ , so this is how we started, again non-characteristic quasi problem.

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Here  $k = (k_1, \dots, k_n)$  and  $s = (s_1, \dots, s_{n-1})$ . Thus a solution to IVP (2.6), (2.8) is an integral surface  $z = z(x)$  in  $\mathbb{R}^{n+1}$  which pass through the lifted manifold  $S$ . Now the system of ODE (2.7) can be completed by adjoining the equation for  $z$ . Since  $z(t) = z(x(t))$ , we get

$$\frac{dz}{dt} = \sum_{i=1}^n \frac{\partial z}{\partial x_i} \frac{dx_i}{dt} = \sum_{i=1}^n a_i(x(t), s(t)) \frac{\partial z(x(t))}{\partial x_i} = a_0(x(t), s(t)). \quad (2.10)$$

The IVP (2.6), (2.8) can be solved as follows. For any point in  $S$  which is given by  $(k(x), g(x))$  for some  $x \in V$ , solve the complete ODE system (2.7), (2.10) with initial values  $x(0) = x_0, s(0) = s_0, z(0) = z_0$ . To complete the analysis, we need to invert an algebraic system as in two dimensional case. That is, for  $x \in \Omega_k$ , a neighbourhood close to  $S$ , we have to find a time  $t$  and  $s \in V$  such that

$$x(t) = x. \quad (2.11)$$

The non-characteristic condition (2.9) together with inverse function theorem ensures the above claim. Let  $\eta(x), s(x)$  solve (2.11), then  $z(x) = z(\eta(x), s(x))$  solves the IVP (2.6), (2.8). Thus, we have the following theorem.

**Theorem 2.5 (Existence and uniqueness)** Consider the IVP (2.6), (2.8) where  $a, a_0, g$  are real valued  $C^1$  functions. Let  $S$  be a hyper-surface of class  $C^1$  in  $\mathbb{R}^n$  which satisfies the condition (2.9). Then, for a sufficiently small neighbourhood  $\Omega_k$  of  $S$ , there is a unique solution  $z \in C^1(\Omega_k)$  of (2.6), (2.8).

$F(x, z, p) = a(x) \cdot p + a_0(x)z - f(x)$  and in the quasi-linear case, we have  $F(x, z, p) = a(x) \cdot z$ ,  $p = a_0(x, z)$ . Thus, in the either of the cases  $a = \nabla_p F$ . This together with the analysis of general case in two variables, motivates us to define the characteristic curves  $x(t)$  as the integral curves of the vector field  $\nabla_p F$ . That is, define  $x = x(t)$  as

$$\frac{dx}{dt} = \nabla_p F. \quad (2.13)$$

Now, adjain the equation for  $z$  as

$$\frac{dz}{dt} = \sum_{i=1}^n \frac{\partial z}{\partial x_i} \frac{dx_i}{dt} = p \cdot \nabla_p F. \quad (2.14)$$

Notice that in the linear case  $\nabla_p F = a(x)$  does not depend on the unknown  $z$  and hence (2.13) is a complete system. In the quasi-linear case  $\nabla_p F = a(x, z)$  and thus (2.13) together with (2.14) is a complete system. In the general case  $\nabla_p F$  and  $p \cdot \nabla_p F$  may depend not only on the unknown  $z$ , but also on the  $n$  derivatives  $p = \nabla_x z$ . Hence, we need to derive  $n$  equations for  $\frac{dp_i}{dt}$ . Now,  $p_i(t) = p_i(x(t))$  and compute

$$\frac{dp_i}{dt} = \sum_{j=1}^n \frac{\partial p_i}{\partial x_j} \frac{dx_j}{dt} = \sum_{j=1}^n \frac{\partial^2 F}{\partial x_j \partial x_i} \frac{\partial F}{\partial p_j} = \sum_{j=1}^n \frac{\partial^2 F}{\partial x_i \partial x_j} \frac{\partial F}{\partial p_j}. \quad (2.15)$$

On the right hand side, we have the undesired second derivatives  $\frac{\partial^2 F}{\partial x_i \partial x_j}$ . We need to eliminate it.

The same theme as, as in the case of two variable continues. So, you again describe the characteristic equations. And now, you will have  $n$  equations coming from the -- for the  $x$  variables and one from the solution and then there are  $n$  equations coming from the first derivatives. This was not there even in the quasilinear case. So, these are additional things. Now, there are two  $n$  plus 1 equations, and two  $n$  plus 1 unknowns.

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$\frac{dy_i}{dt} = \sum_{j=1}^n \frac{\partial y_j}{\partial x_i} \frac{dx_j}{dt} = \sum_{j=1}^n \frac{\partial^2 F}{\partial x_i \partial x_j} \frac{\partial F}{\partial y_j} = \sum_{j=1}^n \frac{\partial y_j}{\partial x_i} \frac{\partial F}{\partial y_j}$  (2.13)

On the right hand side, we have the undesired second derivatives  $\frac{\partial y_j}{\partial x_i}$ . We need to eliminate it. Differentiating (2.12) with respect to  $x_i$ , we get

$$\frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial z} \frac{dz}{dx_i} + \sum_{j=1}^n \frac{\partial F}{\partial y_j} \frac{\partial y_j}{\partial x_i} = 0.$$

Thus, we arrive at

$$\frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i} - \frac{\partial F}{\partial z} \frac{dz}{dx_i}$$
 (2.14)

for  $1 \leq i \leq n$ . Hence, we have a system of  $2n + 1$  equations given by (2.13), (2.14), (2.15) for  $x(t), z(t), p(t)$ . Thus, we are solving not only for the unknown, but for the derivatives  $\frac{dy_i}{dt}$  as well, exactly what we have seen in the two dimensional case.

The set-up is similar for IVP as in the quad-linear case like defining  $S$  where the initial values are defined and then lift it to the initial surface  $\tilde{S}$ . Now important issue is the identification of a initial condition for  $p(t)$  for the system (2.14). This is done as follows. Recall  $x(0) = x(0, s) =$

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$$\frac{\partial z}{\partial x_i} = \sum_{j=1}^n \frac{\partial z}{\partial y_j} \frac{\partial y_j}{\partial x_i} = \sum_{j=1}^n p_j \frac{\partial y_j}{\partial x_i}$$
 (2.17)

$\uparrow$   
 $\downarrow$

But I would like to draw your attention is to this strip condition. And so, this is this trip condition, and we have this a -- see this variables pi. So, we have obtained ODE's for them, but there are no initial conditions for them, pi. So, for initial conditions for this x, we can take it a point on the given surface, and initial condition for z comes from the initial condition imposed.

So, initial condition for x and z there is absolutely no problem. But for p, there is no initial condition. So, we cannot really solve this set of two n plus 1 ODE and we are assuming that there are functions which qualify to be initial conditions for the t equations. These t equations. And this is a restriction on the function f. So, that I want to write it.

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Nonlinear Eqn  $F(x, u, p) = 0$

Initial conditions on  $p_i$ 's

$$F(x(s), u(s), p(s)) = 0$$
known                      unknown

$$\frac{du}{ds} = \sum_{j=1}^n p_j(s) \frac{dx_j}{ds}$$

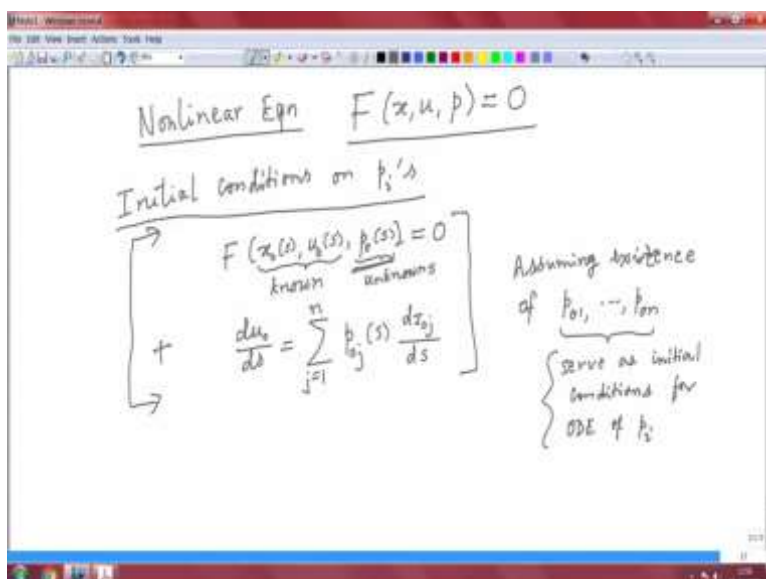
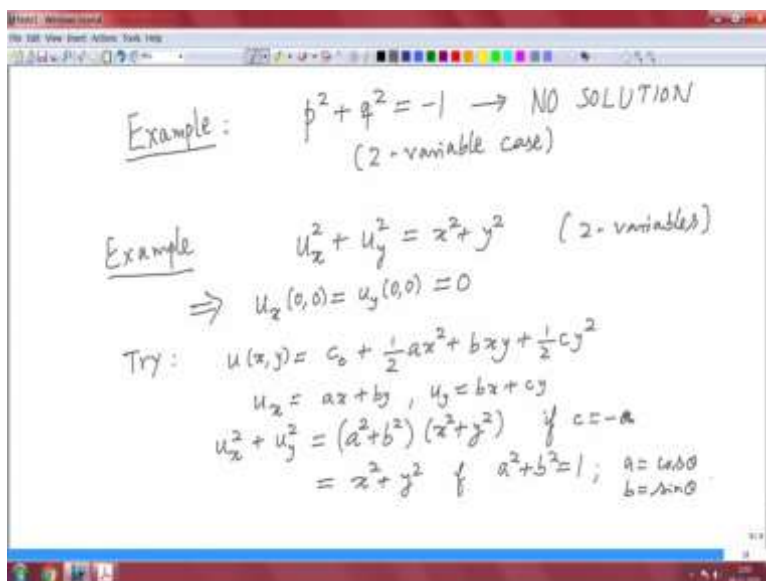
Assuming existence of  $p_{01}, \dots, p_{0n}$

serve as initial conditions for ODE of  $p_i$

So, here the initial conditions. So, I want to make a comment on this, initial conditions on  $p_i$ . So, they are assumed in the form this  $x_0, s, s$  is parameter on the surface, and  $u_0$ , and then  $p_0$ . So, this you should remember. We are assuming -- so, these are known, these are unknowns. So, we are making an assumption that these functions exist, so that this equation is satisfied, plus this trip condition, that is also important.

So, namely  $du_0$  by  $ds$  equal to  $p_0 dx_0$  by  $ds$ , these are -- this  $n$  case, so just let me write it in one go. Summation  $p_0, j, s, dx_0, j$  by  $ds$ . We are assuming the existence of  $p_{01}$ , and these  $p_{01}, p_{02}, p_{0n}$  they serve as initial condition for ODE of  $p_i$ . So, then the system of characteristic equation complete and in general it is very difficult. Given a situation -- in some situations, we can easily decide whether such functions exist or not, but in some other case it is very difficult.

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So, I would like to illustrate with some examples. So, this example is very simple. So, I take in two variables, just two variables. So, instead of  $p_1, p_2$ , I am using  $p, q$ . So, we can immediately conclude no solution. As we are seeking only real solution we can immediately solve two variable case. Let me mention that. Let me illustrate with another example. So, where it becomes so difficult why a certain equation has solution. And so, this depends on the initial condition also.

So, just remember here, your equation is in one, and something else is also. So, another example. So  $u_x$  square again two variable. Let me write it. So, this you can extend it to more than two variables, I am just taking a simple equation. I will come to the initial condition

little later. So, this implies just by looking at the equation  $u_x$  at  $(0, 0)$  equal to  $u_y$  at  $(0, 0)$  equal to 0. At the origin, so this right hand side is zero and then these are squares, they also must be zero. So, we can try a function of the form whose first derivative is at the origin vanishes.

So, this is some quadratic, so we can try that. So, constant  $C_0$  we can add that will not affect the equation at all. So, the  $C_0$  is a constant, and we will assume a quadratic function. So, at least these conditions are satisfied by this. So, then we have a simple calculation. So, this is just  $ax^2 + by^2$ , and  $u_y$  is  $2by$ ;  $a, b, c$ , are constants. So, we can see that this  $u_x$  square and  $u_y$  square, if we do not want the cross, mainly the  $x, y$  term.

So, we should have -- so, this is just comes to the, so let me write it simple calculation if  $c$  is equal to minus  $a$ . You take  $c$  equal to minus  $a$ , and you see that the cross the  $x, y$  term gets cancelled,  $x, y$  term comes here;  $x, y$  term comes there, so that get cancelled and you have this simple expression and this is equal to  $x^2 + y^2$ , if  $a^2 + b^2$  equal to one. So, that let us call that  $a$  equal to  $\sin \theta$ ,  $b$  equal to  $\cos \theta$  or  $\sin \theta$ .

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Handwritten mathematical derivation on a whiteboard:

$$u(x,y) = C_0 + \frac{1}{2} \cos \theta x^2 \pm \sin \theta xy - \frac{1}{2} \cos \theta y^2$$

$$u(x,0) = C_0 + \frac{1}{2} \cos \theta x^2, \text{ on } y=0$$

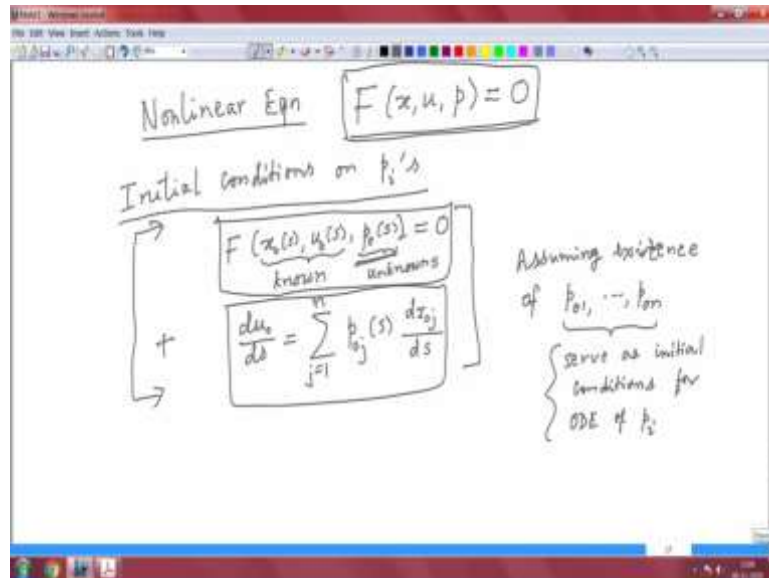
$| \cos \theta | \leq \frac{1}{2}$

Can we take  $u(x,0) = C_0 - x^2$ ? or  $C_0 + x^2$ ?

So, you can check that. So  $u, xy$  is a solution, constant half  $\cos \theta$ ,  $x^2$ , in fact you can now take plus or minus  $b, x, y$  minus half for  $\sin \theta$ ,  $y^2$ . And suppose, we -- so there are two solutions. You see immediately there are two solutions  $C_0 + \frac{1}{2} \cos \theta x^2$ . And just observe this coefficient here. So, if the absolute value is always less than or equal to half.

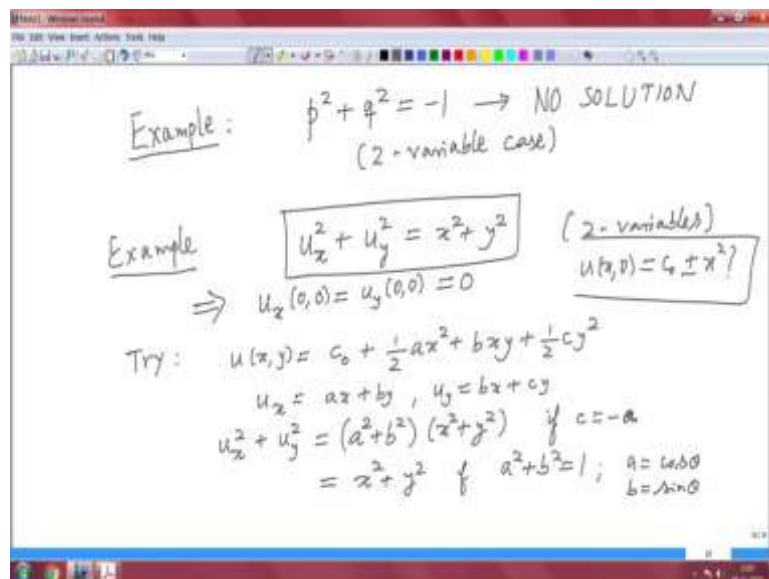
So, this on y equal to 0. So, suppose -- so, this solution satisfies this initial condition. So, my question now is, can we take  $u(x,0)$ , some constant say, any constant, so constant we see that is not bothered minus  $x$  square or  $c_0$  plus  $x$  square, answer is not easy. So, I want you to study this example in carefully. And you see, see this is decent looking equation and decent looking initial conditions.

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Yes, it is difficult to see whether it has a solution or not. So, it only shows that the method of characteristic still has some mystery which are not revealed in full. So, in particular the study of first order equations, there are still some gaps.

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So, because we are not able to say completely that such an equation, very decent looking equation, with some conditions like this initial conditions whether it has a solution or not. So, with that, I come to an end of this discussion on non-linear first order equations, non-linear equations are really difficult. So, you have to study them very carefully. And even in simple examples, you see we face difficulties. So, try to do these things by method of characteristics and see the difficulties yourself. Thank you.