First Course on Partial Defferential Equations – 1

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Lecture 12

First order equations in more than two variables - 6

Hello, everyone. I am P. S. Datti, former faculty at TIFR - Centre for Applicable Mathematics, Bengaluru. Professor Nandakumaran of Department of Mathematics, Indian Institute of Science and myself, are giving this First Course on Partial Differential Equations. Professor Nandakumaran has already explained in great detail about our plan in this course, in this video course. Essentially, we will be covering many chapters of our recently published book on partial differential equation that Professor Nandakumaran has already shown you.

Apart from that, normally, when we have given such a course in the classroom, we have not bothered to say anything about the prerequisites, essentially, multivariable calculus and some analysis. But in this video course, since students might come from very different backgrounds, we have tried to list a number of topics from multivariable calculus and in some other things that are required in the study of partial differential equations.

So, I urge you to again go back to your multivariable calculus course and try to refresh your memory. So, that they are not only useful in this video course, but in your future study on partial differential equations. So, partial differential equations they need -- they require lots of tools from other branches of mathematics, especially analysis, differential geometry, and the demand is so much you want new topics like distributions and other things are created for the purpose of analysis of partial differential equations.

Apart from that, partial differential equations essentially arise in all branches of science, physics, chemistry, biology, engineering, and the interaction between the sciences and this part of mathematics have benefited each other. They have -- this interaction has enriched both mathematics and also the other sciences.

Especially, you might have heard that now there is a discipline called mathematical biology, which concentrates on population models and even epidemiology models, and they are on either ordinary differential equations or partial differential equations. And thus the analysis of partial differential equation becomes very important in the study of such topics.

So, though, we have chosen some simple topics from our book, so obviously in such a short course, we cannot cover the entire book. There are many-many topics in the book. But nevertheless, it is important for you to follow it seriously and work out all the problems and all the details in the theoretical part also from our book. So, with this thing, I begin my lectures and today's topic is first order equations.

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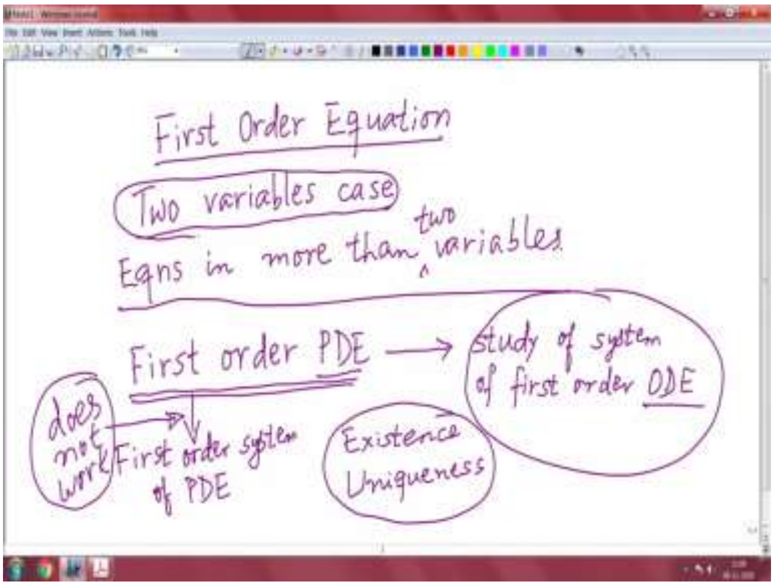

So, when we say equations, it is always a partial differential equation. So, you have already heard. Professor Nandakumaran speaking about this first order equation in great detail confining to two variable case. But most of the ideas and tools are already there. So, what we are going to do in a couple of lectures, we are considering the equations in more than two variables.

So, it is essentially, the same as the two variable case except for some more equations and more algebra, more computation. So, it is only that kind of work. I want to stress here one point. The study of first order PDE, this is important and this also brings in a certain difference between PDE and ODE. The study of first order PDE essentially reduces to study of system of first order ODE.

So, I want to stress this point. And this part, the study of first order ODE is quite well established. So, we have for example, existence uniqueness results, whether it is one equation or system of several equations, these results on existence and uniqueness are more or less the same. So, that is -- but here we are considering first order equations volume.

And we do not have the luxury of extending it to first order system. Whereas in the case of ODE, if you study one equation, the same method works for a system of first order ODE, but here, it does not work. This passage does not work. So, I want to highlight that. So, the study of first order system of PDE is very, very complicated and still there are many things to be known.

For example here, so any higher order equation can be converted to a first order system or first order ODE. So hence, once you know the existence result for a first order system, so we know at least existence uniqueness for ordinary differential equations of arbitrary order. So, for example, here we cannot even just go from first order PDE to second order PDE that also you see in this course.

So, the methods, for example, second order PDE are so different from the one we study for our first order PDE. So, let us fix some notation. Once so we fix the notation, so whatever you have learned in two variable case just carries over and there is absolutely no difficulty.

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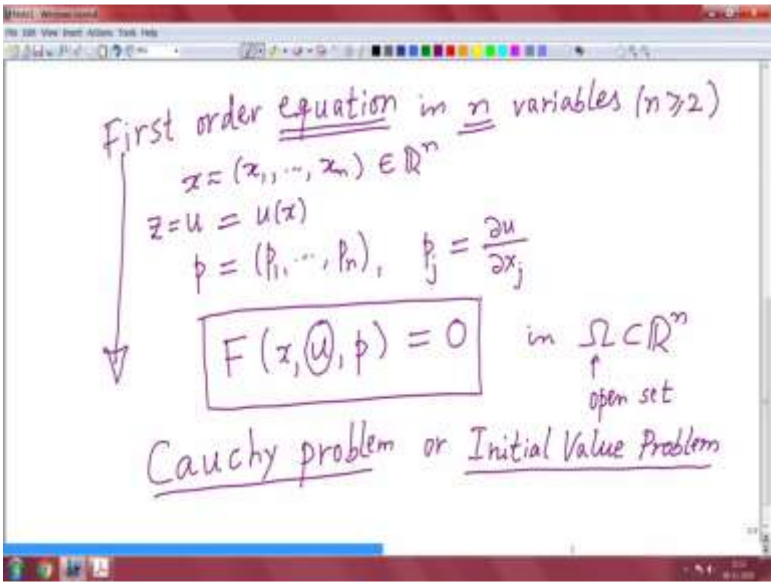

So, first let me write a general first order equation in n variables. So, just in n variables. So just in n variables. So we denote a point in Rn by x1, x2, xn, so standard notation, Rn. And we use for the unknown function u, so this is a single equation. Let me just stress that single equation, interchangeably we also use Z. So, this is a function of x. So, in some textbooks they also use Z.

Then for the derivative, so now since u is a function of x, u has n first order derivative. So, let us call that as p1, p2, pn. So, in case of two variables, we do not say the variables were denoted by xy. And instead of writing the other vector, we used p and q, p for the x derivative and q for the y derivative. But here, since we are dealing, so n is atleast 2, because it is a PDE.

So, this pj is dell u by dell x. So, with these notations, so first our general first order equation is of the form F, so it depends on x, u, and p, equal to 0. In some domain, open set -- open connected set, just open set, but the description is not essential, but it is a at least an open set. So, that the derivatives and other things make sense. It could be whole of Rn.

So, as we have already seen even in the case of ODE, even in the linear case, we cannot write a general solution for such an equation, that is ruled out. So even for ODE, that is how the analysis of ODE is started and then the analysis of PDE also started. So, just giving an equation, so this first order because we are using only the first order derivatives. And then we will see further classification into linear, quasi linear, and fully nonlinear, just like in two variable case.

So, there absolutely no difference, so basic problem and important problem we want for ordinary differential equations. So, always a PDE is associated with a Cauchy problem. So, we should be able to solve Cauchy problem or also called Initial Value Problem. So, it always comes with that. So, we should be able to find always a method to solve such a Cauchy value, Cauchy problem.

So, for that thing, we need to prescribe an initial value on the solution. So, this u need to satisfy certain given initial condition. But in this setup, n variable setup, where should we prescribe the initial values, that is the first question. So, once you understand the statement of the Cauchy problem then it is becomes easier to add up the method of the two variable case and do analysis.

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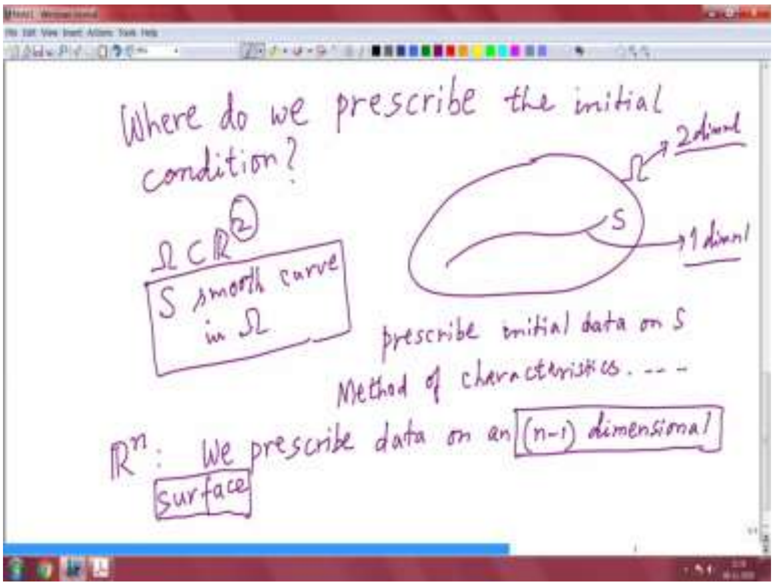

So, that is the next question, where do you prescribe the initial condition and that constitutes the Cauchy problem. So, again let us go back to the two variable case. So, omega is in our to now an open set and there we described the initial condition on a smooth curve, S. So, omega now in our two and S is smooth curve in omega. And the requirement on the curve was, it should be transversal. So, here we get a different terminology, but it is transversal. Let me come to that a bit later, so maybe just.

And then we prescribe initial data on S. Then the method of characteristics will follow. Method of characteristics – In order to analyse this solution and also prove the existence and in some cases uniqueness also of the initial value problem. So, here just to count the dimension, omega is in our two and S is a smooth curve, so it is one dimensional less. So this is just like a line. So, this is two dimensional and this is one dimensional.

So, similarly, in case of n dimensions in case of Rn, so we prescribe data on and n minus one dimensional surface. So, this has already been explained to you in the plenary topics, but it is also good practice again to recall so that you see in the context of PDE. So, once we understand this the notion of n dimensional one surface, then it is easy to formulate the Cauchy problem.

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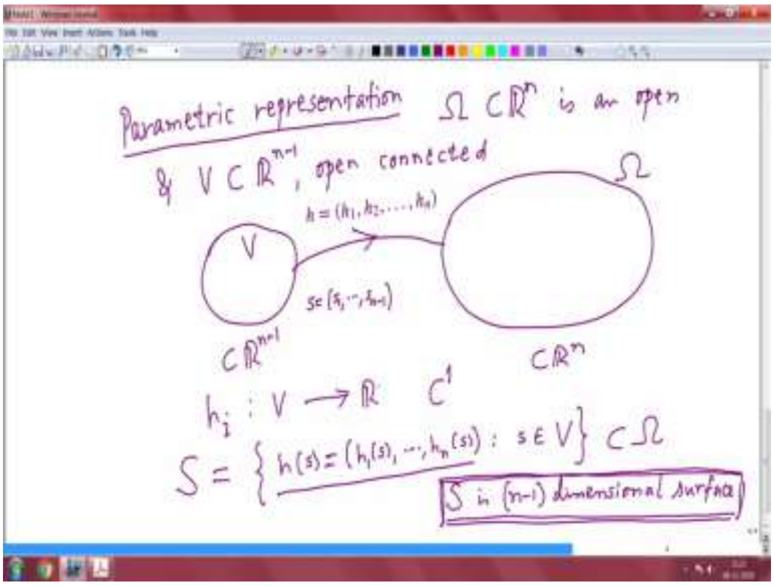

So, let me just recall. So, a very convenient way of describing the surface parametric representation. So, this actually is part of classical differential geometry. So, again omega is an open set and V in Rn minus 1. So, one dimension less open connected. So, we do not want to consider union of for example, two bars, just a single thing. And so, again I write that, I cannot write the main dimension, so just again write omega here and I want to define a mapping.

So, this is in Rn minus 1 V, so this is part of Rn minus 1, and this is in Rn. So, I want to define a mapping. So, here there are the variable we denoted by S1, Sn minus 1 and since we want to go from Rn minus 1 to Rn, so we need n functions. So, hi are from V to R. So, we need at least C1, they are all C1 functions. So, I want to say that this image of, so you collect all these points.

So, this is just set up of hx, hs. So, again let me call it. So, this is h1s, hns, n components. So, this is in Rn as S varies in V. So, this sits in omega. So, this is a subset. So, I take the image of all these functions, that is what I am doing here, just image of this set V, that sits in omega. So, we want to say that this s is n dimensional -- n minus 1 dimensional object surface.

We want to call it. So, what conditions on h, h1, h2, hn imply that this object is n minus 1 dimensional and not lower. It cannot be more because it is only in Rn minus 1. So, we cannot expect it to be, n dimensional. So, it can be lower than n minus 1. So, but what qualifies it to be called and n minus 1 dimensional surface. Let me just explain that.

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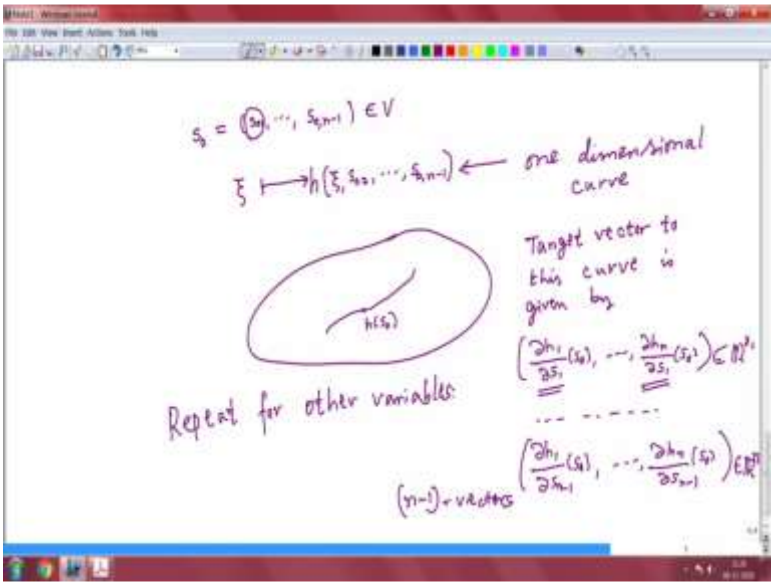

So again, you will fix s not, so call it s not 1, s0 n minus 1 in V you fixed and now, you just fix all the components except the first one, so you consider a mapping from z to zs02, let me s0 n minus 1 and the requirement is that as z varies, so this point should vary in V so that I can take operate h on that. So, this is now one dimensional curve.

So, we have again omega. So, this is hs0 and now I am just passing a curve. So, this is the curve. And again from calculus, we learned that the tangent vector to this curve is given by dell h1 by dell s1 and s0 of course, let me not repeat it. So, sometimes I write it as column vectors, sometimes row vector, so that is -- so a vector in Rn now, because there are the n components, but the partial derivatives only with respect to s1. Because we are varying only the first coordinate.

And you know you repeat it for all the other variables. This process and we get, so dell h1, the tangent vectors they are in different directions, sn minus 1, these are only sn minus 1. These are all vectors in Rn, so that is you remember. And how many are there? There are n minus 1 vectors in Rn. So, here comes the definition of the hyper surface Rn minus 1 dimensional surface.

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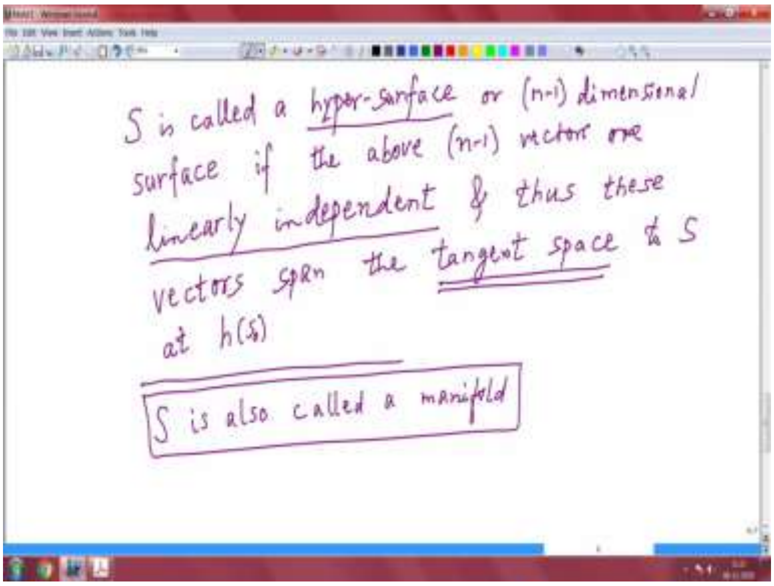

So, s is called a hyper surface or n minus 1 dimensional surface, if the above n minus 1 vectors are linearly independent. So, that is the requirement on the function. So, they are obviously such requirement depends on the functions h1, h2, hn. So, this both n minus 1 should be linearly independent and thus these vectors span the tangent space to S at hs0.

Sometimes we just write s0, it could be image of, so tangent space to s. So, let me just show you that, so here. So, these vectors are in different directions they formed the tangent space. So, it is very difficult to draw the figure but you can just in two dimensions and three dimensions we can try that.

So, these n minus 1 vectors should be linearly independent in order that s to be called a hyper surface or n dimensional 1 surface. It is also called a manifold. Manifold is a more general object than surfaces. So, surface is an example of a manifold, but manifolds are more than surface. Locally they look like a surface but, s, it is also called a manifold, n minus 1 dimensional manifold.

But manifold is a lower general geometric object so different terminologies. So, once we understand what a hyper surface -- so, just when we say surface, hereafter it means only n minus 1 dimensional surface as far as first order PDE is concerned. So, now, we can state the Cauchy problem and then try to analyse it.

So, all these things are done in two variable case. So, I just recall them and it is more like a copy and paste feature. So, we will first -- what we will do in the next class -- so this condition of the n minus vectors being linearly independent this is more algebraic. And now, we state this in analytic terms. And then we will go for the statement of the Cauchy problem. Thank you.