First Course on Partial Differential Equations – 1
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Lecture 14
Partial Differential Equations - 1

So, welcome back. So, before taking up the quasilinear case, so let me just show you some slides.

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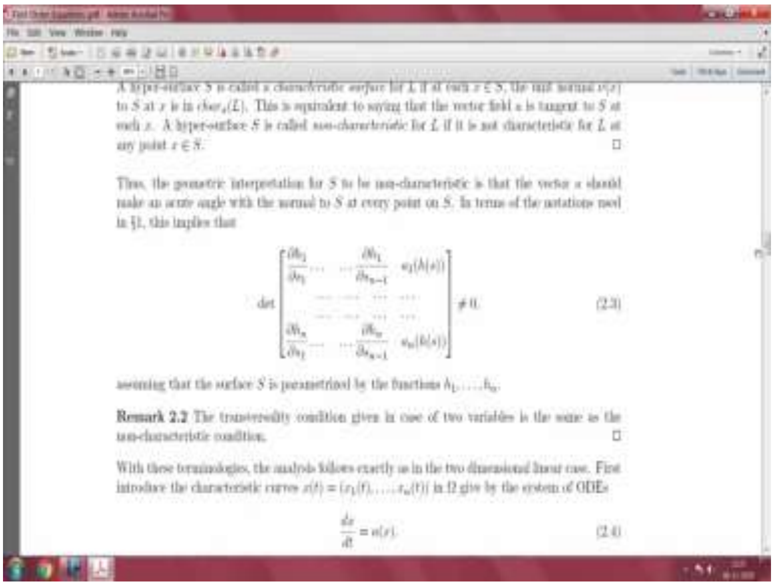

So, here I collected all the things I shared about the linear case so that non-characteristic condition on the surface, so this characteristic form and then we define this characteristic set or variety and so defined what the characteristic surface and non-characteristic surface. Again, if you go back and see the condition that non-characteristic condition, so this unit normal should not be perpendicular to ax, that vector a, and then that you translate into the this parametric functions h1, h2, hn defining the surface s and you arrive at this determinant condition.

So, this determinant, so you praise the -- so, this is n by n minus 1 matrix. So, you just add one column here a1, a2, and again restricting the values to the surface because hs is on the surface. So, now you get an n by n matrix and this determines should not be 0. So, this is the analytic statement of s being non-characteristic.

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And then you just form these characteristics and then the given differential equation also is converted to an ODE on the characteristic and you derive -- you show that these are all -- you can solve all these ODE’s and then again you get back your solution using inverse function theorem, using the non-characteristic condition.

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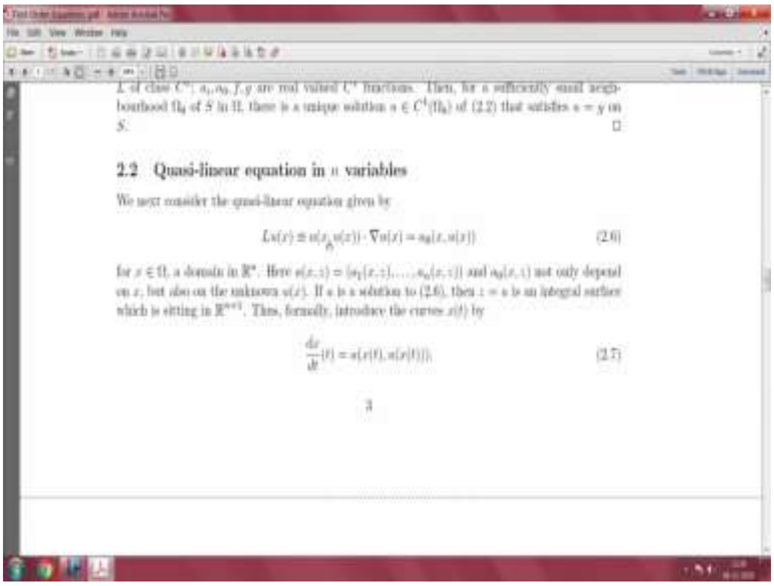

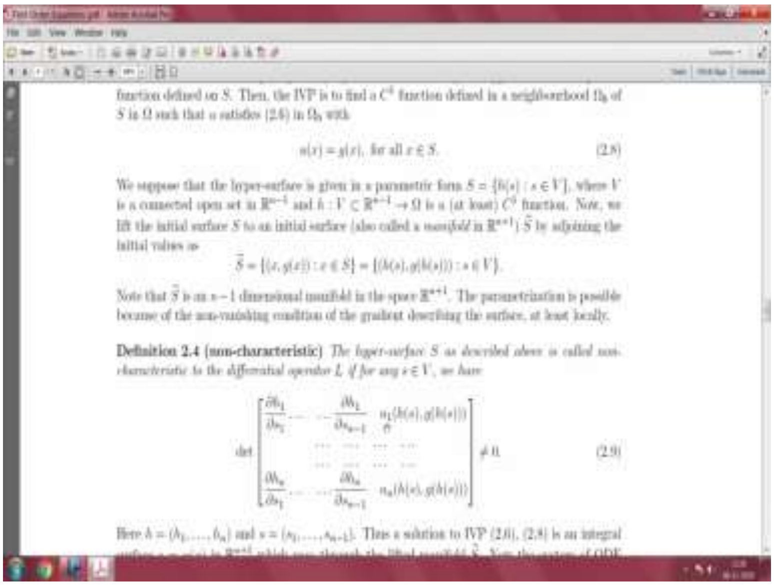

Similar thing for quasilinear equation, it is almost same but now only thing is the co-efficients ai they may also depend on u. So, that is how – that is why this system of characteristic equations first order system of ODE’s is incomplete because the right hand side also depends on u. And so, we have to somehow ad-joint that and that procedure is called again given the surface you lift it to the n plus one dimension and there you define s tilde, and again the non-characteristic condition becomes.

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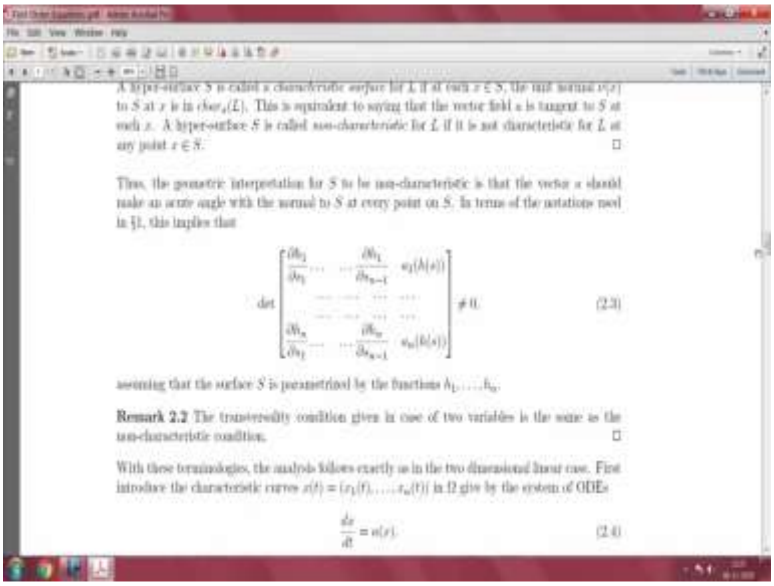

So, same thing as you see here. But now, these are also functions of u, so, that you are added and this condition is on s tilde. But in any case, s tilde is obtained by s by just lifting and you get again this non-characteristic condition and again the same procedure, you solve a system of ODE, and again apply implicit function theorem to get back the solution.

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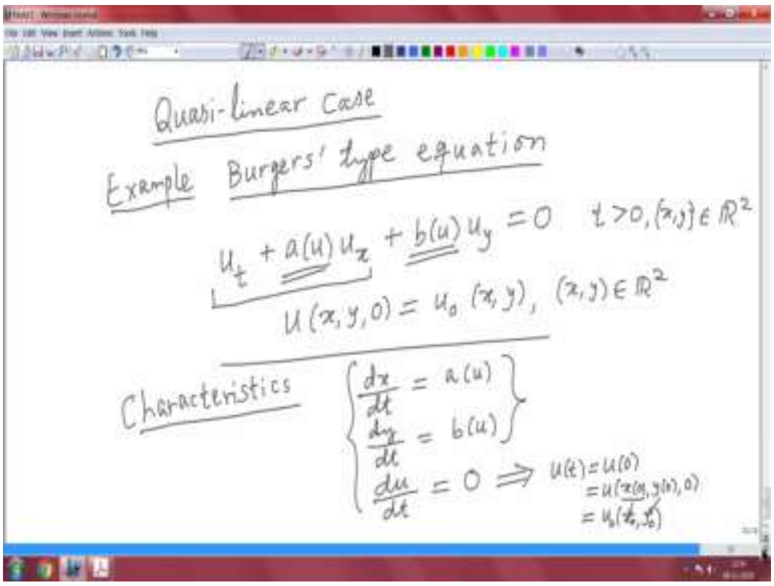

So, exactly similar to the two variable case. So, let me just explain this procedure quasilinear case using some examples, examples are always better. So, quasilinear case: So, the first example is Burger type equation. So, again let me use the variables as t, au, ux plus bu, uy. You have already seen this part that means only in two variable and you have seen that.

For example, if you take a equal to u, so in comparison with linear equation, so however smooth initial condition is, we see that the solution develops singularities at a finite time. So, similar phenomena we observe even in this two dimensional case. So, xy belong to R2. So, this is quasilinear equation because a and b they depend on the co-efficients, depend on u.

So, let me again use the variable t itself as the parameter for the characteristic. So, whenever convenient we should do that instead of going for the s. Of course, to develop a general theory we have to develop in that way, but in a typical example like this, if some parameter -- some variable if it is possible to use one of the variables itself as a parameter, we should do that.

So, given this thing, so I would like to increase value and try to find solution for all t positive and let us see, just like the burger equation in one d, whether this solution develops singularities or not. So, here the characteristics again. So, dx by dt is equal to au, dy by dt equal to b. Of course, since u itself is unknown, we cannot solve. So, that is how we have join that u also. In this case, you immediately see that du by dt along the characteristic 0.

So, this is very easy to solve. So, this is just u equal to -- because it is a constant u0. And so, that means again just see that we are abuse of notation y0, 0 and that is given as u0 let me call this as x0 and y0. So, this is x0 and that is y0. Since u is constant along the characteristics, we can also integrate these two equations. So, let me write that.

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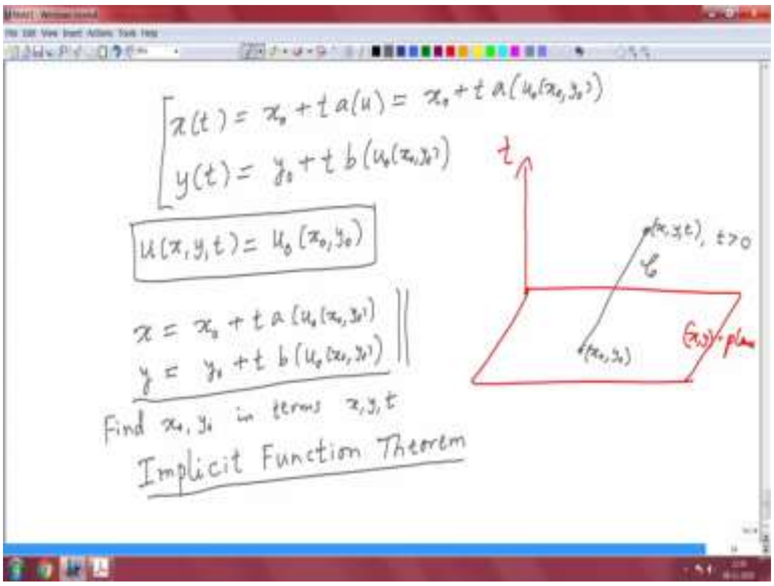

So, xt at t equal to 0, it is x0 plus t au. So, let me write that because that is a constant and we know that what that constant is a of u0 x0, y0. And similarly, y of t is equal to y0 plus t b of u0 at x0, y0. Our task is -- so, again by this also a straight line. So, let me just draw that. So, this is the xy plane and this is the tx. Now, our next task is -- so, this starting anywhere x0, y0. So, this is the xy plane and that is t equal to 0 there, so this is a straight line.

So this is c, characteristics. Now, our task is to find the solution. So, given any x, y, t, t positive, we should be able to draw a characteristic meeting the xy plane at x0, y0 and then we should be able to actually figure out what x0, y0, and in that case, we have x, y, t, because u is constant along the characteristic. So, given this x, y, t, if this x0, y0 unique then we do get the solution at x0, y0, just like the burger's equation in the one dimensional case.

So, our task is, so now consider this equations, ta, u0, x0, y0, and y, 0. So, task is to find x0, y0 in terms of x, y, t and these are nonlinear equations. So, we will not be able to find them explicitly. But the implicit function theorem comes to our rescue, whether such a thing is possible or not even that we are not able to decide just by looking at these two equations.

So, we have to apply implicit function theorem in order to find out whether x0, y0 from this two equations are expressible in terms of x, y, t. So, we want to solve these two equations for x0, y0 in terms of x, y, t and that those are given to us. So, given a point x, y, t, we want to find the x0, y0 and then the solution is given by this. Once we find the x0, y0, the solution is given by that.

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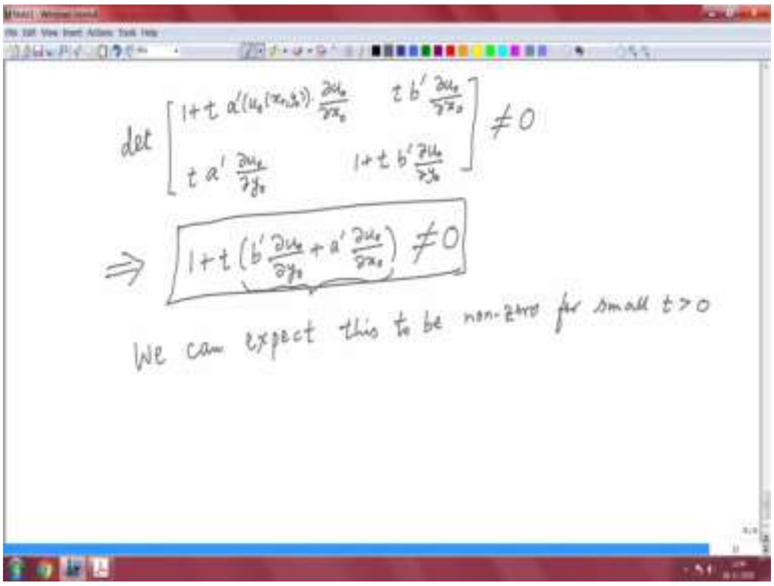

And what does the implicit function theorem tells, so you look at the determinant of the Jacobian. So, this is a 2 by 2t. So, let me just write it. So, 1 plus t, a prime at u0 x. So, everything we are writing at x0, y0. So, let me just try it once and then I leave it. So, dell u0 by dell x0, t b prime. So, here the derivative a prime, b prime are with respect to that corresponding variable namely da by du, dv by du and this is dell u not by dell x not, and similarly, here t, a prime dell u0 by del y0, 1 plus t, b prime.

So, that should be b prime dell u0 by dell y0 should not be 0. Then the implicit function theorem tells us that the x0 and y0 are expressible in terms of x, y, t and what does this condition -- so, this is just 2 by 2 matrix, so you can easily determine and this is 1 plus t, b prime, dell u by del y0 plus a prime x0 should not be.

In case of one variable, we have only had one term, now both the terms are combined here. And that t equal to 0 this determinate is 1, so certainly not 0. So, we can expect this to be non 0, this means this left hand side for small t positive. Of course, if this term in the bracket remains non-negative, so similar to the one dimensional burger's equation, this always true.

But when that changes sign, then there is a finite time when this becomes 0 and the solution becomes singular there, namely it is derivative when so, similar to the burger's equation in one dimension a theory can be developed for this also and even higher dimensions same procedure, but the single equation. And so, you just like burger's equation we continue at the solution for all time as a week solution all those things, are all that theory can be developed here also.

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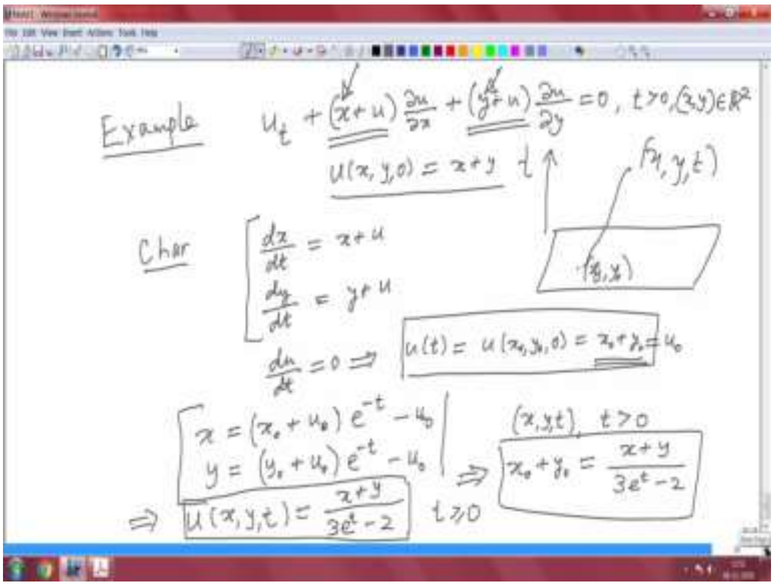

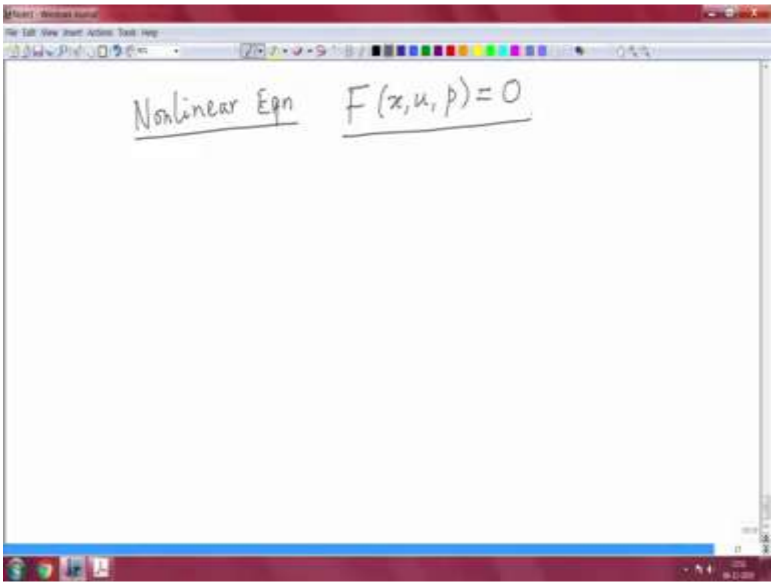

My second example again a quasilinear equation similar to burgers equation, but now we have something different there. So, x plus u, du by dx plus y plus u, du by dy equal to 0. So, examples tell us there are many different situations, so we saw in that burgers type equation similarities will develop, but here we see in this particular initial value problem, again t equal to 0 is a non-characteristic here.

So, we should take different initial conditions and try to see what happens. So, this is also quasilinear, because the coefficients depend on the solution u. But now, there is an x there,

there is an y also. So, the coefficients also depend on x and y. So, again the characteristics, so dx by dt is equal to x plus u and dy by dt is equal to y plus u. And similar to the previous example, we also have du by dt equal to 0.

So, that implies, again like previous thing, so x0, y0, 0. And this is by the initial condition, this is just x0 plus y0. So, x0, y0 are the initial conditions for these two equations. So, plugging in this extra information, we got in these two equations and integrate them. So, it is very easy to do that. So, we get x is equal to x0. Let me just directly computation, so just I write it here x0 plus -- so let me call this as u0 or simplification.

So, u0, e to the minus t, minus u0. So you integrate this -- so, it is no more a constant, but there is an x term. So, that is how you will get that exponential term. So, and similarly, y is equal to y0 plus u0, e to the minus t, minus u0. So, again given x, y, t, a point, with t positive, we want to see -- now, this is no more a straight line, it is an exponential thing here.

So, it is a characteristic curve and we want to find out the characteristic curve meeting the xy plane. So, this is t; x y t, t positive, wherever it hits. From these two equations, it is easy to solve for x0, y0, but to do not need that. What we need is, x0 plus y0 together, that is much easier to obtain. So just little algebra, just leave it.

So this implies x0 plus y0 is equal to x plus y divided by 3e to t minus 2 and therefore, u, x, y, t, so this is also quasilinear equation. But we are able to find an explicit representation of the solution, which is valid not only for all x y, but all t bigger than equal to 0. So, just because a equation is quasilinear, we should not say that it will develop singularities at finite time. So, here you do not see any singularities, very smooth. So, obviously, some effect is coming from addition of this two.

So, now, let me again from this move on to the fully non-linear case. So, non-linear equation, of course, this even in two dimensions you have seen is quite complicated. So, it required the introduction of more geometrical objects like Monje cone, characteristics strips, and so many other things. So, you can expect same difficulty to continue you want in higher dimensions. So, this x, u, so this is how we started, again non-characteristic quasi problem.

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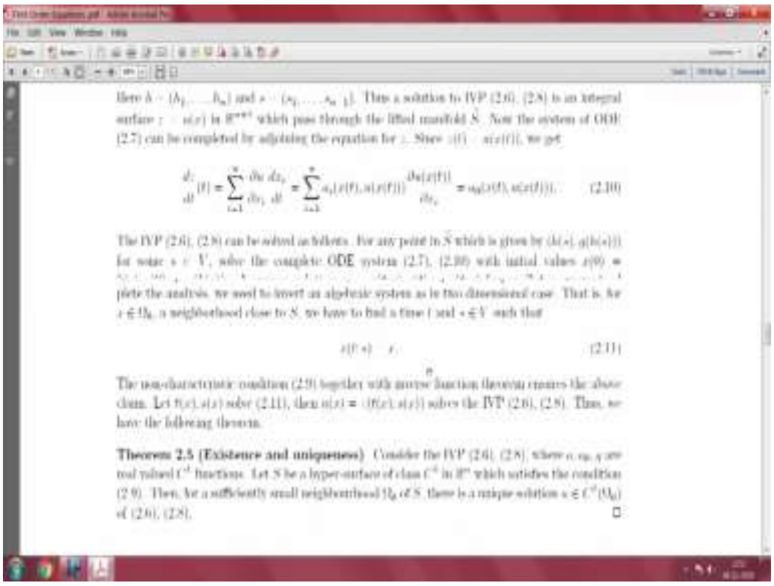

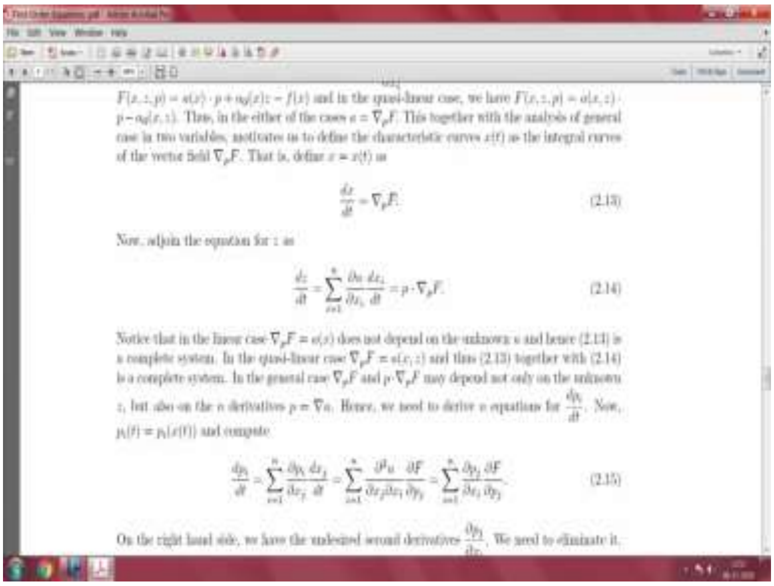

The same theme as, as in the case of two variable continues. So, you again describe the characteristic equations. And now, you will have n equations coming from the -- for the x variables and one from the solution and then there are n equations coming from the first derivatives. This was not there even in the quasilinear case. So, these are additional things. Now, there are two n plus 1 equations, and two n plus 1 unknowns.

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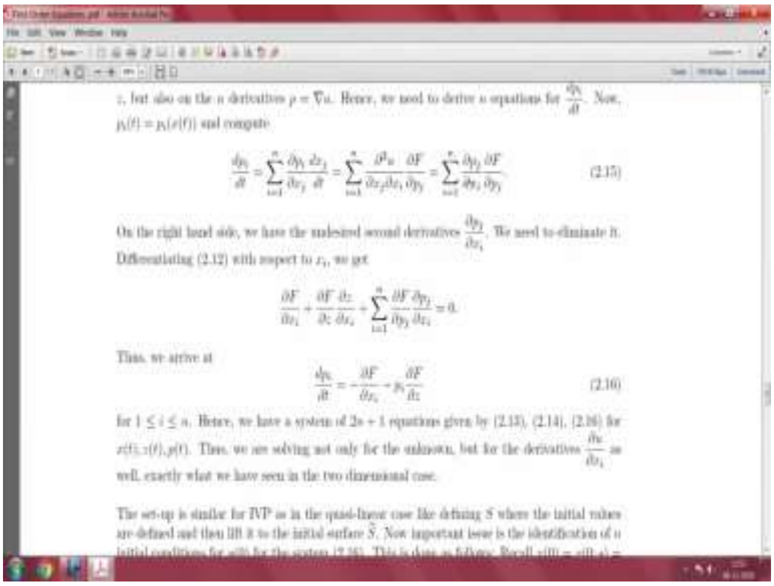

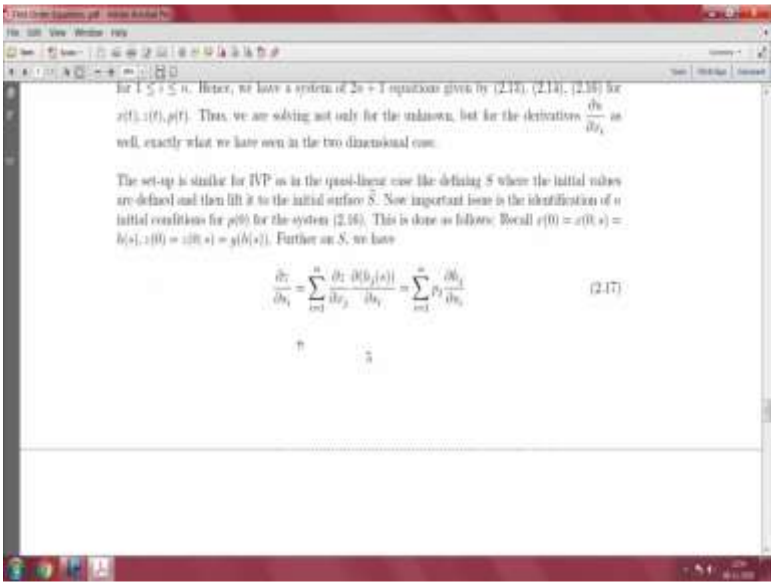

But I would like to draw your attention is to this strip condition. And so, this is this trip condition, and we have this a -- see this variables pi. So, we have obtained ODE's for them, but there are no initial conditions for them, pi. So, for initial conditions for this x, we can take it a point on the given surface, and initial condition for z comes from the initial condition imposed.

So, initial condition for x and z there is absolutely no problem. But for p, there is no initial condition. So, we cannot really solve this set of two n plus 1 ODE and we are assuming that there are functions which qualify to be initial conditions for the t equations. These t equations. And this is a restriction on the function f. So, that I want to write it.

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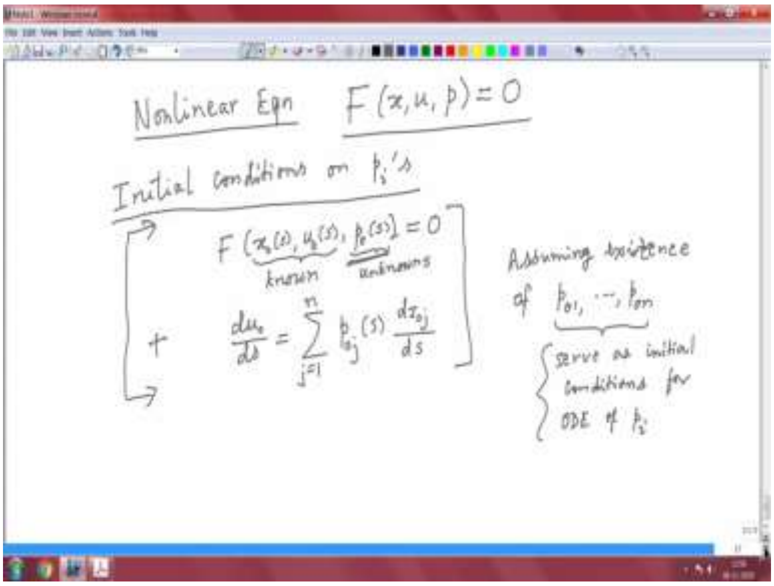

So, here the initial conditions. So, I want to make a comment on this, initial conditions on pi. So, they are assumed in the form this x0, s, s is parameter on the surface, and u0s, and then p0s. So, this you should remember. We are assuming -- so, these are known, these are unknowns. So, we are making an assumption that these functions exist, so that this equation is satisfied, plus this trip condition, that is also important.

So, namely du0 by ds equal to p0 dx0 by ds, these are -- this n case, so just let me write it in one go. Summation p0, j, s, dx0, j by ds. We are assuming the existence of p01, and these p01, p02, p0n they serve as initial condition for ODE of pi. So, then the system of characteristic equation complete and in general it is very difficult. Given a situation -- in some situations, we can easily decide whether such functions exist or not, but in some other case it is very difficult.

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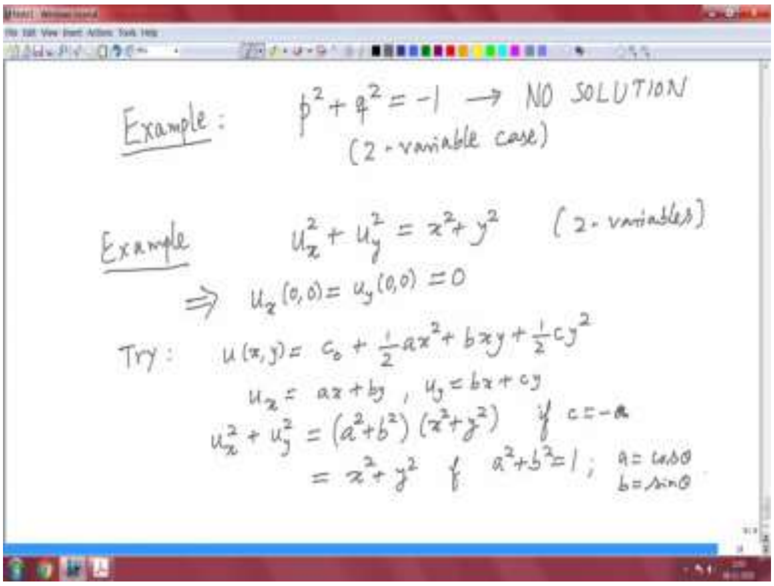

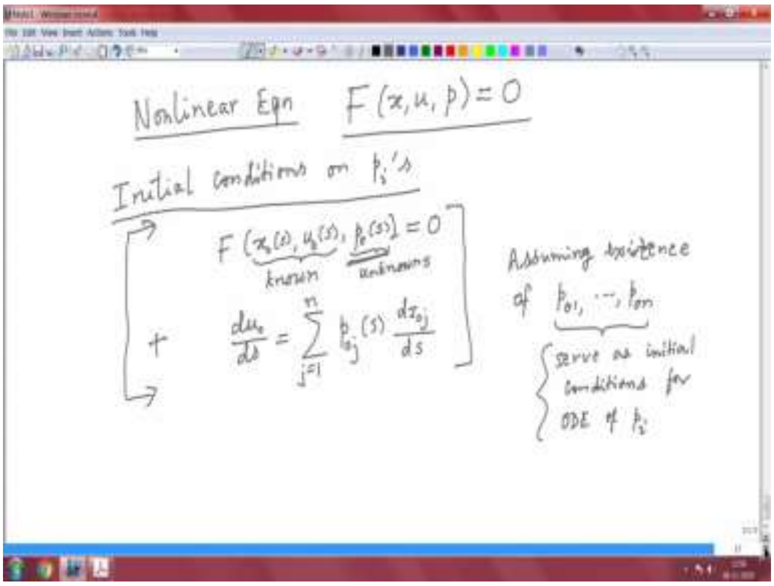

So, I would like to illustrate with some examples. So, this example is very simple. So, I take in two variables, just two variables. So, instead of p1, p2, I am using pq. So, we can immediately conclude no solution. As we are seeking only real solution we can immediately solve two variable case. Let me mention that. Let me illustrate with another example. So, where it becomes so difficult why a certain equation has solution. And so, this depends on the initial condition also.

So, just remember here, your equation is in one, and something else is also. So, another example. So ux square again two variable. Let me write it. So, this you can extend it to more than two variables, I am just taking a simple equation. I will come to the initial condition little later. So, this implies just by looking at the equation ux at 0, 0 equal to uy at 0, 0 equal to 0. At the origin, so this right hand side is zero and then these are squares, they also must be zero. So, we can try a function of the far whose first derivative is at the origin vanish.

So, this is some quadratic, so we can try that. So, constant I can add that will not affect the equation at all. So, the c0 is a constant, and we will assume a quadratic function. So, at least these conditions are satisfied by this. So, then we have a simple calculation. So, this is just ax plus by, and uy is dx plus cy; a, b, c, are constants. So, we can see that this ux square and uy square, if we do not want the cost, mainly the x, y term.

So, we should have -- so, this is just comes to the, so let me write it simple calculation if c is equal to minus a. You take c equal to minus a, and you see that the cross the x, y term gets cancelled, x, y term comes here; x, y term comes there, so that get cancelled and you have this simple expression and this is equal to x square plus y square, if a square plus b square equal to one. So, that let us call that a equal to sin theta, cos theta or sin theta.

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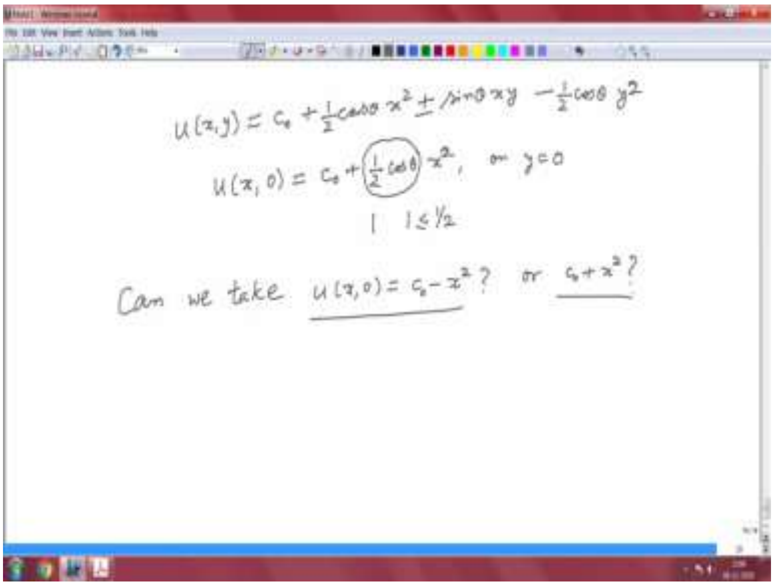

So, you can check that. So u, xy is a solution, constant half cos sin theta, x square, in fact you can now take plus or minus b, x, y minus half for sin theta, y square. And suppose, we -- so there are two solutions. You see immediately there are two solutions x0, c0 plus half plus sin theta x square. And just observe this coefficient here. So, if the absolute value is always less than or equal to half.

So, this on y equal to 0. So, suppose -- so, this solution satisfies this initial condition. So, my question now is, can we take ux0, some constant say, any constant, so constant we see that is not bothered minus x square or c0 plus x square, answer is not easy. So, I want you to study this example in carefully. And you see, see this is decent looking equation and decent looking initial conditions.

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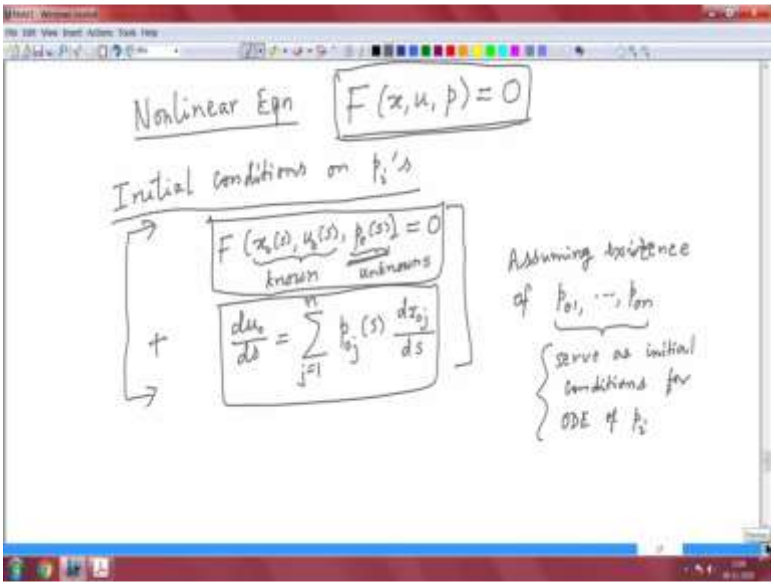

Yes, it is difficult to see whether it has a solution or not. So, it only shows that the method of characteristic still has some mystery which are not revealed in full. So, in particular the study of first order equations, there are still some gaps.

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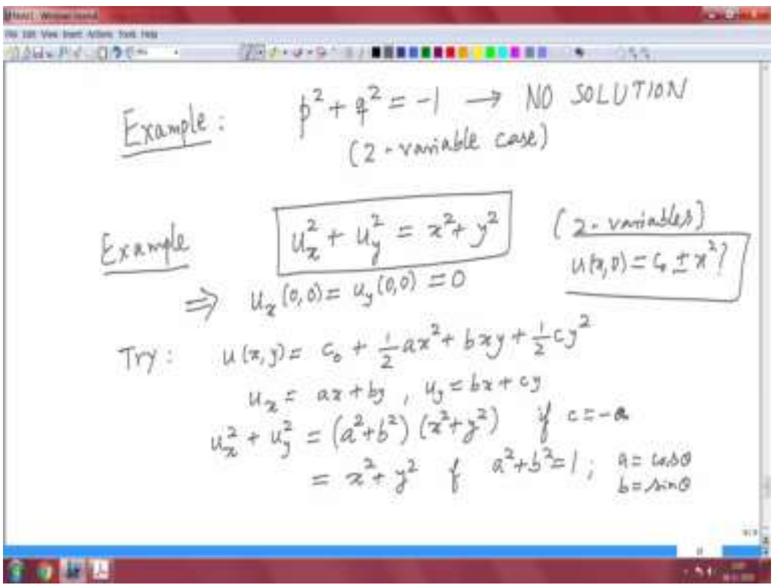

So, because we are not able to say completely that such an equation, very decent looking equation, with some conditions like this initial conditions whether it has a solution or not. So, with that, I come to an end of this discussion on non-linear first order equations, non-linear equations are really difficult. So, you have to study them very carefully. And even in simple examples, you see we face difficulties. So, try to do these things by method of characteristics and see the difficulties yourself. Thank you.