First Course on Partial Defferential Equations – 1
Professor A. K. Nandakumaran
Department of Mathematics,
Indian Institute of Science, Bengaluru
Professor P. S. Datti
Former Faculty, Tata Institute of Fundamental Research - Centre for Applicable Mathematics
Lecture 13
First order equations in more than two variables - 7

So, welcome back. So, we will now state the definition of surface analytically, since expressions are a little longer. I would like to show you some PDF file.

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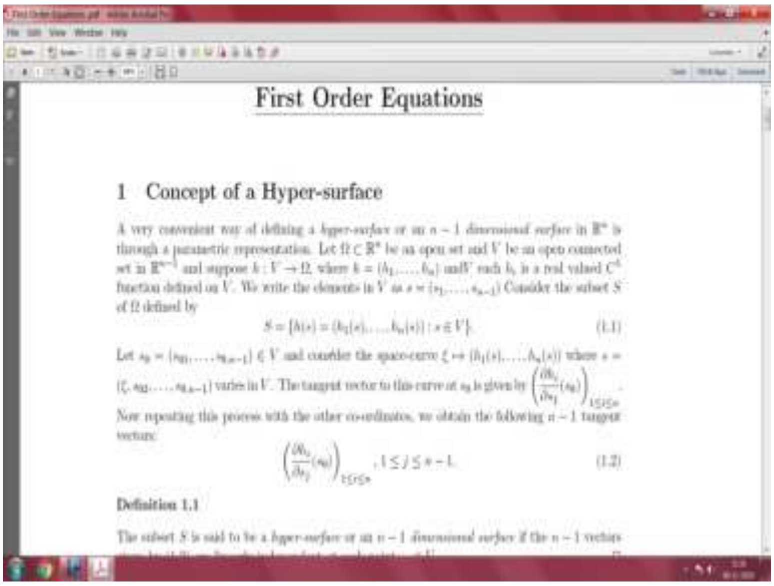

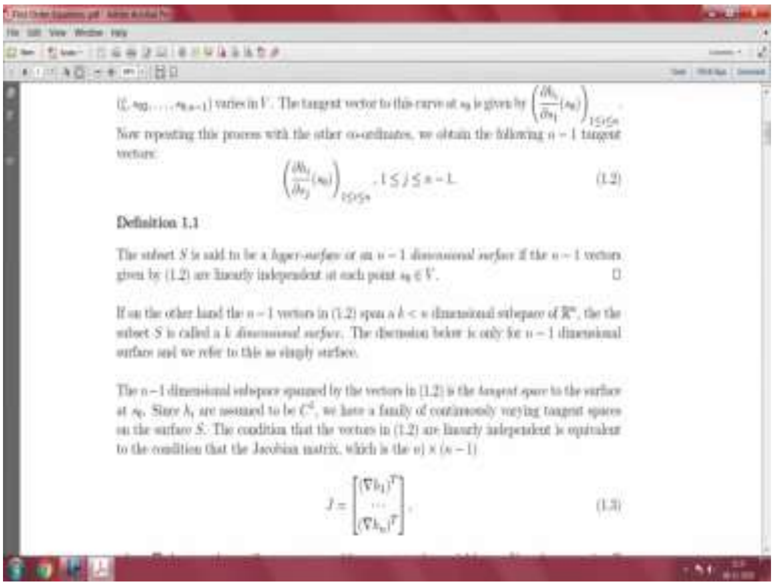

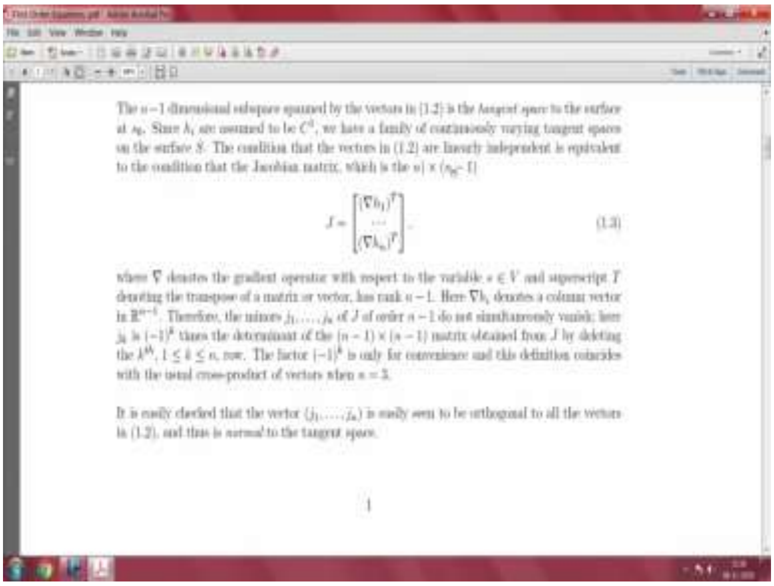

The $n-1$ dimensional subspace spanned by the vectors in (1.2) is the *tangent space* to the surface at x_0. Since h_i are assumed to be C^1, we have a family of continuously varying tangent spaces on the surface S. The condition that the vectors in (1.2) are linearly independent is equivalent to the condition that the Jacobian matrix, which is the $n) \times (n_0 - 1)$

$$J = \begin{bmatrix} (\nabla h_1)^T \\ \cdots \\ (\nabla h_n)^T \end{bmatrix}, \qquad (1.3)$$

where ∇ denotes the gradient operator with respect to the variable $x \in V$ and superscript T denoting the transpose of a matrix or vector, has rank $n-1$. Here ∇h_i denotes a column vector in $\mathbb{R}^{n-1}$. Therefore, the minors $j_1, \ldots, j_n$ of J of order $n-1$ do not simultaneously vanish; here j_k is $(-1)^k$ times the determinant of the $(n-1) \times (n-1)$ matrix obtained from J by deleting the k^{th}, $1 \le k \le n$, row. The factor $(-1)^k$ is only for convenience and this definition coincides with the usual cross-product of vectors when $n = 3$.

It is easily checked that the vector $(j_1, \ldots, j_n)$ is easily seen to be orthogonal to all the vectors in (1.2), and thus is *normal* to the tangent space.

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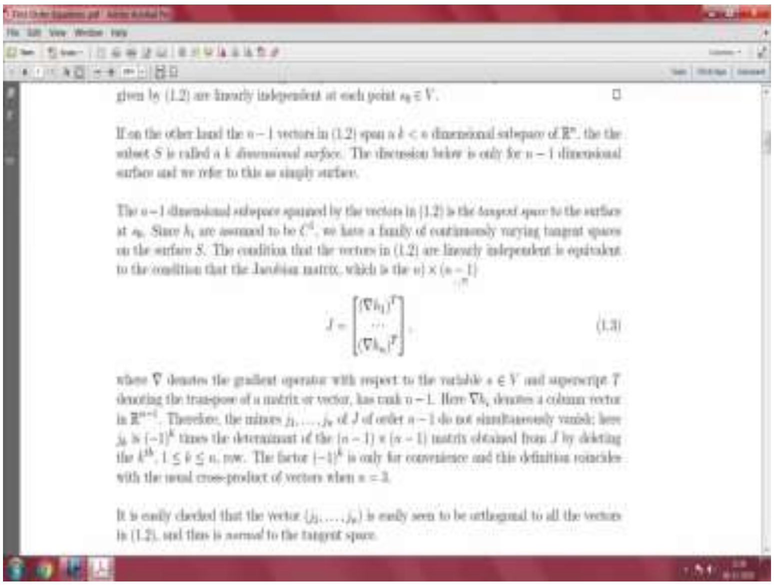

given by (1.2) are linearly independent at each point $x_0 \in V$. □

If on the other hand the $n-1$ vectors in (1.2) span a $k < n$ dimensional subspace of $\mathbb{R}^n$, the the subset S is called a *k dimensional surface*. The discussion below is only for $n-1$ dimensional surface and we refer to this as simply surface.

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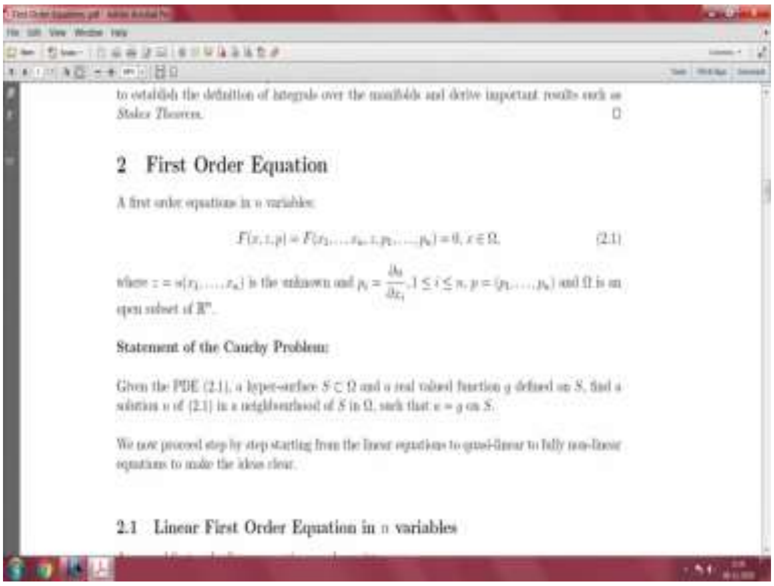

to establish the definition of integrals over the manifolds and derive important results such as *Stokes Theorem*. □

2 First Order Equation

A first order equations in n variables

$$F(x, z, p) = F(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0, \; x \in \Omega \qquad (2.1)$$

where $z = u(x_1, \ldots, x_n)$ is the unknown and $p_i = \dfrac{\partial u}{\partial x_i}$, $1 \le i \le n$, $p = (p_1, \ldots, p_n)$ and Ω is an open subset of $\mathbb{R}^n$.

Statement of the Cauchy Problem:

Given the PDE (2.1), a hyper-surface $S \subset \Omega$ and a real valued function g defined on S, find a solution u of (2.1) in a neighborhood of S in Ω, such that $u = g$ on S.

We now proceed step by step starting from the linear equations to quasi-linear to fully non-linear equations to make the ideas clear.

2.1 Linear First Order Equation in n variables

So, here more details are given. So, this is this is more like geometric definition. So, if these vectors for our n minus 1 vectors are linearly independent then we say that s is a hyper surface are simply a surface. And using linear algebra, we can convert this statement into an analytic statement and for that you form this Jacobean matrix. You take all these gradient vectors of each function h1, h2, hn, so they are the defining function of the surface. You take that and you write them as rows.

So, there are n rows and there are n minus 1 columns because there are only n minus 1 derivatives, s1, s2, sn minus 1. So, you formed this matrix which is n by n minus 1 and from linear algebra we see that -- so, the rank cannot be -- because it is a rectangular matrix rank is utmost n minus 1. And it qualifies to be n minus 1 rank is n minus 1, if all the -- there are n minus 1 rows linearly independent or n minus 1 columns linearly independent.

And that we connect to the determinant and those are called minors. So, you take all the minors of this n by n minus 1 matrix of order n minus 1. So, there are n such matrices and you take the determinant of all those matrices and the rank condition is that these, I call the minus at j1, j2, jn, there are the n minus or either n minus 1.

The rank condition say that at least one of them should be non-zero. So, that is the analytic statement of the this object s being a surface. And you can also check that again this is linear algebra, you should take this vector, now this vector is in Rn. All these gradient vectors are in Rn -- gradient vectors are in Rn minus 1. So, this is in Rn and this is orthogonal to all the vectors in one point.

And thus it is -- so since they span the tangent space, so this vector j1, j2, jn is in the direction of the normal to that surface at that point. So, everything is local. And for example, you take the case of n equal to 3 and work out, and this j1, j2, j3 you get can be expressed as the usual cross product of vectors in R3. So, that is the determinants precisely come to be cross products.

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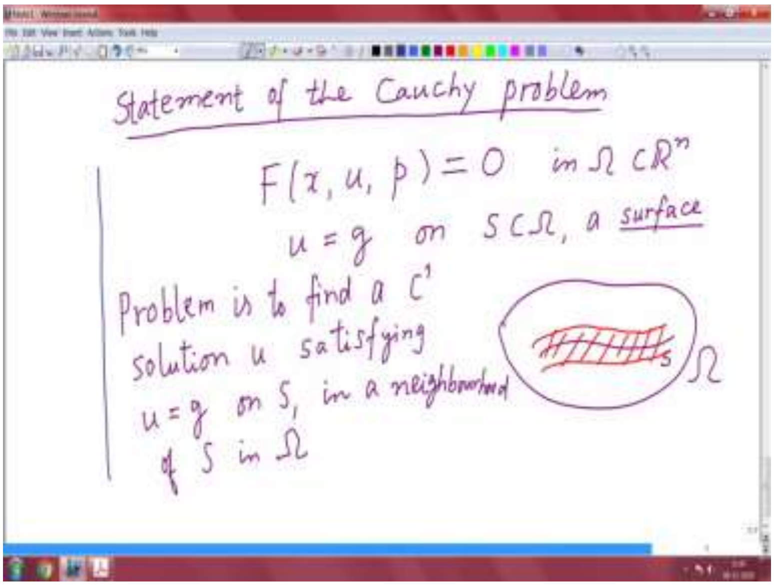

So, once that is done now here is the statement of the Cauchy problem. So, let me just state it for the general case and then we will come to the individual cases of linear equation, and Cauchy linear equation, and then fully non-linear equation. So, this is our first order equation, and so this is in omega in Rn and we are given u equal to g on s. This is in omega, a surface.

So, again let me repeat that surface for us is that hyper surface n minus 1 dimensional surface. So, this is omega again and this is s. The problem is to find a C1 solution u, we require the first derivative only, solution u satisfying u equal to g on s. And where do you find the solution? In a neighbourhood, we are satisfied, even if we could find a small neighbourhood of s in omega.

We will see even this restriction is essential even for the linear case. So, here let me use the different – s is linear. So, this is -- that neighbourhood. That is sufficient for us. So, even for the linear case we will not be able to find, for example, if omega is Rn, we may not be able to find a solution in the entire region Rn.

So, there are some restrictions. And this is the most general statement and obviously, we need to have some restriction on s. So, in this set up any arbitrary surface will not be able to do find a solution of this Cauchy problem. So, once you understand this problem, then let us see how to go about it.

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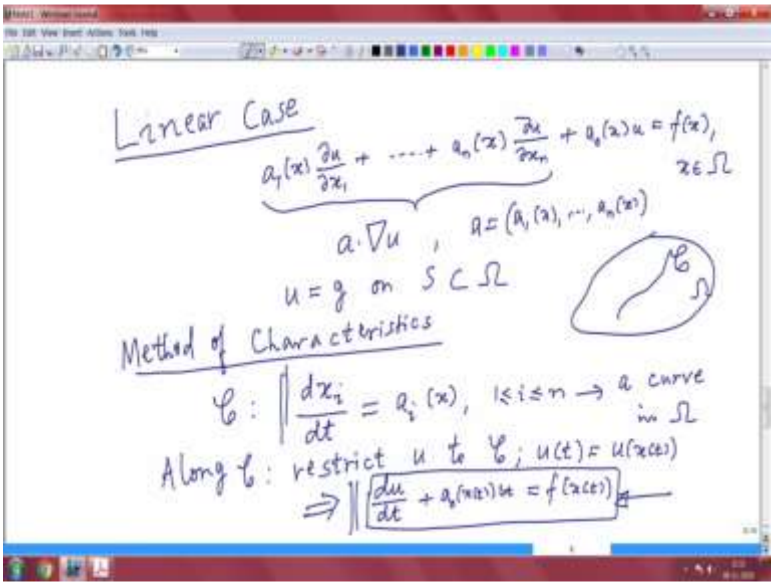

So, let us start with the linear case. So, that is very special here. So, let me just simply consider this a1 x, du by dx1, anx, du by dxn, you can take the lower R terms also, let me write that. And even some intermediate step. So, this is conveniently written as a dot product, scalar product, a dot grad.

So, A is the vector, vector field. So, for each x in omega you get a vector in Rn and that is how you get field of vectors. So, this x is in omega. So, the Cauchy problem again let me just write it, u equal to g on s. So, s is your surface. So, this as we did it in two variable case. So, we approach the problem with method of characteristics.

So, the characteristics in the simplest case, just compare this with two variable case. So, here there are -- so, t the parameter on the characteristics ai, x, 1 less than or equal to i less than or equal to n. Of course, we have to prescribe some initial conditions. So, let me call it characteristic C.

So, this is -- it describes geometrically a curve in omega. If I start the initial point in omega, so this will form a curve in omega. Along c, so it is that means you restrict u to c. Actually, I should define another function but by a use of notation we just write ut, so this actually means u of xt, u is not a function of the t but it is a function of x. So, this abuse of notation.

And then if you use the given linear equation, what you see is by chain rule, du by dt plus a0. Again, let me write xt, because we are restricting the all functions to c only, so they are just functions of u equal to f of xt. So, this is what I was tracing. So, the study of the first order PDE radio's is to the study a first order system of ODE, remember just these are ODE's.

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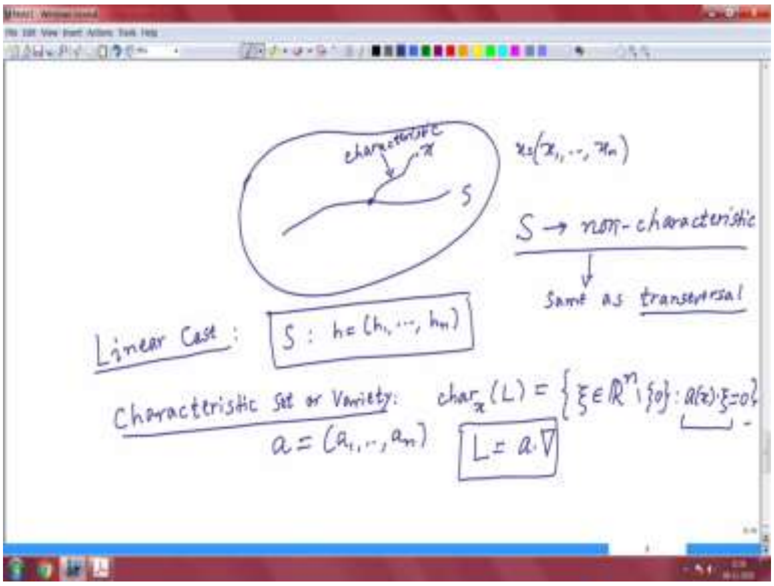

And once we are able to solve this, the problem is again as you observed in two variable case, the problem is -- so, given a point x, x1, x2, xn, if we are able to draw a characteristics and that is where the initial surface and initial conditions come into picture. And so, this is say that s and this is characteristic variable to do that. And we pick the data from here and we already solved the differential equation this one and you pick up the data from the initial surface and that is how we will be able to obtain a solution of the given PDE.

And that is what you have learnt in a two variable case and this is where the relation between this characteristic and this initial surface comes into picture. The s has to be non-characteristics. Let me just briefly mention what that non-characteristic is. So, it in two variable case, it is same as transversal to the given PDE. So non-characteristic with respect to given PDE.

Let me just quickly explain what that non-characteristic means and why the non-characteristic Cauchy problems are somewhat easier and characteristic Cauchy problems in general demand more conditions on the initial data as we see an example, even in just two dimensions. So, what is this non-characteristic? Again let me just start with the linear case. So, this will come one-by-one. So, again let s be described by the function h1, h2, hn, we saw that. Just remember that, so this parametric representation of the surface.

And now we introduce what is called a characteristic set of the given PDE or characteristic variety, characteristic set or variety. So, this can be done even for more general equations and even higher order equations. So, let me just denote it by char and it changes from s-t-s. So, this is let me call it L in this case just ax dot ax.

Just the confusion, dot next set of all J in Rn and you omit 0, because 0 is always in the set, such that ax dot J is equal to 0. So remember, A comes from the PDE, those coefficients, a1, a2, an. And if you take the union of this caar xL, so L is the operator a dot dell, first order linear operator. So, you take only the terms involving derivatives and this is just formally replacing this brand operator by the vector J.

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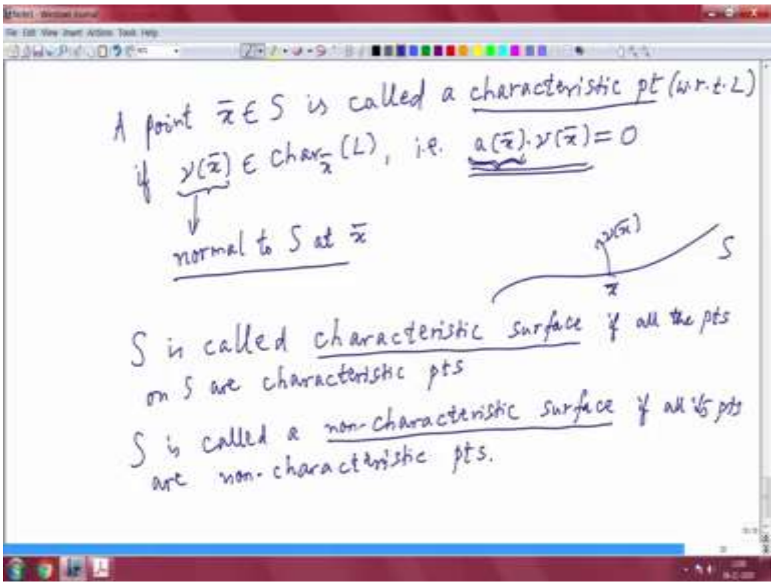

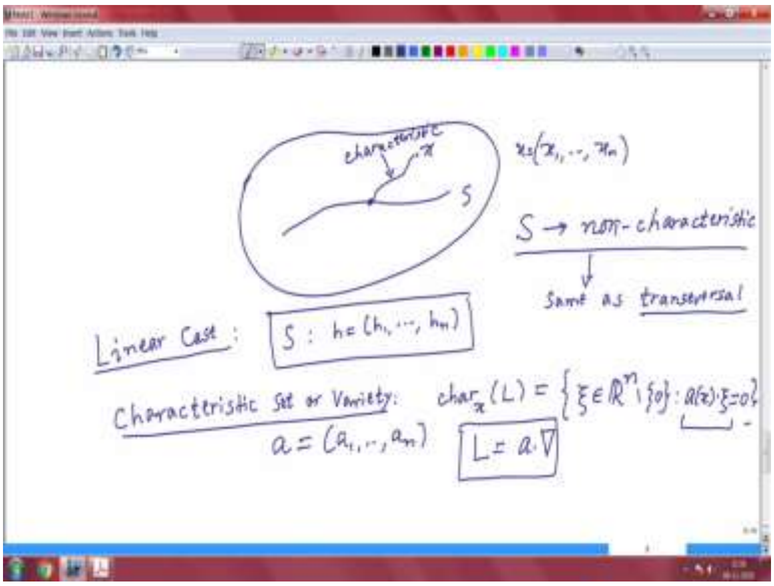

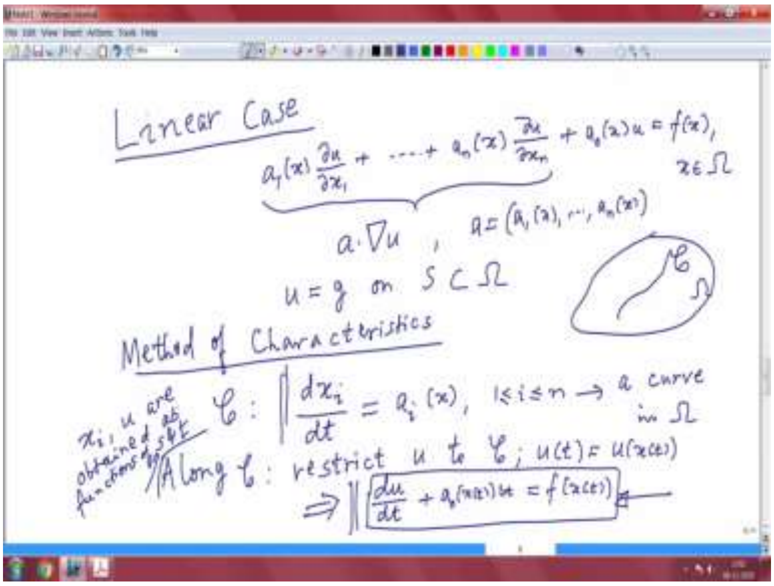

So, here is the definition. A point x bar in s is called a characteristic point, of course with respect to PDE, with respect to L, that L has to be there. If mu x bar belongs to char x bar of L that is ax bar dot mu x bar is 0. So, what is mu x bar? So, this is not normal to s at x bar. So, here is s, here is x bar and this is mu x bar. What he says is you call that point x bar a characteristic point.

If the normal at that point is orthogonal to the given vector A, so this comes from PDE. And this relation say that mu is perpendicular to ax bar – a, at that point. So, s is called a characteristic surface, if all the points are characteristic points. So, S is called a non-characteristic surface, if all this points are non-characteristic points.

That means nowhere on S, the unit -- the normal at that point, nowhere is perpendicular to S. So, it always makes an acute angle with the vector a, characteristic points. Analytically, this condition is required -- as you again recall the two variable case, we want to apply the inverse function theorem in order to do this thing or just let me –

So, given a point in omega we would like to draw a characteristic meeting the initial surface at some point. In order to do this thing, so we want to find the characteristic variable t and the variable that describes s, namely s. Then because what we have obtained is here this xi and u are obtained as functions of s and t. So, we have we are not yet given the initial condition.

So, these initial conditions will be on the given surface functions of s and t. So, s is the variable describing the surface initial surface and t the parameter that describes the characteristics. So, we want to obtain s and t in terms of xi, x1, x2. And for that, you require implicit function theorem and this non-characteristic condition is precisely that. Let me just mention that.

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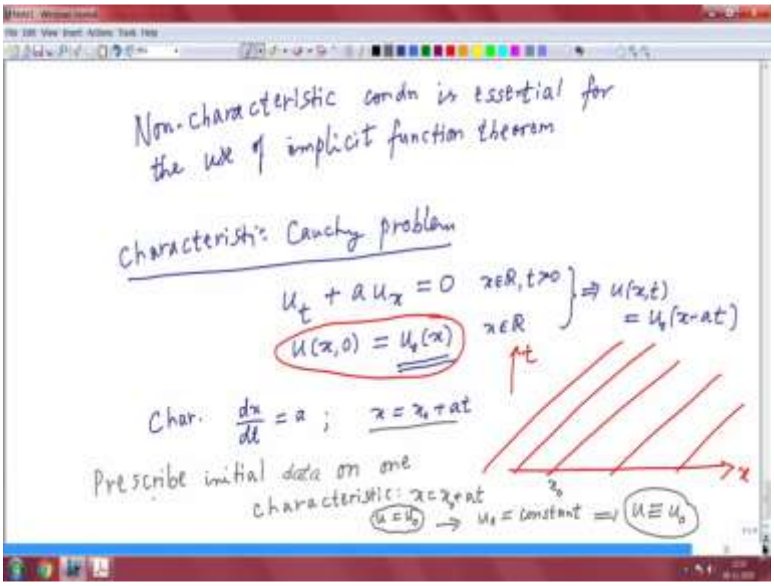

The non-characteristic condition is essential for the use of implicit function, for applying implicit function. And in two variable case, it was called transversal condition. So, that is the only terminology is different, but this one is applicable in the general setup and also for higher order equations. So, this terminology is more used in the context of PDEs.

So, what about the characteristic Cauchy problems? So, let me just give you an example, very simple one, which you already seen it. In some way characteristic Cauchy problem and what are the difficulties. So, consider the very simple first order equation. So, a is a constant. So, if we are given at t, t equal 0, is a non-characteristic here in this case. Because we know what the characteristics are.

u 0 x, I am considering this in x in R t for t, so this is simple transport equation and the solution is u0 – u of xt. So, here u0 is any c1 function, arbitrary c1 function and we are able to solve this equation explicitly in this case and uniquely. So, what are the characteristics here? The characteristics are simply given by dx by dt equal to a. So, they are straight lines. So, x is equal to x0 plus a.

So, let me just draw that. Suppose a is positive, so these are the straight lines with slope, so this is the x axis and this is t axis. Now, suppose we give the initial condition on one of these characteristics and that is called characteristic Cauchy problems. So, you take any characteristic of the given equation and you prescribe the initial condition on that.

So, suppose I prescribe initial, instead of the line t equal 0 this is line t equal to 0 initial condition on one characteristics, any characteristic. So, this one passing through some x not here. So, I prescribed the initial condition on this. What happens? U equal to u0. So can u0 be arbitrary. So, in this case first of all the differential equations tell that u is constant along the characteristics.

So, I cannot arbitrarily prescribe u0, u0 has to be a constant along this. And then what about at other points, the other points do not intersect this characteristic anywhere. So, it is totally arbitrary. I can prescribe any value at any other point. So, there is first of all u0 has to be constant, so restriction. And if you want this u to be a C1 function that u has to be constant through. I cannot take another constant and demand that it is continuous and differentiable.

So, if you prescribe the initial condition on a characteristic, we face problems and there is only one solution namely u is equal to constant and nothing else. And even higher odd equations, such problems arise if you want to prescribe the Cauchy problem on a characteristic. That is why we avoid Cauchy problems on a characteristic and we just consider non-characteristic Cauchy problem.

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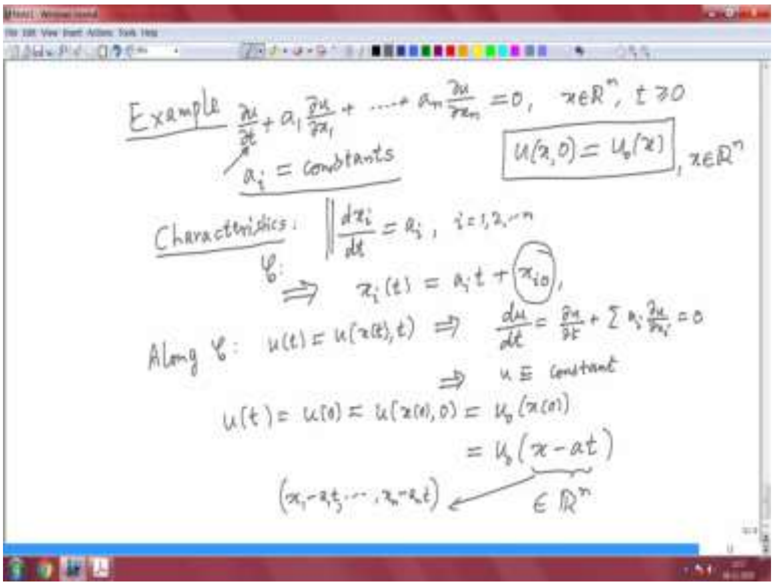

So, let us go back to linear case and one simple example. So, example again. So, let me take the constants as coefficients instead of xn, maybe I will just add one more variable, just for simplicity, otherwise I have to divide by -- so I am actually considering -- so, x is in Rn and t I can just take positive.

So, this is a transport equation in more than one dimension. So ai is our constants, for simplicity I am taking. And let me use the variable t itself for the characteristics. So, I will prescribe -- come to that. So, dxi by dt is equal to ai. And t variable is there, so dt-by-dt equal to 1, so that is a -- and you easily check that t equal to 0. So, these are again straight lines even in n dimension, so it is difficult to draw the picture here.

So, the solutions immediately we see that the xi, t is equal to ai t plus xi0. So, where xi0 are the initial conditions for this n equations. So, if you take constant it is a very easy to integrate this differential equation and you call this as again a characteristic. So, along C, again by at least after notation, so if you write this u of t, now it is u of xt, t, because I am using t itself as a characteristic parameter and this you immediately see that du by dt, again by implicit differentiation, you see this is du by dt plus summation ai, du by dxi, and that is 0.

So, that means, u is identically constant. So, that means what u of t is equal to u of 0. That means, u of x0, 0. And so, this I give the initial condition, this is u0 of x. So, u0 is a given function and t equal to 0 is a non-characteristic. And if you use that initial condition -- and these are the initial condition, that xi0 these are all xi0. So, if I expressed that one, so you just get u0, x minus at.

So, exactly similar to the one dimensional case that means n equal to 1, but only thing is now you see that this is in Rn, x is in Rn. So, that what this equation describes is just transportation. So, whatever initial profile is transported in time in different directions, if you expand this whole thing there are n components x1 minus a1t comma xn minus ant.

So, the initial profile that u0 is transported in time in different directions. So, it is with there are speed -- different speeds, they are given by this coefficients a1, a2, an. So, similarly, in generic linear case you can just follow this procedure and the non-characteristic condition helps us to you the implicit function theorem, and in principle, we can solve the first order linear equation without any problem, provided we prescribe the initial condition on a non-characteristic surface.

So, the for example, here what is the surface, surface is just t equal to 0 that is n dimensional space Rn. So, here we are in n plus 1 dimensions, so we are prescribing the initial condition on an n dimensional. In fact, here it is a subspace. So, this is also an example of a hyper surface. So, with that, we come to an end of this discussion on linear equations. And in the next class, we take up the study of Cauchy linear problems and then briefly the fully non-linear equations. Thank you.