First Course On Partial Differential Equations - I
Professor A.K. Nandakumaran
Department Of Mathematics
Indian Institute Of Science Bengaluru
Professor P.S. Datti
Former Faculty, TIFR-Cam, Bengaluru
Lecture-10
First Order Equations In Two Variables-4

Welcome back to this quasilinear situation again. So, we were discussing about the quasilinear equations, and then we are introduced, we have introduced what are called the characteristics curves.

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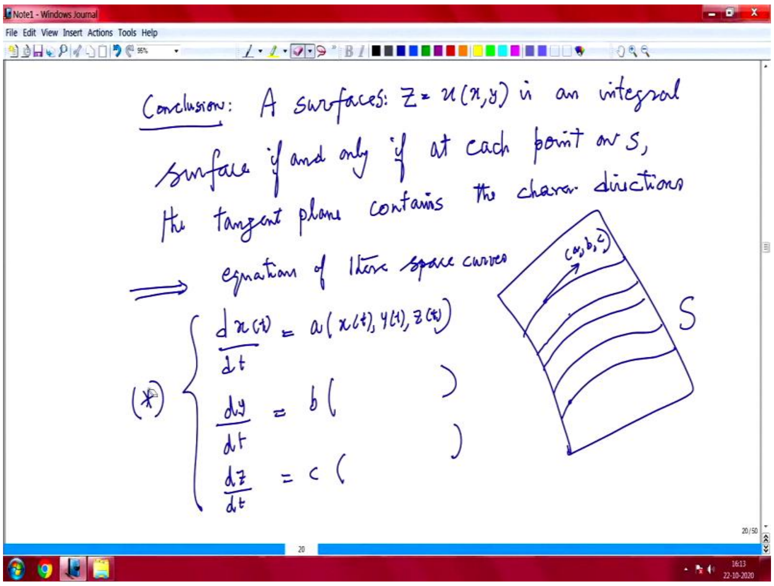

So, let have this system called, d x t by d t equal to a, d y by d t equal to b and d z. I call it, that star.

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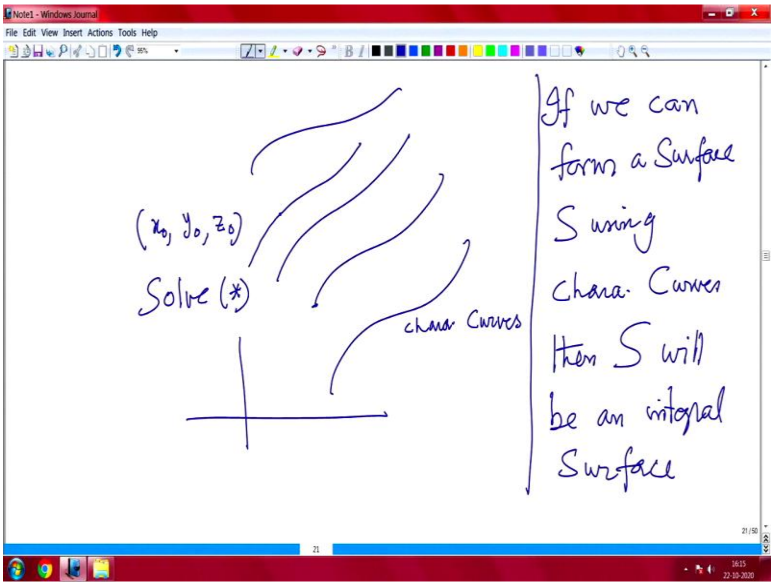

So, basically, look at any point x naught, y naught, z naught; x naught, y naught, z naught in R3. So you have a point here, and then solve star. Solve star, that is an ODE, is an ODE system with 3 components. And then you can uniquely solve for every point, you will have a curve here. This is in R3, you have to understand that. So, you will have curves all along. So these are called the characteristic curves if you want it. If you want to put it, characteristics curves. So, every point passing through, so the whole region R3 can in principal filled with that curves.

So, what we are looking is that, so integral surfaces, so if we can form a surface, if we can form a surface S, using these, using characteristic curves, then S will be an integral surface, then S will be an integral surface. That is what we are telling. The converse is also true. So, you have your region R3. In the region R3, you are looking for any point and solve that ODE to get a curve. So, vary that point. So, you will have a kind of, because curves are in some sense is a one dimensional object.

So, you have some two dimensional class of. I do not want to call dimension, but you can see that kind of two parameter or whatever it is. You will have a set of curves filling the R3. From that, I can form a surface using these characteristic curves. Then that surface will become an integral surface, because these curves have the property, its tangential directions are a, b, c. So if you form a surface, using only 1 family of characteristic curves; then it will the only thing for a surface to be an integral surface, the only condition to be satisfied is that, the tangent plane contains that tangential direction. That is exactly this thing.

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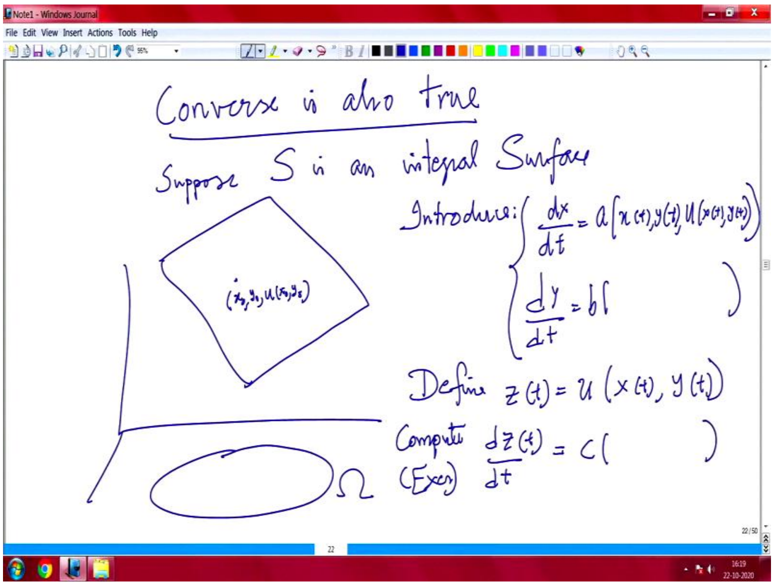

Only conversely, the converse is also true. Converse is also true in the sense, suppose you have an integral surface. Suppose S is an integral surface. What does that mean? You have a surface here; you take any point. Now the point will be x naught, y naught, u of x naught, y naught. You are to see that. Because it formed an integral surface. You have your omega here.

Now what I do is that, I introduce an ODE; introduce d x by d t is equal to a of x t, y t, u of x t, y t. Because u is known to you now. Because the surface is an integral surface. Earlier situation, the previous thing is that given a surface, you are forming integral surface. When that surface becomes an integral surface. So the surface is formed from the integral curves.

Here I am telling that, the integral surface you start with it, you can decompose into integral curves; and d y by d t. Now this is a complete system, because the third component is known to you. Now you define the third part, z t is equal to u of, this is you can determine x t, you can write z t is equal to u of x t, y t. Then if you compute, if you compute d z by d t; do this computation. This is a small exercise. If you compute this, you can see that this is nothing but c. You just do the computation, use the PDE, you get it.

So in conclusion, what we have done is that for every point you can have an integral curve. In R3, if you can form a surface using these integral curves, you get an integral surface, which will be an integral surface. Conversely if you give an integral surface, then I can decompose that surface into integral curves.

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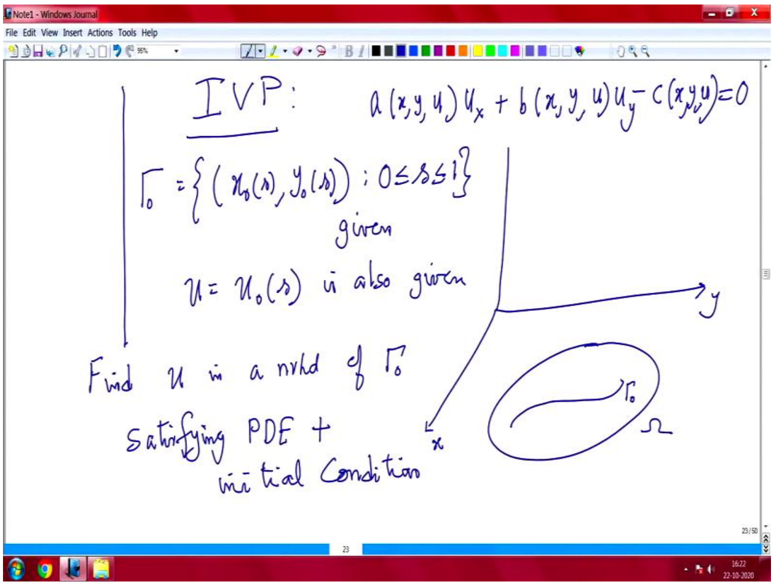

Now, with this information, we will go to Initial Value Problem for quasilinear equation. So now let us have a initial. So the Initial Value Problem is the same, the interpretation the geometry behind the quasilinear equation and the complications arising due to the nonlinearities are different. So what is your PDE? So let me find, once more call it; since I am introducing my quasi x, y, u, u x, b x, y, u, u y minus c x, y, u equal to 0. This is your PDE.

So, you have your domain here. In 2 dimensional domain. So this is not I want to do. So you have your 2 dimensional domain here, omega. This is your x, this is your y, and then you are looking for, and then you have an initial curve. So the initial curve gamma naught. So you have to understand that. So you have your gamma naught. Gamma naught is given as the initial curve, x naught of s, y naught of s; 0 less than or equal to s less than or equal to1.

That is initial curve, this is given, and u equal to u naught of s is also given. So, what is the initial? So this is the complete data. PDE is given, your initial curve gamma naught is given, u is given, and find u in a neighbourhood of gamma naught, satisfying PDE plus initial condition. So that is it. That is your Initial Value Problem. Now how do you solve it? How did we solve it in the linear case? In the linear case you are try to find the characteristic curve along the curve gamma naught; which was not possible. So what I am going to do is lift this curve.

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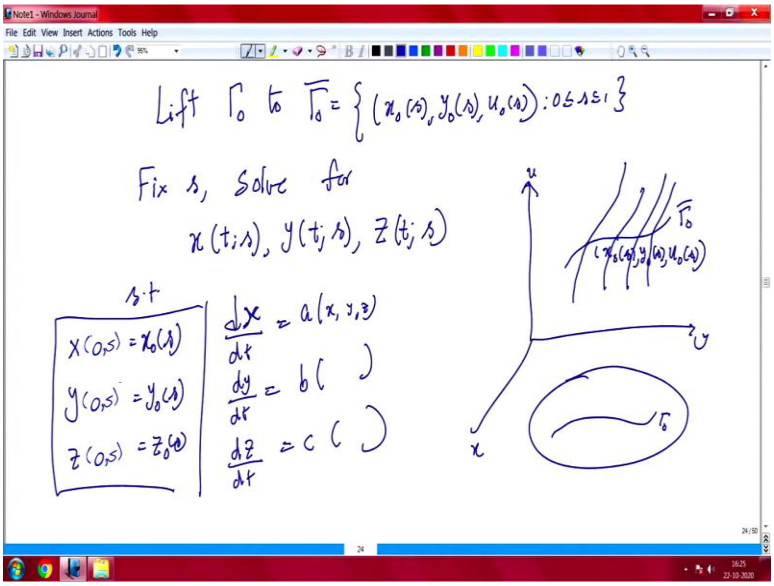

So the method; lift gamma naught, lift gamma naught to gamma naught bar by adjoining my initial conditions, initial value x naught of s, y naught of s, and add u naught of s here. So, this will be a space curve. What did I do basically? You have this; you have your initial condition, you have x here, y here, you have your u here. So you have your domain here and you have your gamma naught here. On gamma naught, I will get a space curve gamma naught bar. Because I do not know how to find my characteristics here.

So, what do I do here, is precisely the problem is that for each point; now I find it. Because now I know that how to find a space curve as an initial curve. So each point here, so the any point here will be of the form of x naught of s, y naught of s, u naught of s. Along that initial curve, so you solve your equation. So you are solve here 3 dimensional equation exactly that.

So, what you do is that you fix s, and solve for x t, the curve, but that depend on s; so you write s here. y t s, z t s; z is u in end of it; s satisfying such that d x by d t, d x by d t is equal to a of x t, y t, x t s; everything; x, y, z. d y everything in t variable. d y by d t equal to b of that; and d z by d t is equal to c of that; with the initial condition x at 0, s is equal to x naught of s; y at 0, s is equal to y naught of s and z at 0, s is equal to z naught of s, using you solve this.

So you have a system of 3 unknowns x, y, z with initial condition. So when you s, you change it; you have a characteristic curve. As I said that, last 2 hours we are discussing that, there is a question of inverse; given a point there will be a characteristic curve which passes through that. Yes we need a condition, now it is time to state the theorem.

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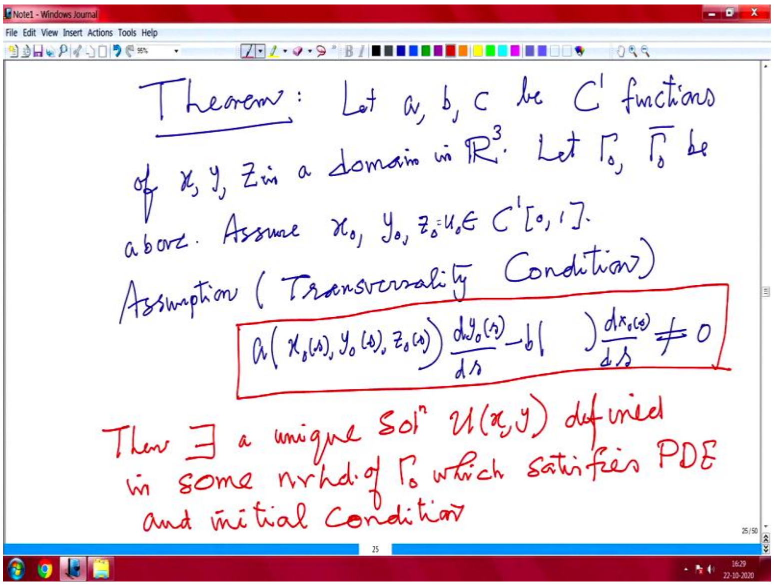

And then I, proof is just the inverse function theorem. So I will give the theorem here. Let a b c. So let me put conditions; a, b, c be C1 functions of, be C1 means continuously differentiable; C1 functions of x, y, z in a domain R3. Let gamma naught, gamma naught bar be as above, the initial curve given by the problem; gamma naught is the initial curve given by the problem, and gamma naught bar is the (())(13:50).

And x naught, y naught, z naught is the curve describing the initial value. Assume x naught, y naught, z naught is in C1. It is defined in one variable. And the important thing is about the assumption for the condition of to apply, in assumption this is called the transversality condition, transversality condition.

What is the transversality condition? You look at the value at the initial curve. This is a x naught of s, y naught of s, z naught of s into d y naught. That is the tangents to the. So you need to understand, what this condition meaningful; minus b, again same argument d x naught s by d y naught of s.; sorry d s. This should be not equal to 0. I will tell you, that you have the physical interpretation. I exactly tells you.

So this is, you have transversality condition. This is the additional condition. In addition to the smoothness of the theorem, you have to satisfy this addition. Then there exists a unique solution u x, y; defined in some neighbourhood of gamma naught, which satisfies the PDE, and initial condition. So that is the theorem. So what is this transversality condition means? So go to the next page.

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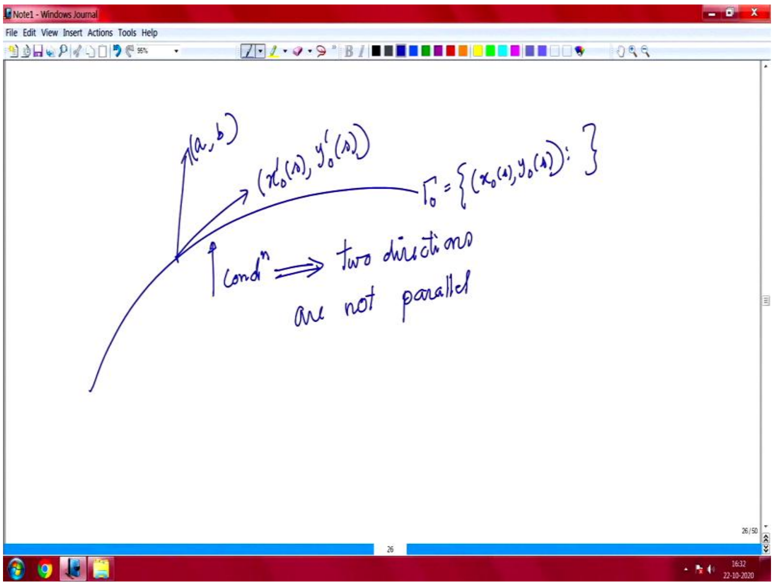

So, you have your initial curve. You have your initial curve gamma naught, and then at this point you have your tangent, that is nothing but x naught; d x naught by d s. I am using 2 different notations. That is all. y naught of s and then a, b is your direction. This is some a, b will give you a direction here, given by a of x naught of s, y naught of s, z naught of s.

The condition tells that, these 2 directions, condition implies the transversality condition implies these 2 directions are not, are independent. 2 directions are not parallel, independent. That is what it is. In other words, you are given in terms of the orthogonality. So if you look at the back the condition. So d y naught minus d x naught will be the normal, and that should not be orthogonal to that.

So, you have that direction here. So you, this is given by x naught of s, y naught of s. And this is a, b is part of your characteristic curve direction; that means this gamma naught should not be a characteristic direction. It should not be at any point of time your gamma naught has to, it should not be a characteristic direction, it should not indicate the characteristic direction. Because a, b gives you the direction to us the characteristics. So the condition is precisely your gamma naught has to be a non-characteristic. That is what.

You will learn a more and more about as the course proceeds, you will learn more about your gamma naught. This is exactly what we have observed in some examples. If your initial curve happened to be a characteristic curve, the other characteristic curves will not come and intersect the characteristic to transport the values. And what we have done is a precise definition of that.

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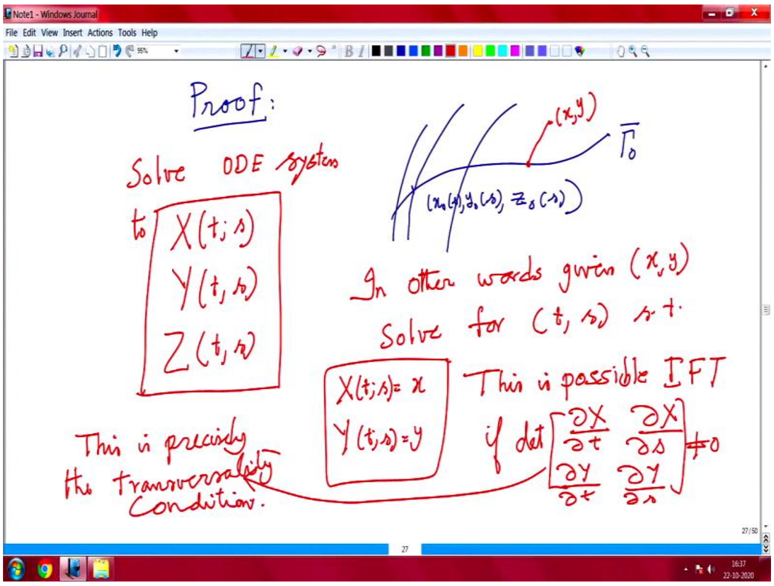

So, let me give you a quick proof of it. Because you almost done everything. So it is a one line proof, it is application of, it is just an application of that. So we have done almost all the job. So you have your gamma naught bar. So you have your gamma naught bar, and you know that every point you have a characteristic curve. So, this point is x naught of s, y naught of s, z naught of s. And then what do you want? So you have every point. You can have characteristics.

Now, the issue is that, you take a point here. You want to know, does there exist a characteristic curve. So you want to know. So the given, so you solve your ODE, solve ODE system, to get your solution of the ODE; which are let me denote that solution by x; it depends on s; I told you. x, y t, s and z t, s. And this is a solution to s.

Now, what do you want? Starting with an x t, you want to find a characteristic which meets your characteristic initial curve, lifted initial curve. That is what you want. In other words, what you are looking for? A characteristic curve, described by the parameter t and then, here, this is not a t, because we use the t; this should be y.

Given x, y; you want in other words, given x, y in a neighbourhood of gamma naught, wherever it is neighbourhood of gamma naught. Solve for t s, such that x is equal to, x t s is equal to, this is what you want to solve it. You want to see that, there is a characteristic curve; y t, s.

This is possible by Implicit Function Theorem, or Inverse Function Theorem. Both are fine. Inverse Function Theorem; if the corresponding determinant is non-zero. So d x by d t, d x by d s, d y by d t, d y by d s not equal to 0. These condition of determinant of d x by d t, all that not equal to 0, is precisely your transversality condition. This is precisely the transversality condition.

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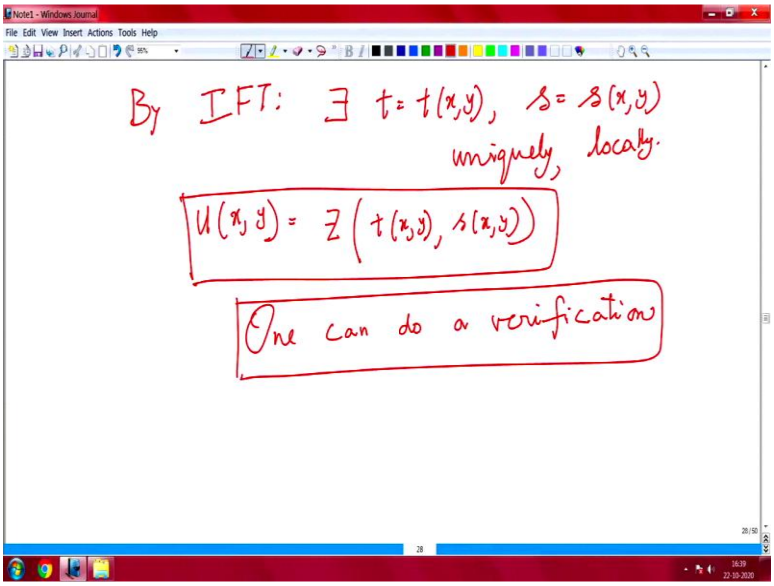

So, by Inverse Function Theorem, there exist t, that t will depend on now x and y. You can solve for t, s is equal to s x, y; uniquely, locally. Everything is local, uniquely of course, local. Then you can compute your u x, y; as given x, y. You have done this here. You got, where you have that. u x, y; you have your z of, you solve it, z of t of x, y; you have solved z x, t.

So we have substitute s x, y. You can actually do a verification. One can do a verification. Verification that, this is indeed a solution. I will not do that one. I will, you can do that. You can compute your u x. u, x will be said corresponding to that all, that you can process; you can do that one and you can show that, it is indeed a solution.

So, that way using the, you see that immediately the effect of Inverse Function Theorem. Using that you can actually solve in a, it is all local result. So, the theorem is local, but this method of solving using characteristic, quite often; you can produce a bigger solution as a global solution. So, the theorem is a local, but it is possible that while solving your differential equation, you will get a global solution.

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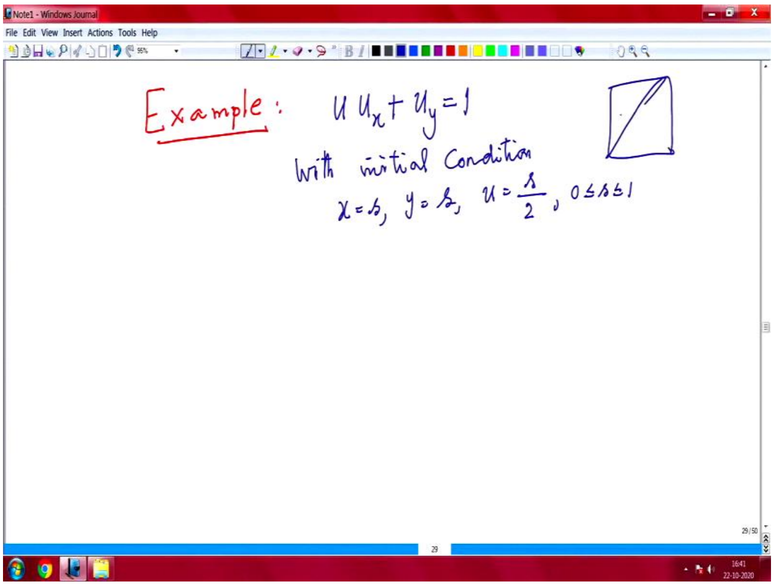

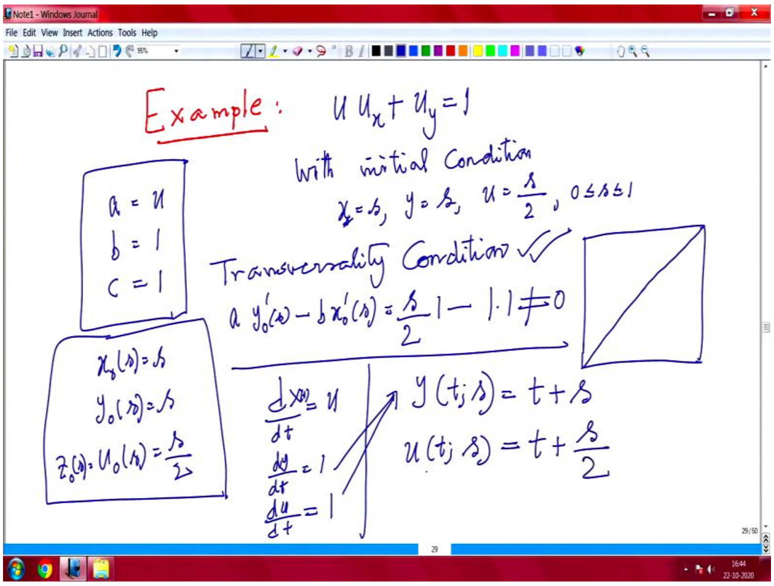

So, with 1 example may be, with 1 example, I will; example. So last I think. So let us consider a equation, u u x. I will not do all the calculations, but you can do the calculation, u y equal to 1. What is the initial conditions? With the initial conditions, computations you can do it, I will tell you the method. And but try to work out more example. Initial conditions x equal to s, y equal to s. What is that curve? You should understand that curve, I am giving it in that format. x equal s, y equal to s and your u equal to s by 2, and I am solving 0 less than equal to s.

So, your values in diagonal is given here. So, if you look at it here, it is given along the diagonal. So x equal to s, y equal to s and your values are given along the diagonal. The values of u are given along the diagonal. Let us do, so what is your a; a is equal to u, your b equal to 1, and c is also equal to 1. That is very clear.

So, first verify transversality condition. What is your transversality condition? a along the curve, so your curve is this one. So, a along the curve, a is nothing but u, and u is s by 2. So, a into y naught prime s, everything in s, minus b x naught prime of s. What is this? a is s u and along the curve it is s by 2, that is s by 2.

And what is y naught of s? This is x naught, along the curve. This is your x naught. So, let me write that also. Your x naught of s is equal to s, y naught of s is equal to s, and your u naught of s, something you can also use it as z naught of s, if you want, equal to s by 2. So, this is your initial condition.

So, this is s by 2, your y naught prime is equal to 1 minus, b is equal to 1, so that is 1. And x naught prime is also equal to 1. So, 0 less than s less than 1. So, this is actually, it can never be 1. This can never be 1. So this not equal to 0. So, the transversality condition is satisfied.

Now, try to solve it. What do you want to solve it? Now solve. d x by d t equal to u, d y by d t is equal to 1 and d u by d t or d z by d t is equal to 1. With this initial condition, x naught of s is equal to 1. So, what you will do? You can solve these 2 equations, directly. That is easy to solve it.

What do you get it? If you solve it, your y t s will become, it is 1, integrate. So, you get t, y t s will become, if you integrate this one, it will become t plus a constant will come. But y at 0 is equal to y naught of s. So that is equal to s. So you will get s here. And then you will get u t s is equal to or z t s is equal to; again here you will get t plus then the constant is s by 2. So you get s by 2. So you have to solve this equation now because you determine. So let us do it.

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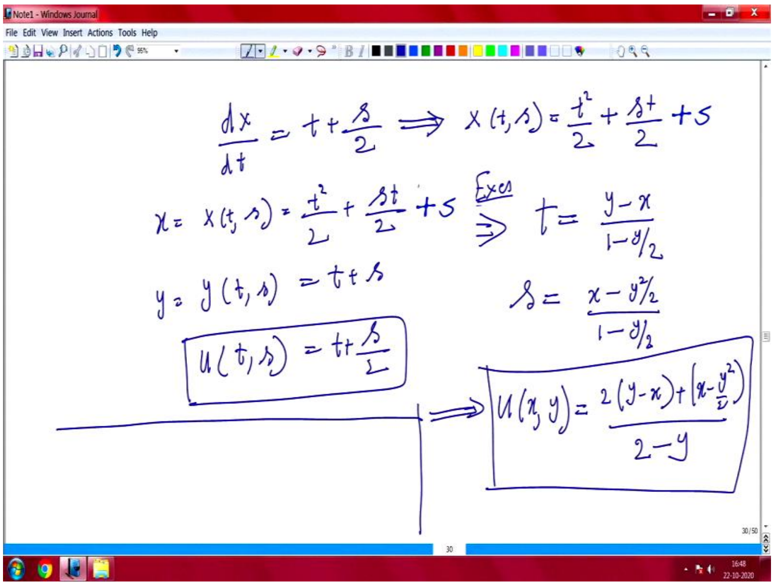

So, you have your d x by d t, t plus s by 2; that implies x t s is equal to t s integrate, that will give you t square by 2, this will give you s t by 2. So, you have everything in terms x t s. So there is an invertibility. So, you have an x t s. So let me write once more, x t s is equal to t square by 2, so you can do some work, s t by 2. And y t s is equal to t plus s, and then z t s is equal to, you have to invert now, or u t s is equal to, t plus s, t plus s by 2 may be. Yeah t plus s by 2.

So, you want to invert this. Which one is x equal to x t s, you have to invert; y equal to y t s you have to invert. So if you invert, that means you have to write down your t and s in terms of thing. I do not do the calculation, but I will write down, you get t is equal to. So, all these equations, exercise. This is a simple exercise to solve it.

You get your t is equal to y minus x by 1 minus y by 2, and you get your s is equal to x minus y square by 2 by 1 minus y by 2. Substitute, that imply, so you substitute in the values of u, t, s. Here, you substitute here, and then that will give your u yu x y. Try to work out more and more problems; 2 into y minus x. Please do this computation, small computation. It is not a difficult computation; x minus y square by 2 by 2 minus y.

So, that completes what I want to discuss about the quasilinear equations. My plan was even to complete the non linear case in this half an hour course. But I could not make it. So, maybe we will take another half an hour. So, the 2 hour plan may be another half an hour, we will do the general non linear equations.

And then that geometry is little more complicated. Since, we have already done this one, try to get, before going understanding step by step, before going to the quasilinear case try to get to understand the linear case properly. Try to solve few problems from the linear case. After understanding the linear case the situation you go to non linear case. Understand that non linear case geometry behind it, you immediately see the different individual from the plane you have to go to space.

But then when you go to non linear case now you cannot go further up because there is only two variable and unknown, so three space. You have to still remain in the R3 space. But the geometry is little more complicated, you have to understand the thing, there are concepts like Monge cone, and characteristics strips and all of that. So, we will take up that quickly in the next half an hour and try to tell you the geometry. Once this 2 dimensional things are completed we will go to n dimensional complete. So thank you. We will complete this. Bye.