First Course on Partial Differential Equations – I
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Lecture – 11
First Order Equations in 2 Variables - 5

Good morning and welcome back to the study of first order equations. So, in the last classes, few classes we were discussing about the first order equation in 2 variables, and we considered the case for linear equations and quasilinear equations. And we have introduced what is known as characteristics, and we have introduced the concept of characteristic. Now in this lecture, we will do the same thing for the general nonlinear equation with the 2 independent variables.

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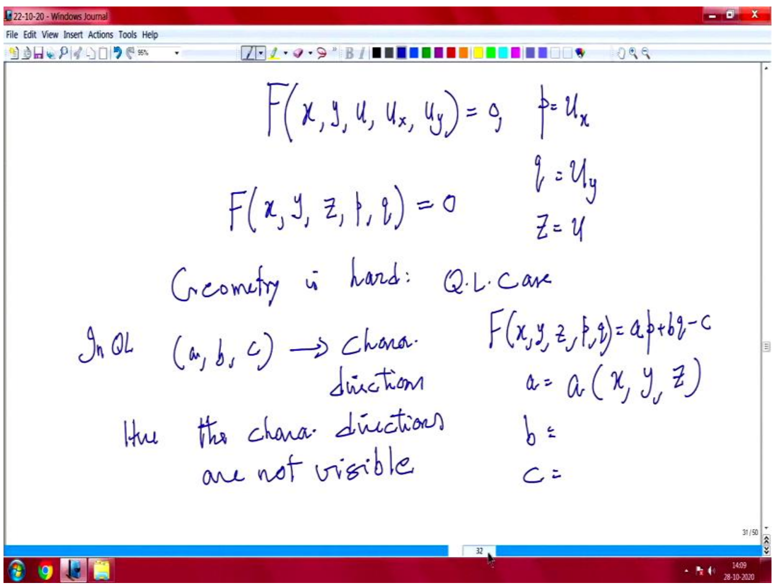

So, a general equation in 2 variables looks like that F of x, y, u, u x, u y equal to 0. So, we introduce some notation which is a customary notation. So, we will have p is equal to u x, q is equal to u y, and sometimes we can leave u also is equal to, z is equal to u. So, in the algebraic form, you can write this equation F x, y, z, p, q is equal to 0.

So, the geometry is more hard. That is the first thing, geometry is hard. In the quasilinear case, quasilinear case, we have seen that F of x, y, z, p, q is equal to a u, a p plus b q minus c. That is what F x y, where a is equal to a b c etcetera are functions of x, y is a function of, sorry, a is a function of x, y z, and it is linear in p and q. Similarly, b similarly c.

The thing is that, if you recall the quasilinear equations a b c is the, in quasilinear case, in quasilinear, a b c gives characteristics directions. That is what we have seen it. But, the one of the difficulty now here, the characters here, the characteristic directions are not visible. So, we have to find out that. Directions are not visible. That is the difficulty here. So, we need to do something more about the geometry of that one.

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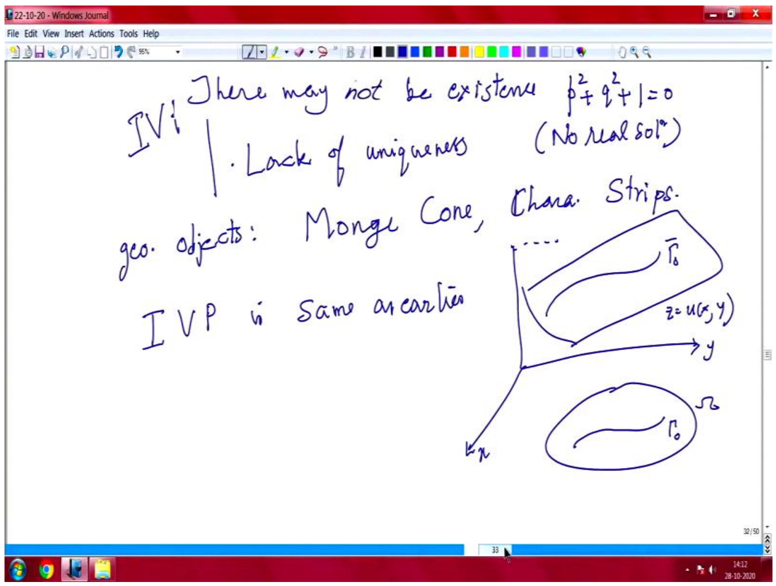

But then there are more difficulties. There may not be existence. You may not be able to solve. So, like the equation, suppose you need this to solve; p square q square plus 1 equal to 0. No real solution, because you cannot find something square plus 1 equal to 0. No real solutions. There is also a difficulty, there may be lack of uniqueness. I am talking about Initial Value Problem.

For the given Initial Value Problem all these things can happen, as you see in this example. And there are more geometrical objects, geometrical objects, like Monge cone, which I am going to introduce; Monge cone, Monge's characteristic strips, characteristics strips etcetera. You have to do such kind of geometry.

However, the Initial Value Problem is similar, Initial Value Problem is same as earlier, same as earlier. For example, what is the Initial Value Problem? You have a domain in XY space, and you have a domain here, Omega. And then on that you have an initial curve gamma naught. Then what do you look for is that, you look for a characteristic.

So, you have a lifted initial curve gamma naught bar, and you look for a surface z equal to u x y. That is what is called the integral surface. These are the things we have done. So, we have to exactly find a characteristic surface passing through a lifted initial curve. So, the, as far as the Initial Value Problem is concerned there is no issue. So, what is the difficulty?

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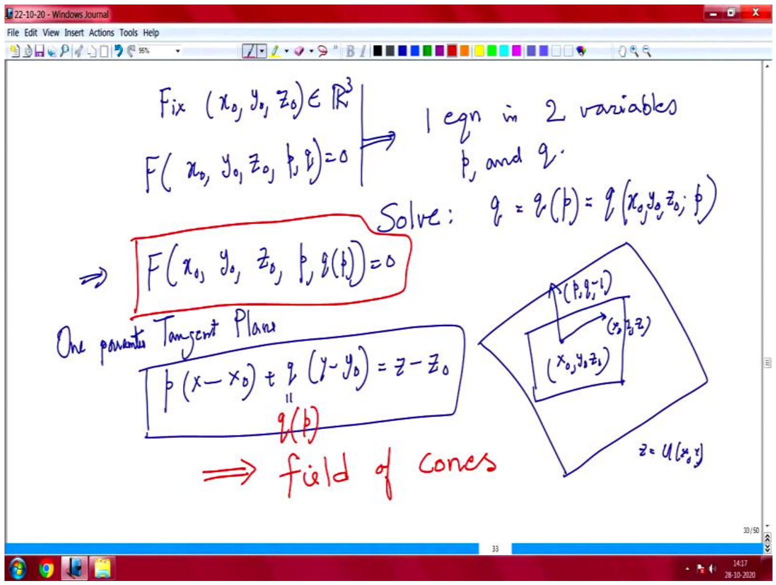

So, let us say, understand the geometry little bit, geometry. So, you fix a point x naught. So, let us try to understand, fix a point x naught, y naught, z naught in R3, and look at this equation F of x naught, y naught, z naught, p and q. Think it as a function of these 2 variables p and q. So, this will provide, this is one equation in 2 variable, one equation in 2 variables, variables p and q.

Thus, in principle assume that you can solve it. So, solve, you know the conditions under which it can be solved. Solve q in terms of p. Assume that there is a unique solution. So, need not be, there may not be a solution like you have seen p square plus q square plus 1 is equal to 0. So, given p you may not be able to find it like that, q square which is minus.

But assume that there is a solve, in fact assume that you can solve it uniquely. That is also need not be correct all the time. So, but since we are trying to understand the geometry, try to say that there exists a solution, there exists a unique solution. So, every p there is a q by solving this non homogeneous equation. This algebraic equation, sorry.

So, let us write z naught is fixed. So, we will not write again and again. So, you will basically have, so that implies F of x naught, y naught, z naught, and p, and q you solve it as in terms of p. So, this is a one parameter family.

So, given any p, so to understand that, so you have a point here, so you think it as an integral surface. If it is an integral surface z equal to u x y. If you take a point x naught, y naught, z naught, imagine this geometry. So, in this case if you are working on, x naught, y naught, z naught, is an arbitrary point. But if you take x naught, y naught, z naught, and if you think that these are all known, its derivative, this is its normal; p, q, minus 1 is the normal. This was the picture again.

So, once you have a normal, there will be a tangent surface. So, you have a tangent plane, one parameter family tangent plane. One parameter family of tangent planes. And what is the equation of this plane? So, if you take a point x naught here, so you a take point x, y, z here. So, you have x minus x naught, this is a direction. So, you are, when the equation of a plane passing through x naught, y naught, z naught and whose normal is p, q minus 1. And then, it is the tangent plane is given by p into x minus x naught plus q into y minus y naught equal to z minus z naught.

So, you have 2 equations which is a one parameter family, because q you can think it as a, q you view it as a function of p. So, you have one parameter family of tangent planes. So, just imagine now the geometry. So, you have a point fixed x naught. On that point you have a plane. That is equation of a plane.

So, when you have a plane, and p changes, you will have another q, you have another tangent plane. So, there will be a plane, there will be planes, passing through this point. As p changes, you get a one parameter family of planes, and it will that, if you look at this envelope, or this one parameter family of planes produces a cone, a generalized cone, need not be right circular cone.

So, this will give you, so the geometry is that, for each point your equation F x naught, y naught, z naught, p, q produces a cone. And if the point varies, you get a field of cones. So, this gives you, a field of cones. You get that right. So, you have a plane for fixed p, you have a q, and you have a plane. As p changes you will have a family of planes, one parameter family of planes all passing through x naught, y naught, z naught. It produces a cone. So, you get that thing.

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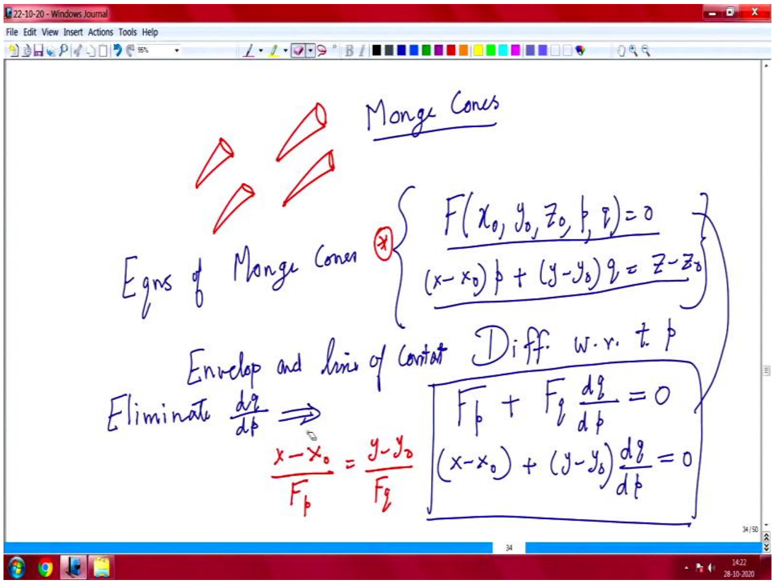

So, what will happen is that for every point you will have a cone. You take the point; you will have a cone. So, let us say imagine little more about the geometry. So, these are called the Monge cones. Monge Cones. So, what will happen, if it is a tangent plane? You want this plane to be a tangent plane. Equivalently, you are looking for a surface.

So essentially, what you are looking for? So, if you are looking for an integral surface, each point you want this cone to be a tangent to the integral surface. So, if you can produce a surface in such a way that at each point, the Monge Cone is tangent to that surface, then that surface will become an integral surface. The converse is also true.

So, you are basically looking at the line of contact between the, that will give you the characteristic direction. So, I am looking for the Monge Cone, and then looking for its, your surface, at each point it will be tangent to the Monge Cones. So, let us try to write down the equations of Monge Cones.

Equations of Monge Cones. So, let me write down all the equations. You have your equations F x naught. So, these are the 3 equations; F x naught, y naught, z naught, p, q equal to 0. You solve this, and then you will have x minus x naught into p; this is the equation of the plane plus y minus y naught into q equal to z minus z naught and then you have. So, these are the 2 equations, you get it.

So, you want to find its envelope, the line of contact. Envelope or, and line of contact. How do you do this one? To do that one, you view these are all functions of p and q. So, differentiate with the respect to p. So, the first equation, f x naught, y naught, z naught, if you differentiate with respect to p you will get F of p plus F of q into d q by d p. You are viewing it as a function of p, that is equal to 0.

If you differentiate the second equation with respect to p, you get x minus x naught plus y minus y naught into d q by d p is equal to 0. So, you have 2 equations, for the. So, these 2 equations, together with these equations basically, so you eliminate now. So, eliminate d q by d p. If you eliminate d p by d q, you get this will give you, so let me use a different color; you eliminate the d q by d p, you get x minus x naught by F p is equal to y minus y naught by F q.

So, you see, so you have these 2 equations; star together with plus u star plus this equation. So, that will give you, your equations of Monge cone and these 2 equations; x minus x naught p plus y minus y naught q equal to z minus z naught; together with this x minus x naught by F p will give you the line of contact.

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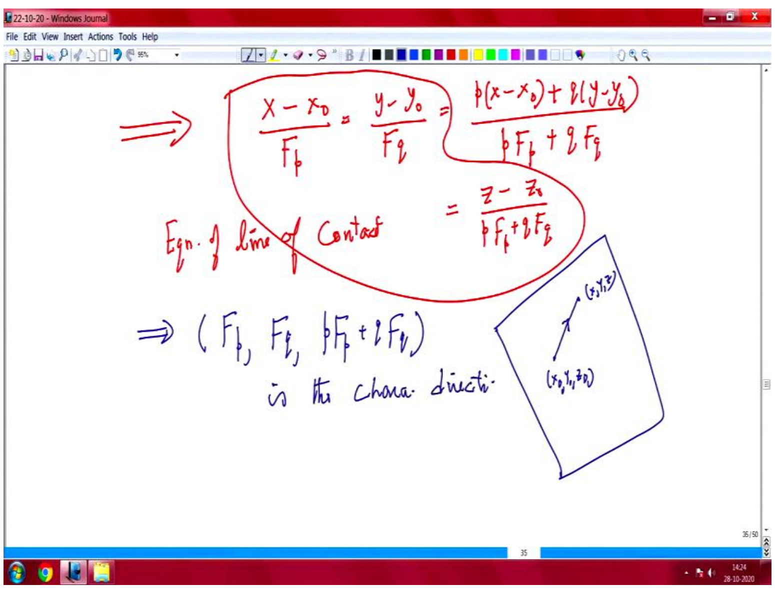

So, I will combine that, if you combining this will imply, implies you get this equation x minus x naught by F p is equal to y minus y naught by F q is equal to, I can multiply and do that one; you will get p into x minus x naught; I can write this, plus q into y minus y naught. So, I am multiplying and dividing. This is standard thing, which you know it, you can do that; this will be equal to q F q.

But what is p into, if you go to the free page, p into x minus x naught, all these will be z minus z naught by then. So, this will be same as z minus z naught by p F p plus q F q. This is nothing but the equation of line. These 3 things are together is the equation of the line of contact. Line of contact. You see, so, you got it.

Now, look at one more thing immediately. So, if you look at this tangent plane. So, let me use a different color. So, look at the tangent plane. So, you have a tangent plane, you have your x minus x naught, y naught, z naught and then you have x, y, z in the tangent plane. And you have your, this is your tangential direction.

This is your characteristic direction and this equation tells you immediately F p, F q, p F p plus q F q has the same direction as x minus x naught, y minus y naught, z minus z naught. That means F p, F q, p F p plus q F q gives you the characteristic. This imply the, are the, is the characteristic direction. Once, you get the characteristic direction you can immediately introduce your characteristic curves. So, I will do it in the characteristic curve.

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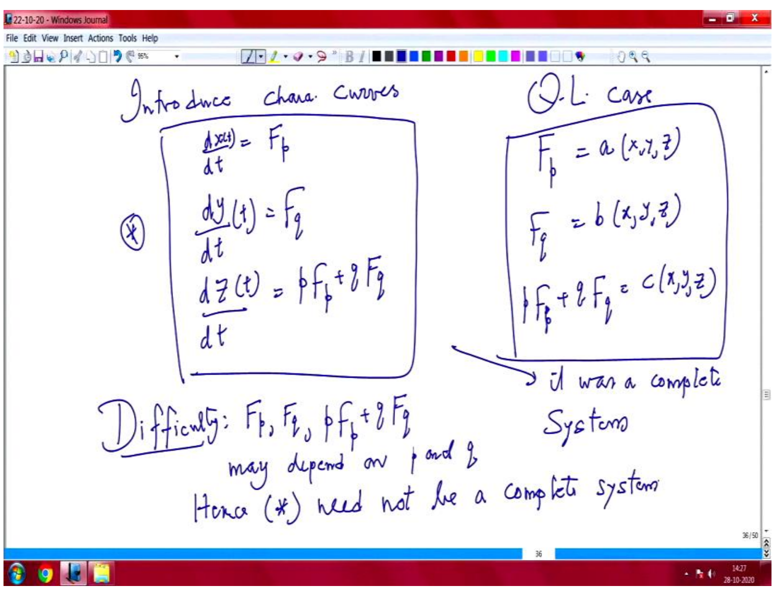

So, introduce, introduce characteristic curves and then we will see the difficulty. Your d x t by d t equal to F. These are all functions of t, is equal to F p; d y by d t, t equal to F q; and d z by d t is equal to p F p plus q F q. So, what are the difficulty compared to your quasilinear case. So, what was in the quasilinear case? So, I am comparing. So, you will understand the difficulty. Quasilinear case F p was a, which is a function of x, y, z; and F q was b, which is also a function of x, y, z; and p F p, you can compute this, p F p plus q F q is equal to c. You can verify all this.

So, that shows, this was a complete system. But here, what is the difficulty? It was a complete system. This one, it was a complete system. Difficulty here, F p, F q etcetera; p F p plus q F q may also depend on p and q, may depend on p and q. And hence, star need not be a complete system. And this also suggest you, you need equations for p and q.

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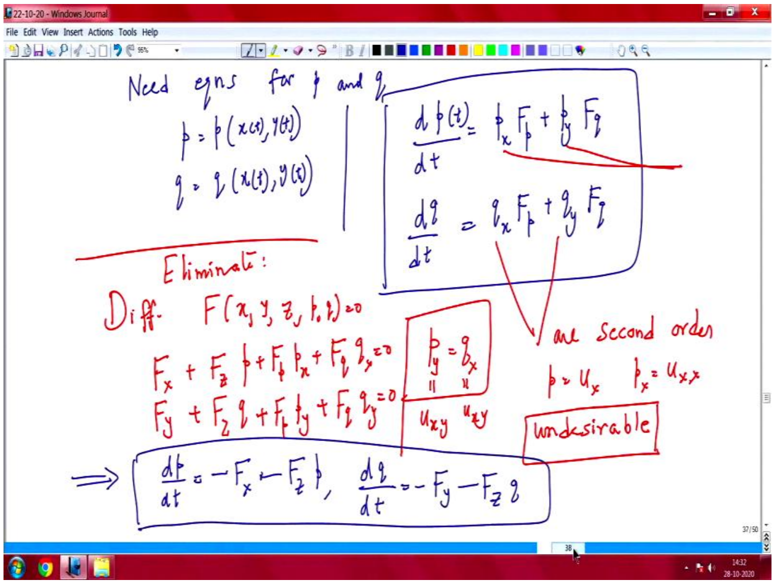

So, need equations for p and q. That we can derive easily p and q. How do I derive? Because p is equal to, p of x t, y t. q is equal to q of x t, y t; p of x t, y t. So, if you differentiate, you get 2 more equations; d p by d t is equal to p x into d x by d t, that d x by d t is equal to F p plus p y into F q.

Similarly, d p by, d q by d t; d q by d t is equal to q x F p plus q y F q. So, you have now 2 more equations. Earlier you already got 3 equations, x; d x by d t, d y by d t, d z by d t together with, now, you have 5 equations. I will tell you, the geometry; you should not think, it is a curve in R5 because our working space is R3 only. So, you should not be seeing, it is a curve in R5. Still, the characteristic curves, you have to see it as a curve in R3 only. But what is this p t and q t are? But you are that, shows that you are unable to solve x t, y t, z t completely. You have to solve x t, y t, z t, together with its p t, q t, x t.

But before that, the difficulty is that, this p x and q; p x q y are second order and that is undesirable. For example, because p is equal to u x. So, p x means u x x; undesirable. You do not want that second order, because you are dealing with the first order equation and you do not want second order terms coming into, second order derivatives coming into the picture. Somehow, you have to eliminate p x and q x, which you can do it. So, eliminate. You have to eliminate. I will not give you the details, and you can work it out the details.

You write down, differentiate the equation F x, y. Differentiate F x, y, z, p, q; with the respect to x and y. If you differentiate, you get F x plus F z. You differentiate F z into p plus F p into p x, F q into q x equal to 0. And similarly, you differentiate F y, you get F z again into q and F p into p y plus F q into q y. So, you have to eliminate 4 quantities; p x, p y, q x, q y. You got 4 equations, but you need 1 more equation, that you can easily check that p x equal to q y; is also true; p x equal to because p x is equal, this is equal to u x or p y is equal to q x. I think so.

So, this is p y equal to q x. So, this is nothing but u x y is also u x y. So, you have 5 equations eliminate and you can write down the equations now. So, this will imply by eliminating, it will imply your d p by d t is equal to minus F of x, minus F of z p, and then you will have your d q by d t equal to minus F of y minus F z. So, you see you have 5 set of equations.

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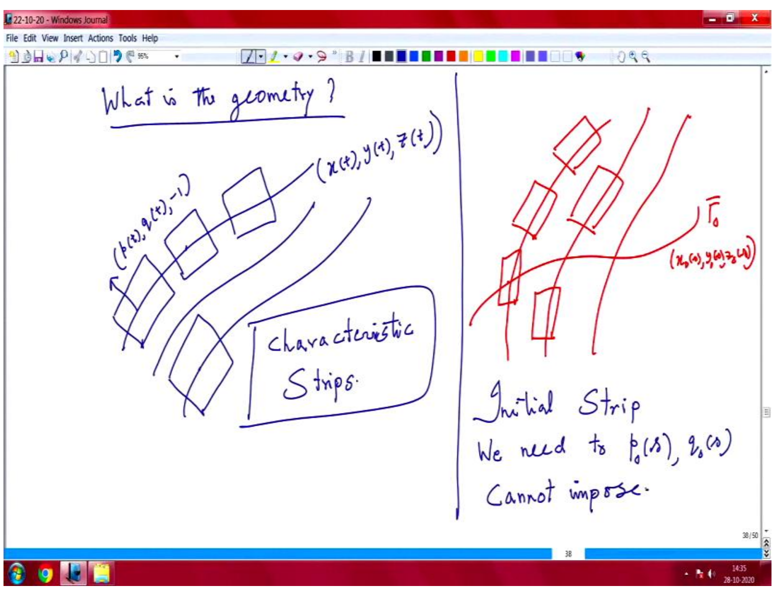

But how do you solve it? And what is the geometry? So, I want to, what is the geometry now? So, you have a characteristic curve, you want to do it. You are not able to solve it, x t, y t, z t.

You want to solve x t, y t, z t; together with p t q t, but p t q t minus 1 is a normal. So, to view this one. So, you are not only solving the curve x t, y t, z t. You are also solving the normal, when you are solving the normal, which is equivalent to saying that you are actually solving the tangent plane.

So, each point, you are solving for the tangent plane also. So, you, have to solve for every curve like that. So, you have this thing. So, every curve you have to do that thing; but then, to solve that, you need 5 initial conditions. But you are given only 3 initial conditions. So, you have to create 2 more initial conditions and these are called the characteristics strips.

So, let me tell you once more how do you, so you have, you have your lifted initial curve, so you have your lifted initial curve. So, let me do it in the different color. So, you have your gamma naught bar. That you know, this is nothing but x naught of s, y naught of s, z naught of s. You have that. On this point you want to solve characteristic curves. Then you can have the same procedure as in. But to solve characteristic curve you also need to solve these things together. So, you have to solve not only the characteristic curve, you have to solve the tangent plane.

So, similarly if you want then you need initial strip basically, you need to have and need to have initial strip. So, in other words we need to get, you can need to get your initial conditions p naught of s, and q naught of s. This cannot be imposed; you cannot impose these conditions. You say that in the problem it should be given, no cannot impose you have to derive that; cannot impose and that is what you are doing. So how do you do all that?

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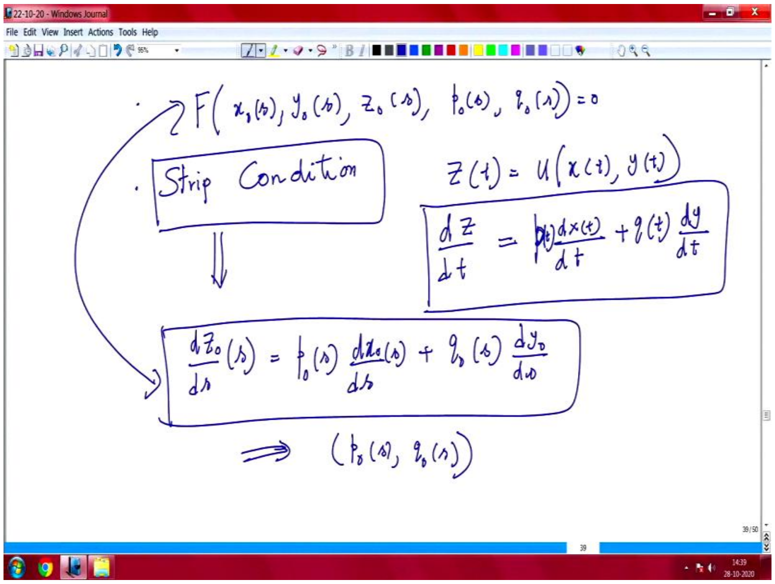

So, first thing is that you need 2 conditions. Now you have to solve for p naught of s, as well as q naught of s. You have x naught of s, y naught of s, z naught of s. But the characteristic curves are solved for x t, y t, z t, together with p t q t. So, you need initial conditions for p naught of s, and q not of s.

But the initial curve lies also on the equation, that also satisfies the equation, because it lies on the; the lifted initial curve lies on the characteristic integral surface. So, that implies F of x naught of s, y naught of s, z naught of s, p naught of s, q naught of s equal to 0. Of course, whether we have a solution etcetera, you need, is a different issue, and so there is one more condition.

So, you need one more equation that is called the strip condition. All this you have to know; strip condition. What is strip condition? You have your u x t is equal to, lies on right. Let me come to the strip condition; z is equal to. So, you have generally, z t is equal to u of x t, y t. So, if you differentiate this with respect to t, you will have d z by d t is equal to u x, which is, you will do, which is p. So, you will have p into d x by d t. This is p t, this satisfy for every t, plus q t that is u y into d y by d t. This strip condition is satisfied, not only the characteristic curve; so the strip condition. This implies this strip condition d z by d t equal to etcetera, also satisfies the initial curve.

So, you will get one more condition for your initial p naught of s, q naught of s. That means you will have your z naught of s, that is nothing but your d z naught by d s. This is also satisfied at s is equal to p naught of s into d x naught by d s. This is along the initial curve plus q naught of s into d y naught by d s. So, you have 2 equations, these 2 equations will give you p naught of s, and q naught of s.

So, in principle if you look at it, you have 5 equations if you go for; 3 equations for x t, y t, z t. Then 2 equations for p t, q t, 5 equations. And then, that 5 equations you have to solve it to get the characteristic strips. Then 3 conditions for initial conditions are given, and 2 initial conditions are; in all this can be solved.

So that is where when you do a particular problem you may face, you may face difficulties, is not that always it can be solved. But then you cannot apply this method and if you look at it, all this thing what you see is that you are only solving ODE. So, the messages is that to get the solutions to your PDE, you are actually solving ODEs.

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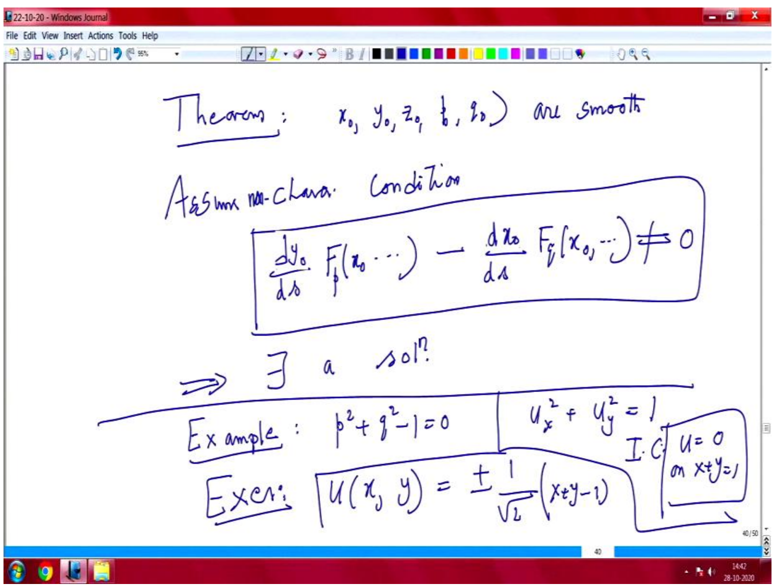

So, let me complete this talk, this particular talk by giving the final theorem. So, you define x naught, these are all has to be smooth, y naught, z naught, p naught, q naught are smooth. There are very precise conditions. If you want to get very precise conditions look into the, our reference book.

And assume your characteristic condition. Assume non-characteristic condition, it is a non-characteristic, you do not want to be a characteristic, non-characteristic condition. This is the condition for solvability; d y naught by d s. A instead of a in the quasilinear case, you have F p. It is everything in x naught etcetera, minus d x naught by d s into F q x naught etcetera. This should be non-zero. This is along the initial curve, lifted initial curve. Then implies, there exist not unique solution, there exist a solution. There is no uniqueness.

So, I will end this talk with showing an example. This theorem can be proved along the similar lines as in the quasilinear equation. Example, so you look at these equations; p square plus q square minus 1 equal to 0. That is the PDE's u x square plus u y square equal to 1. So, this is an exercise which you can work out. There are more exercises in the book. I suggest you go through and do some problems.

And you can see that you can get 2 solutions; u x y is equal to plus or minus 1 by, and what is the initial condition, initial condition. So, the initial condition u equal to 0, on x plus y equal to 1. This is your initial condition. So, you can see that plus or minus 1 by root 2 x plus y minus 1. This is your PDE. This is your 2 solution.

The reason is that when you solve for q in terms of p, you get 2 solutions. You see, you do not have a unique solution. So, for each p, you get 2 separate one parameter family of planes, deriving that 2 different surfaces leading to 2 integral surfaces, passing through the same initial curve. But up to quasilinear equation there was a unique solution passing through that.

So, with this I will end my talks on first order equations in 2 variables. But you will now listen later from the higher dimension multivariable case from next lecture in another 2 hours or something. We will introduce you to similar concepts, but this kind of easy or simple geometry may not be available, but geometry is still there. Understanding the hyper surfaces in R n plus 1. You have to look for solutions, the surfaces in R n plus 1. Just like 2 variables, you look for surfaces in a R3, there you look for surfaces in, is or something like hyper surfaces in R3. Thank you.