Partial Differential Equations - 1
Professor A K Nandakumaran
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Lecture - 01
Introduction - 1

Good morning and welcome to this course on partial differential equations is an NPTEL video course this course actually has two parts PDE 1 and PDE 2. In the first course we will be doing PDE 1 partial differential equations 1 which is a 20 hour lecture and this will be taken by two of us myself A K Nandakumaran from department of Mathematics in the Indian Institute of Science and professor P S Datti, he is a faculty in Tata Institute of Fundamental Research - Centre for Applicable Mathematics in Bangalore.

So, what in my first lecture I will do, half an hour we will briefly discuss or describe the material which we are going to cover not only in PDE 1 and we will also explain to you PDE 2 what we will be doing it. So, both these 2 courses together will cover a one semester course.

So, this first course will be a half semester course and the second one will be another half course, so totally you will have 40 to 42 lectures covering a basic course on PDE. We will not be covering any advanced course but because of it is nature you need to learn some good fundamentals so it is ideal for a Masters student, this course, both courses put together will be an ideal course for Masters student.

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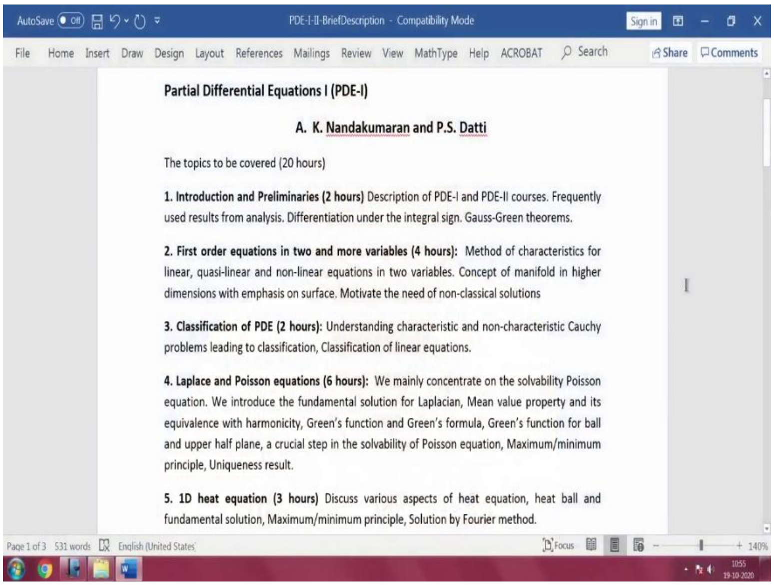
Partial Differential Equations I (PDE-I)

A. K. Nandakumaran and P.S. Datti

The topics to be covered (20 hours)

1. Introduction and Preliminaries (2 hours) Description of PDE-I and PDE-II courses. Frequently used results from analysis. Differentiation under the integral sign. Gauss-Green theorems.

2. First order equations in two and more variables (4 hours): Method of characteristics for linear, quasi-linear and non-linear equations in two variables. Concept of manifold in higher dimensions with emphasis on surface. Motivate the need of non-classical solutions

3. Classification of PDE (2 hours): Understanding characteristic and non-characteristic Cauchy problems leading to classification, Classification of linear equations.

4. Laplace and Poisson equations (6 hours): We mainly concentrate on the solvability Poisson equation. We introduce the fundamental solution for Laplacian, Mean value property and its equivalence with harmonicity, Green's function and Green's formula, Green's function for ball and upper half plane, a crucial step in the solvability of Poisson equation, Maximum/minimum principle, Uniqueness result.

5. 1D heat equation (3 hours) Discuss various aspects of heat equation, heat ball and fundamental solution, Maximum/minimum principle, Solution by Fourier method.

So, let me briefly go through it what we are going to do it in the first 20 hour lecture, so we will start with some introduction and preliminaries that is what we are doing it and this we are planning to cover in 2 hours maybe 3 hours but as I said this requires some substantial amount of preliminaries and I request suggest the students who wants to study this course should go through these preliminaries. Since, it is a PDE course we cannot spend too much time on these preliminaries but I will describe some important aspects of preliminaries.

So, first thing is describing PDE 1 and PDE 2 and that is what introduction. After doing that, the second important thing is about the study of first order equations we will do it in two variables to explain to you the some of the ideas namely the method of characteristics and we will explain in two variable case to bring out the essential concepts behind it.

And that will be separated for linear, quasi-linear and non-linear equations. And then the higher dimensions we will proceed further. So through these lectures, this not only giving some describing the method of characteristics we will also try to motivate you the importance of the studying solutions for partial differential equations in a non-classical fashion. Perhaps, we will introduce the concept of manifold at least in the case of Euclidean space and then we do and this we are planning to cover in 4 hours.

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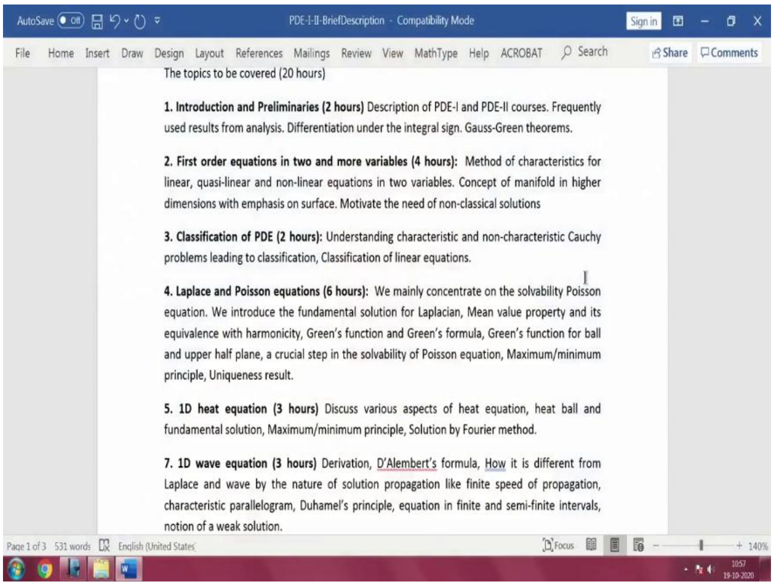
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5. 1D heat equation (3 hours) Discuss various aspects of heat equation, heat ball and fundamental solution, Maximum/minimum principle, Solution by Fourier method.

7. 1D wave equation (3 hours) Derivation, D'Alembert's formula, How it is different from Laplace and wave by the nature of solution propagation like finite speed of propagation, characteristic parallelogram, Duhamel's principle, equation in finite and semi-finite intervals, notion of a weak solution.

After that as you see which I will describe today in my like second lecture the importance of classifying PDE in as it is not easy to understand PDE. So, you have to understand the concept of characteristic and non-characteristic Cauchy problems which will lead to what is called a classification which we will do it in 2 hours. Once classification is done we will move on to study the second order equations, based on the classification we will have three equations basically and namely Laplace and Poisson equations which is a class of elliptic equations which is a representative of elliptic equation.

In that we will mainly concentrate on the solvability of Poisson equation. In the process we will introduce what is called the fundamental solution for Laplacian the important of mean value property and it is equivalence with harmonicity, Green's function and Green's formula and specifically we will construct crease function for ball and upper half plane. And crucial step in the solvability not only of Poisson equation and also other general second order elliptic equation. We also derive maximum and minimum principle uniqueness and all that.

The another class of equations is the heat equation and wave equation and in this first PDE 1 we will discuss about 1d heat equation that means heat equations in one dimension, we will discuss various aspects of heat equation, we will introduce what is a heat ball, fundamental solution again maximum and minimum principle and perhaps solution by Fourier method and we will also discuss, that will do it in 3 hours.

And then we will do it the one dimensional wave equation, we will do the derivation of one dimensional wave equation D'Alembert's formula how it is different from Laplace and wave by the nature of solutions propagation like finite speed of propagation, characteristic parallelogram, Duhamel's principle, equations in finite and semi-finite intervals, notion of a weak solution, again trying to tell you the importance of studying a non-classical solution. This is what we are planning to cover in our this PDE 1 course consisting of 20 hours.

In the second part let me briefly tell you I will not do it in the so we will study little more elaborate way whatever we studied in PDE course 1 this is the PDE course 1 this probably may be offered next year in 2022 or maybe in the later second half of second semester of 2021.

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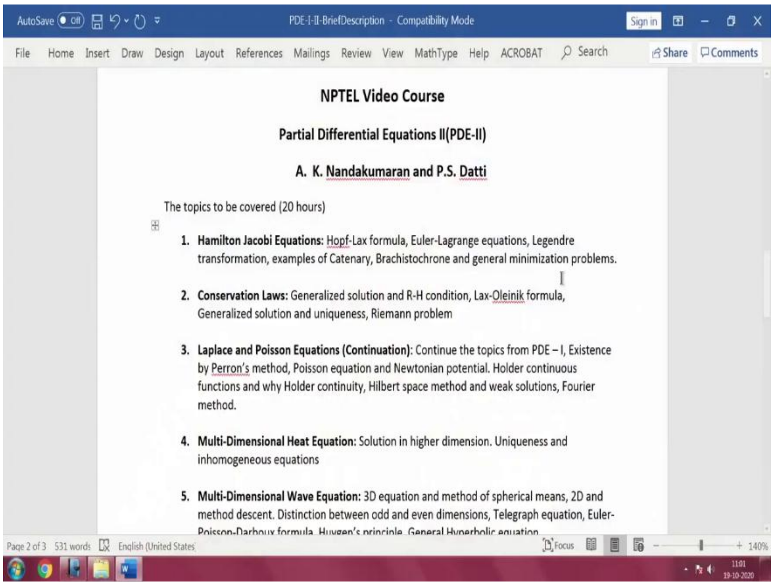

NPTEL Video Course

Partial Differential Equations II(PDE-II)

A. K. Nandakumaran and P.S. Datti

The topics to be covered (20 hours)

1. **Hamilton Jacobi Equations:** Hopf-Lax formula, Euler-Lagrange equations, Legendre transformation, examples of Catenary, Brachistochrone and general minimization problems.
2. **Conservation Laws:** Generalized solution and R-H condition, Lax-Oleinik formula, Generalized solution and uniqueness, Riemann problem
3. **Laplace and Poisson Equations (Continuation):** Continue the topics from PDE – I, Existence by Perron's method, Poisson equation and Newtonian potential. Holder continuous functions and why Holder continuity, Hilbert space method and weak solutions, Fourier method.
4. **Multi-Dimensional Heat Equation:** Solution in higher dimension. Uniqueness and inhomogeneous equations
5. **Multi-Dimensional Wave Equation:** 3D equation and method of spherical means, 2D and method descent. Distinction between odd and even dimensions, Telegraph equation, Euler-Poisson-Darboux formula, Huygen's principle, General Hyperbolic equation

Then we will study two important equations namely Hamilton Jacobi equation and Conservation Law, both are first order equations. Let me warn you we can only touch some minimal aspects of this equations. For because, Hamilton Jacobi equations it iself you can study as a one semester or two semester course, Conservation Laws you can study plenty of hours a one full semester or two semester or three semester course and the amount of literature in Conservation Laws and Hamilton Jacobi equations are enormous.

And the but what we will do some important aspects of Hamilton Jacobi equation namely Hopf-Lax formula, Euler Lagrange equation, Legendre transformations, some classical problems of examples namely Catenary, Brachistochrone problem and general minimization problems. And in the Conservation Law we will introduce the concept of generalized solution and ranking (()) (8:40) condition, Lax-Oleinik formula and some aspects of conservation law.

And this will motivate you for the to study further on these topics and then we will come back to Laplace and Poisson equation which we have already studied in the PDE 1 we will study more about it. In the first part we may studied various aspects of it like maximum principle minimum principles and fundamental solutions and all that. So, and we will try to explain to you some aspects of existence of by Perron's method and then we introduce Newtonian potential for the solvability of Poisson equation.

We introduce another important class of functions and some certainties about studying a Laplace equation and we introduce the concept of Holder continuous functions, why the importance of Holder continuity and then some weak formulation we try to explain what is called the modern method Hilbert space method and weak solutions and all that.

And similarly, we will go further on multi-dimensional heat equation for one d, one dimensional of heat equation you already studied in the first class and first course and then we will do the solutions in higher dimensions, uniqueness in homogeneous situations and all that and similarly multidimensional verification there you will see that subtle difference the important difference of solutions in three dimensional wave equation or and two dimensional wave equation. More generally, you will see the difference in even dimensional equation and odd dimensional wave equation. So, typically we solve study first the 3D equation using the method of spherical means and then 2D the method of descent.

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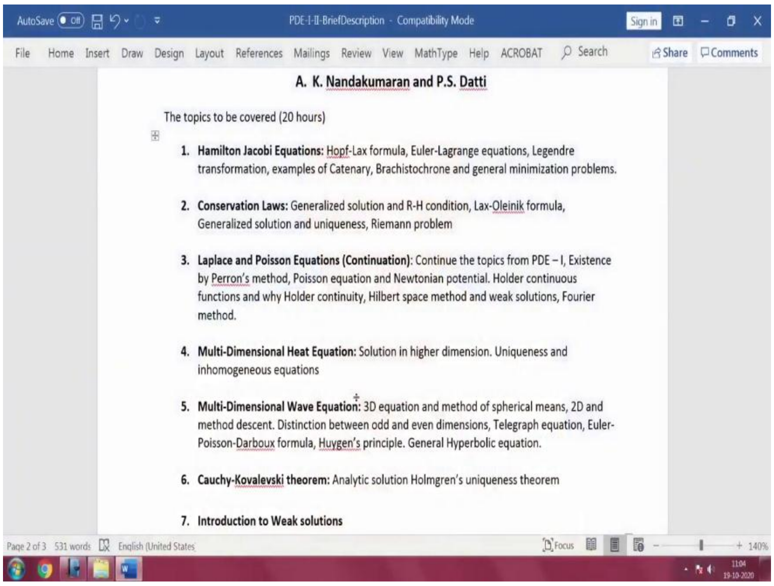

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6. **Cauchy-Kovalevski theorem:** Analytic solution Holmgren's uniqueness theorem
7. **Introduction to Weak solutions**

And distinction between odd and even dimensions, telegraph equation, Euler, Poisson, Darboux formula. We will also introduce what is called the Huygens's principle and maybe some concepts about the general hyperbolic equation. We tried to study one of the classical existence theory what is called the Cauchy Kovalevski theory, maybe we will introduce the concept of analytic functions and then analytic solutions with analytic coefficients Holmgren's uniqueness theorem probably more little about, if time permit is we will do the fundamental solutions of general differential operators.

After that maybe a quick discussion about the introduction to modern weak solutions and the whole PDE for the last 50 to 70 years is studied based on this modern concept which requires a larger amount of functional analysis so we will not be able to cover extensively about the weak solutions. What we will be doing is the, a brief introduction to motivate you to study PDE in a more general setup. So, this is the brief about our course, two courses the first course on PDE 1 and PDE 2.

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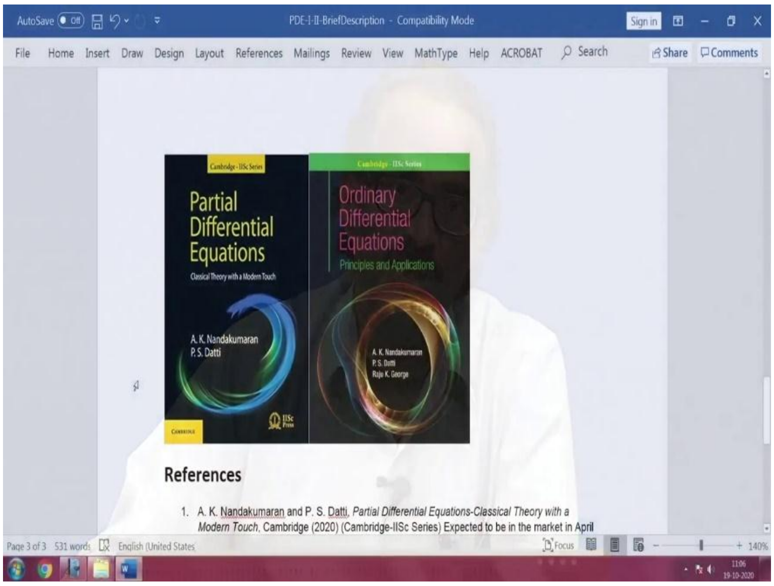

So, let me give you some references, so let me briefly tell you our aim is to essentially cover our recent book on partial differential equations Classical Theory with the Modern Touch. This is a company published under IISc press series. It is recently published in 2020. We are not sure we can cover the entire syllabus and their topics in partial differential equation but I am sure in these two courses we will try to finish some 70 to 80 percent of the material mentioned in that and that will prompt you to study all the material thing. And this is our previous course which we have given again is there as an NPTEL video course on Ordinary Differential Equations, this may be useful for some prerequisites, not all prerequisites there is a video course and there is also a book on ordinary.

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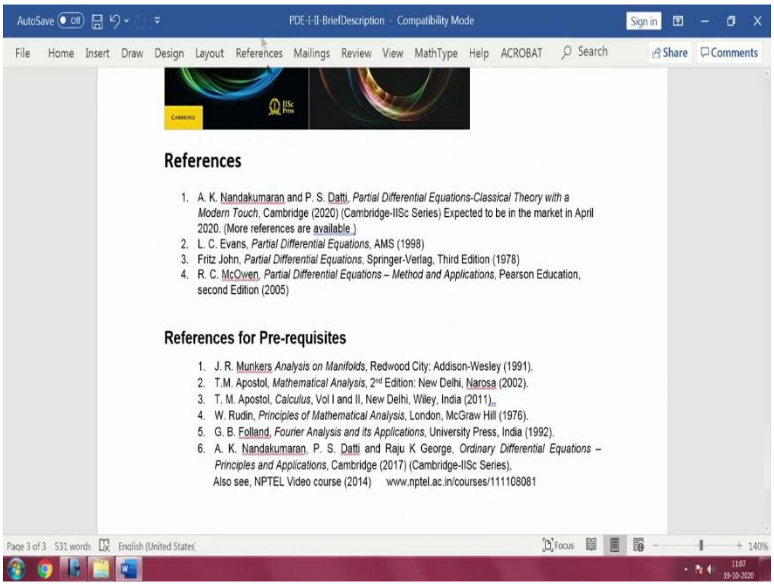

So, our main reference is the recently published book by myself, two of us who is giving this course and partial differential equations. And there are plenty of references given in our course in our book so you can refer. There are many books to refer, there are many good beautiful books so one of the few references I have given L C. Evans Partial Differential Equations, Fritz John these are all some classical references which you already know it probably and McOwen and there are pinch over and there are many books so you whichever you found is suitable you can cover that.

So, for the prerequisites as I said you will soon see probably in the third lecture or something, we will do the, some prerequisites in one or two hours that means from third to six half an hour lecture schedules. And what you have to understand is the, some multivariable calculus. And then you have to understand some Fourier analysis if necessary not too much and then you also have need to know about the surface measure and surface integrator.

This is a very important deep concepts because in a PDE you will be dealing with partial differential equations in a boundary domain and boundary value problems. So, how do you interpret the boundary values, how do you integrate a functions on the boundary, how do you introduce a appropriate measure to integrate. Because you need a measure to integrate all that basics you have to, definitely as I said it is not possible to cover all these things but we give some glimpses of what you require to study. And there are other books like Rudin you can see and then G. B. Folland and our last book is a reference if you want to know more about the ODE concepts.

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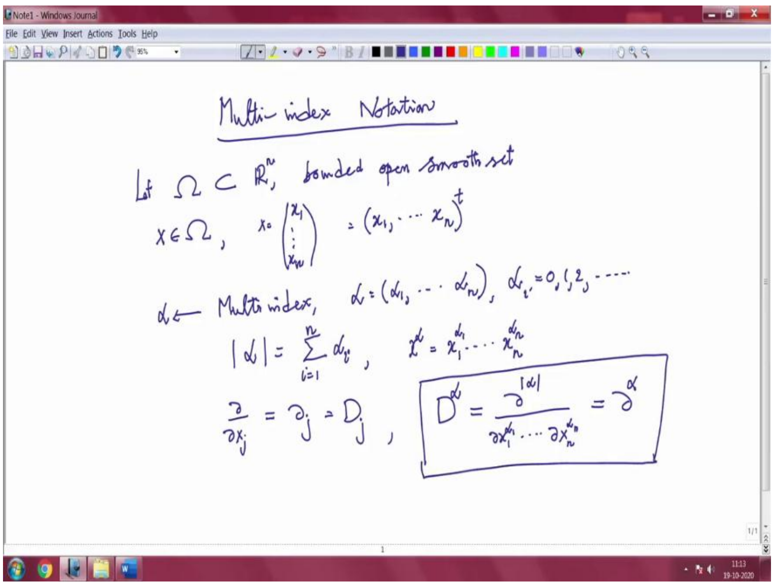

Now so let me tell you so little bit let me start with the course now so what I am going to do it will start with this little bit we will explain to you more. So let us put at one of the important thing what we require is a multi-index notation. So, let me give you multi-index notation. So, let me fix so this is normally this will be for throughout the course we follow omega subset of Rn a bounded open region, bounded open smooth set, you can talk about smoothness in different ways but let me not get into all that.

So, this is where so a typical point will be denoted by x is in omega so that means later we will not repeat this again we will write x equal to x 1 etcetera x n this is what we generally follow but we for the convenience we will also write here like this, this is for the convenience. But so, it is basically transpose but we will try to write that one so you have to understand when I write x either write it as a column vector it is considered as a column vector but we always write it as a for the convenience of writing.

So, this is how you put data. Then for a multi-index alpha, alpha is a multi-index so it is a multi-index. What is a multi-index? Where alpha consists of non-negative integers, alpha where alpha i is equal to 0 1 2 3 etcetera. This is why so this is basically in n power n union 0 you can use that one. And with that we will define mod alpha please keep this in mind there, summation so i equal to 1 to n alpha i. This is to make your partial differential operator in a very convenient way.

So, if you have an x, I will be, you can define x power alpha is nothing but x 1 power alpha 1 into etcetera x n power alpha n. So, you use this notation, you can also have your operator d by d xj this d we will put it in this notation we use various notation so you use this notation sometimes. We also use this notation Dj all it will be the same for us and the more important thing is that you can also put D power alpha. What is D power alpha? D power alpha is the derivative you take the derivative up to mod alpha times, so that is advantage of putting this multi-index notation.

So, d x 1 power alpha 1 etcetera d x n for alpha n. So, you can also put d j power whatever it is one so use these notations here. So d power alpha sometimes we also use it, so if you read different books people use different things, so you can actually use this notation. So, that is the about the multi-index notation. So, keep this in mind there is nothing complicated about it, this allows you to write my derivative in a very convenient way so we will do that one.

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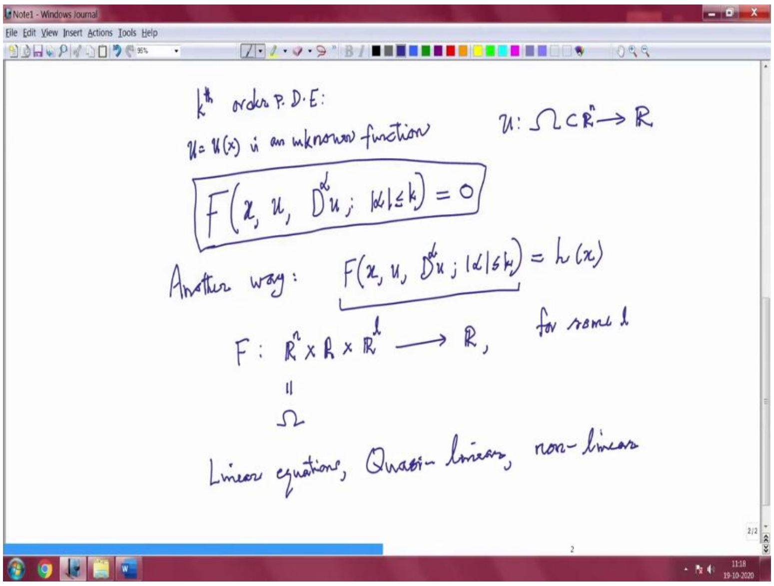

We go to the next page so let me write down so what is a kth order differential operator equation, kth order differential equation. So, it is a relation so there will be an unknown function u equal to u of x is an unknown function, unknown function. So, let me delete this unknown function so it is a relation, the relation is considered as a function which is its variable so you can write it D and you have the derivatives of order alpha and that order goes up to mod alpha less than or equal to k.

So, the up to kth order derivatives at least one kth order derivative appears then it will becomes a kth order PDE. This will be equal to 0. So, we can also write it in a slightly different way, same way there will be certain terms in F which will not have the unknown. So when you have some terms in the F in that expression and you take all these unknown, that is called the homogeneous term, non-homogeneous term, so collect all the non-homogeneous terms and put it on the right hand side.

So another way of writing, another way it is not exactly another way this is to explain a notion of linearity u u D power same thing modulus of alpha less than or equal to k. That is of the form some non-homogeneous term so you see. So, whenever you have an isolated term in this first expression, whenever you have an isolated term in this first expression and you take it and in this expression there are no non-homogeneous term which is taken to the right hand side.

So, F is some function you can see that, F is some function, it maps from some Rn or in some domain omega this not or this may be in some domain omega so you have. It can be in Rn also that is for the x variable. Then you have your u, u is an unknown function, so u mapping from some omega to Rn and it is a real valued map, so you have your u from omega subset of Rn to Rn.

This will be some R then you will have some alpha it depends on how many terms there coming, so it may be some Rl for some l, for some l depending on how many terms of, you will come to little more when you write it first order equations. So that is the way your first order equations are written, not first order equation, a general PDE of kth order is written like that. So, what I want to describe is what is a linear equation.

So quite often why I am explaining this is what we have noticed from our experience most of the students know how to recognize a linear equation and nonlinear equation but then students generally have difficulty of if you ask what is a linear PDE. I want to clarify not only how to recognize a linear PDE you what is this linearity to do with the linear algebra linearity, is it something different and then we will also explain quasi linear, quasi linear. After that you of course you will have non-linear in general all these equations are nonlinear so you have all that. So, we will go to the next page.

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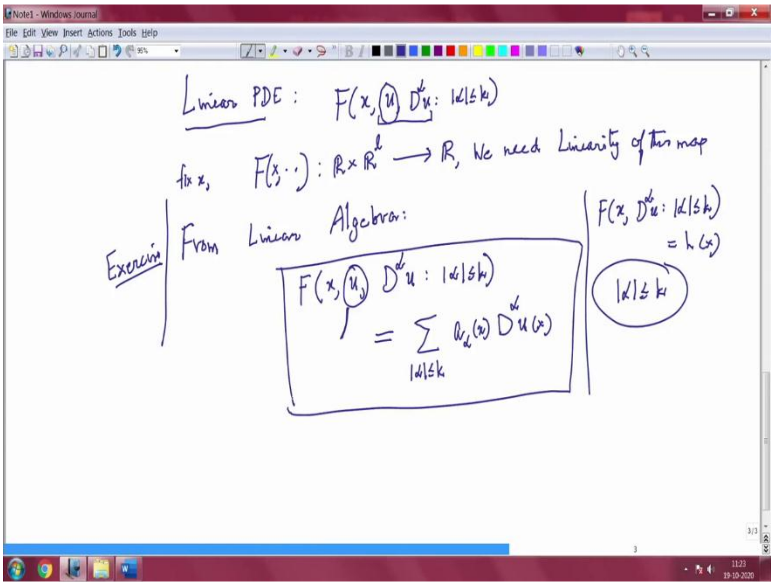

So linear PDE, linear PDE. So, we are looking at the linearity look at this function you have F x u D D power alpha of u with mod alpha less than or equal to k, you look at this thing. what we are asking for, for each x, so F can be viewed as a mapping, so F can be viewed as a mapping fix x then F can be viewed as a mapping for all these variables from R cross some Rl which I told to R.

Then we want the linearity of then, we need, we need linearity of this map. That is a one way there is another way of seeing it of course you are basically looking at the linearity of these two. And that you have a collective linearity. So, this, we need the linearity of this thing. Then from linearity, from linear algebra you can have a form whenever you have a map in a Rl then that can be identified as a dot product basically.

And when you have a linearity looking for each x when you that vector will change it that is all, because you can identified with, because whenever I have a linearity it has to be a Rn every element can be identified with a element from Rn. So, using that linear algebra those who are not familiar, leave this I leave with the as an exercise, those who are not familiar how to write this linearity then this exactly tells you, you can write this F of x u D power alpha of u less than or equal to k.

You can write this as a map from here sigma a alpha of x your D power alpha of u x mod alpha less than or equal to k. Yeah, let me tell you one clarification actually you do not have to include this u here or u here because when I take alpha equal to 0 0 0 etcetera you will get D power 0 of u that is nothing but u. So, it is up to you how you want to write it, so basically alpha equal to 0 it comes here so you have this here so you do not have to add even this term, so the general equation actually can be written in these form.

I only separated it, nothing wrong in the previous way of writing but this is the best way of writing, mod alpha less than or equal to k because when mod is equal to 0 you get D power 0 of u, now that is nothing but u and you get back this term. So, this is what you can write it so from linear algebra that is why, so you can recognize now the moment you are given an equation is given the all the unknown u and all the derivative of u got separated now.

So, there is no mixing no non-linear function of u nothing you will see it is all got separated out. So, I will stop here and then I will continue this linear equations and then I will explain to you a non-linear, a quasi-linear equations. After I explain to you the quasi-linear equations I will write more specifically about your linear equations and quasi linear equations in two variables how you can very precisely write it which most of you are probably familiar when a learn the beginning of your PDE course. So we will see you in the next class, thank you.