Introduction to Algebraic Geometry and Commutative Algebra
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Lecture 09
Universal property of polynomial algebra and examples

So, good afternoon, let us continue our lectures on Commutative Algebra and Algebraic Geometry. So, last lecture we have seen definition of AK algebra and when it is finitely generated etc. And one of the prime example we study in this course is the polynomial algebra over arbitrary commutative ring and even in that case we are we will fix base ring to either to a field or to ring of integers and then it will give us some analogs in algebraic geometry as well as algebraic number theory.

So, that is one of the reason that I am spending more time on the basics of commutative algebra. I still need little bit more but let us see how we can go to geometry and combine algebra with geometry.

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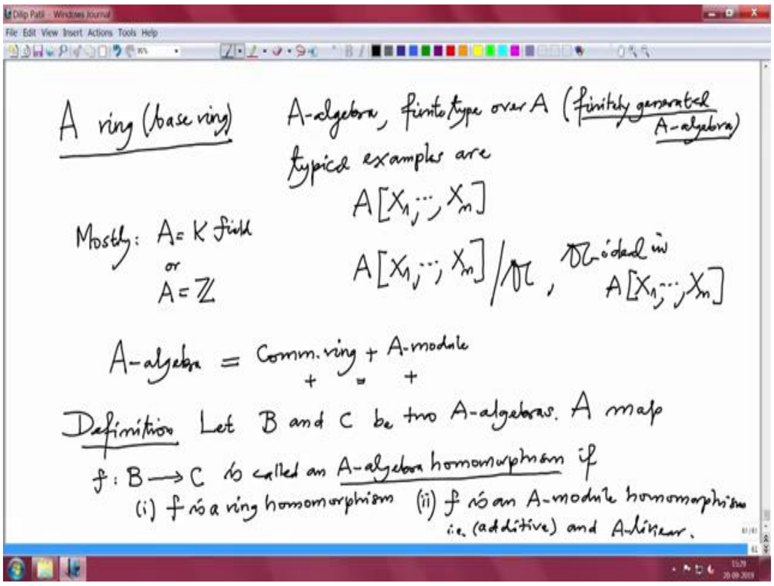

So, in the last lecture what we have done is for any commutative base ring, this is ring, base ring we have defined what is an A algebra, also we have defined when do you say it is finite type over A, finite type over A. This is same thing as also sometimes people use it finitely generated A algebra, finitely generated A algebra, and the typical example of A algebra of a finite type over a commutative ring is, so this is typical examples are polynomial ring over A in finitely many indeterminate X1 to Xn.

And also the quotients of this X1 to Xn module on ideal a where a is an ideal in the polynomial ring. So, these are the typical examples, so therefore, we have to study these rings little bit more for example what happens ideals there or prime ideals or maximal ideals and so on. And mostly in the beginning I will take mostly our base ring we will take A equal to a field or we will take A equal to Z ring of integers.

So, first of all we should say, so I want to also clarify what is the difference between finitely generated algebra and finitely generated module because what is roughly A algebra is what? A algebra by definition I will write ring, commutative ring which is also A module where the plus here equal to plus here and the scalar multiplication of this and multiplication of this ring they are compatible with each other, so that is an algebra.

Now, first of all what should be homomorphism between algebras? So, let us define that, so definition, so A as usual A is fix base ring, let B and C be two A algebras. A map from f, from B to C is called an A algebra homomorphism we need the respect the algebra structure if so how many condition. So, first of all 1, f should be ring homomorphism. So, again I want to remind here under ring homomorphism we are assuming that identity element multiplicative identity goes to multiplicative identity that is very important.

Second, f is an A module homomorphism and what is an A module homomorphism? It is additive, so that is additive that means with respect the addition but the addition is here also. So, it is these additivities checked here also and K linear. So, and respect the scalar multiplication, so K linear I would say together it is called K linear not K, A linear. That is an algebra homomorphism. But you know if I think it is better to write in a different way for.

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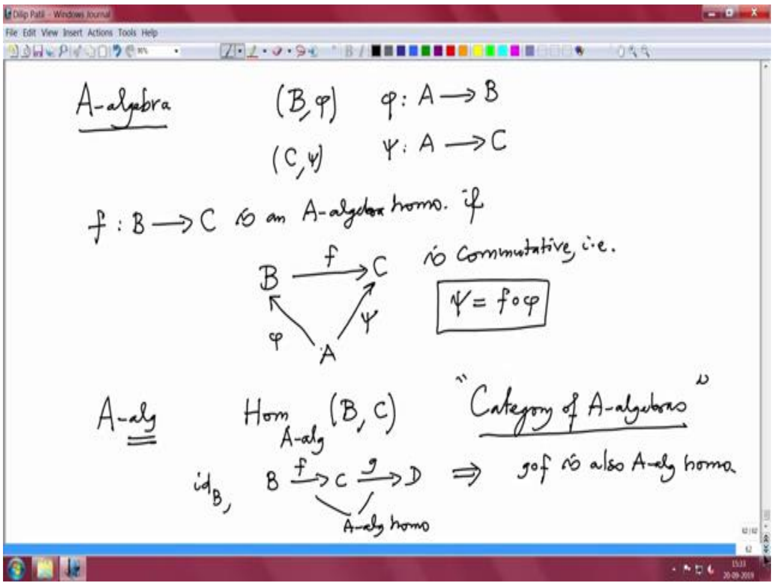

So, algebra I will think now I will not think ring and module and compatibility that is all built-in in the saying that B phi is an algebra, where phi is a structure homomorphism that is from A to B. And similarly, C, psi these are the two algebra given to A to C. So, when do you say map is a A algebra homomorphism f from B to C is an A algebra homomorphism if this diagram B C this is given f and A here and there is a structure homomorphism phi, there is a structure homomorphism here that is psi.

If this diagram is commutative that simply means if I go this way same thing as this, so psi is equals to f compose phi that means the diagram is commutative, so this is much simpler to abbreviate it and it is equivalent to that one that we can check easily. So, that is algebra homomorphism, so as usual the collection of A algebras I will denote by A-alg and homomorphism of one K algebra from one K algebra to the other K algebra is denoted by that set is denoted by Hom A-alg from B to C.

That is this are the set of all A algebra homomorphism from the A algebra B to A algebra C and with this collection it will form a category of A algebras. This again I will come back right now just this correction and this and what is obvious thing that we will need is composition of two algebra homomorphism is algebra homomorphism, identity map is an algebra homomorphism and so on, and we will keep adding more.

Right now identity and identity B and if you have f from B to C and G from C to D if these are the A algebra homomorphisms then G compose f is also an algebra homomorphism that you usually check it for every groups, rings and so on.

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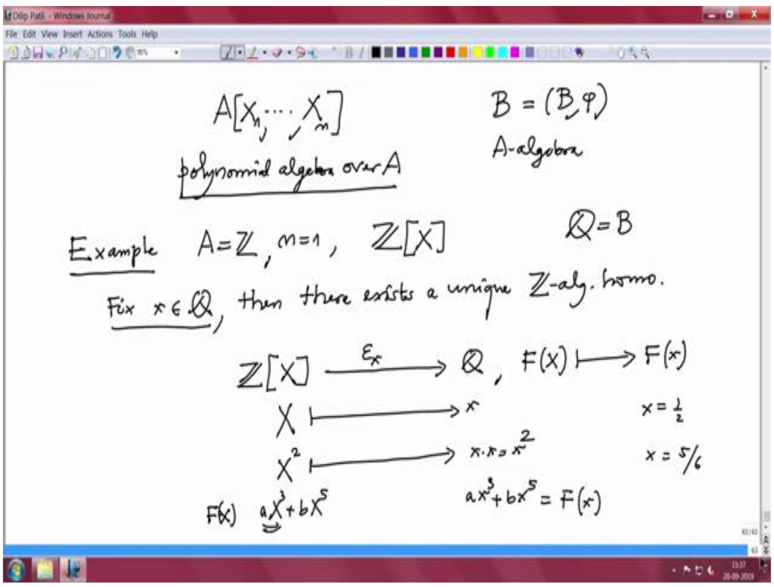

So, now at least I want to describe all A algebra homomorphism such that typical algebra I want to consider is A X1 to Xn this is my one A algebra and the other algebra is B let us say so this is A algebra. So, if you want you can think B, phi and I want to describe all A algebra homomorphism from this polynomial algebra and that is the reason why it is called polynomial algebra.

So, polynomial algebra over A this is also because elements are polynomials. Now, how do you construct. So, let me give first one example and then we will see how to write more general. So, example, let us take A equal to Z and only one variable n equal to 1. So, my algebra is ZX so that is Zx polynomial algebra over Z in one variable and B I am taking it Q this is my B and I want to write algebra, some algebra homomorphism.

So, let us take any rational number x small x, fix any rational number then there exist a unique Z algebra homomorphism from Zx to Q I will denote this by epsilon x which maps X to x. So, this is obvious because all that we have to check is this is a ring homomorphism and this also is a Z linear these are the two thing you have to check but well where will X where go, if you want it algebra homomorphism X where I have no choice it has to go to wherever X goes, wherever X goes time that.

So, this has to go x times small x square. Similarly, all powers will go to the powers to that small x and it should be Z linear. So, if I want somebody like this a X cube plus b X5 then first of all it is additive, so this will individually I can make and A has to go to A and because it is Z linear. So, this, this term will have to go to a X cube and this addition plus b x 5. So, therefore, this map is uniquely determined, this map is in fact any polynomial capital F, FX this will go to F evaluated at X this one is precisely if this is F x this is precisely capital F evaluated at X only wherever you find X your write small x.

So, that is that is why this map is evaluation map, this is evaluation at X. So, you could take X equal to half, x equal to 5 by 6 and so on. So, and all algebra homomorphisms has to be like this because X has to go somewhere and that you call it small x and then everybody is uniquely determined. So, this is called universal property of this polynomial algebra.

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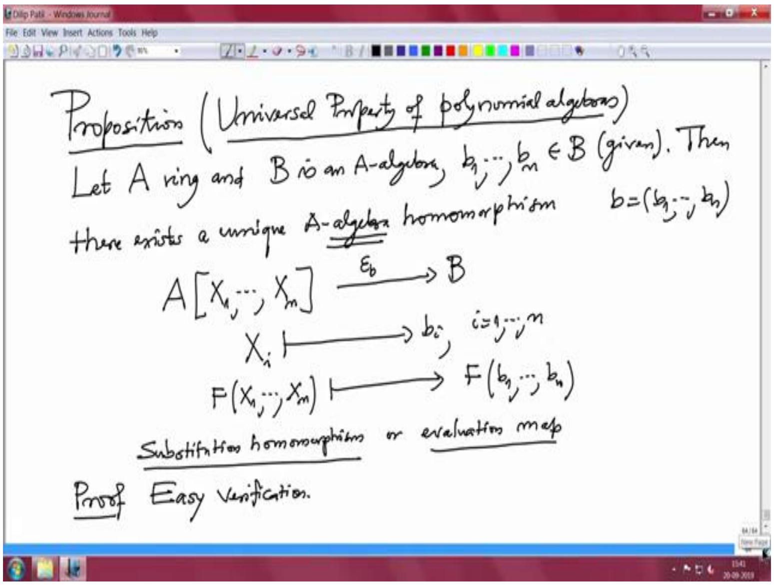

So, we will abbreviate in a, I will formerly recorded, this is a preposition, this is very, very important, it is used very often and mostly it is not said what we are using it but this is what is being use always. So, this is called universal property of polynomial algebras very very important. So, let A be our, general A be our ring and B is an A algebra and b1 to bn they are elements in B they are given elements in b, they are given then their exist a unique A algebra homomorphism.

From the polynomial algebra over A in n variables to B. So, this I will denote the explanation epsilon b, where b is tuple b1 to bn. What is the map? Xi should go to bi for every i 1 to n and then this is uniquely determined because where arbitrary polynomial will go if demand this? Then the arbitrary polynomial have to go to F and substitute for Xi, bi F of b1, etcetera bn.

So, substitution this is also called a substitution homomorphism, substitution homomorphism or also some people call it evaluation homomorphism. Evaluation map because here you are leading at b1 to bn and the proof does not need anything because it is an algebra we want an algebra homomorphism. So, it should respect addition, it should respect K linearity and also it should respect multiplication in the ring.

So, that determine where a polynomial goes. So, proof I will just say easy verification and you should individually you should do yourself homomorphism. So, what does this mean let me spell out what is this algebra homomorphism means I want to spell out clearly.

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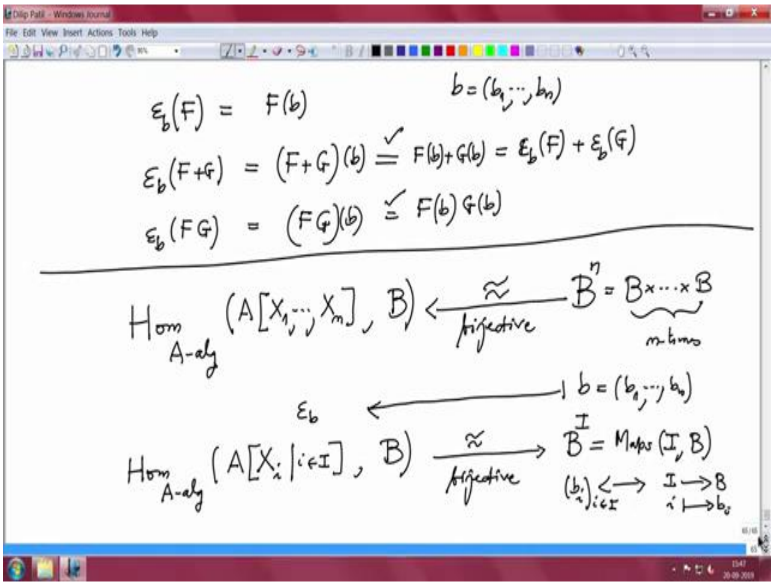

So, that means what? That means if I want to write see remember in the above notation epsilon b of f is not the small f, I should use capital F, F equal to F of b where b is the tuple b going to bn, so you have substituted. So, what is it mean by ring homomorphism? So, first of all epsilon b of F plus g capital G this should be on one end it should be F plus G evaluated at b but this also should be.

So, this should be, because it is a ring homomorphism it should be epsilon b evaluated at F, image of F under epsilon b plus epsilon b evaluated image of G but this is same as evaluation. F, capital F b plus capital G and b. So, this should be equal so that means this evaluating or substituting b is additive, another one is similarly, by the same explanation epsilon b of F time G equal to F of b times G of b.

These are on other hand it is I should write that first, so this is F G evaluated at b this should be same as F b G b this is very very important both these rules are very very important for evaluation of polynomials. So, and what did what does a, what does a above proposition describes? It describes the set of all A algebra homomorphism that we have denoted like this Hom A alge A X1 to Xn, comma B this set we have explained, we have described completely and to describe any homomorphism you need only a tuple of, n tuple of elements from B.

So, that means what? That means that is B power n, other set is B power n, this is B cross B cross B n times n times and what did we do? We gave a map from where to where? We gave a map from here to here, namely tuple b which is b 1 to bn which goes to epsilon b substitution homomorphism by B. So, we gave this map and uniqueness give this map is injective and (())(23:10) is also obvious because given any algebra homomorphism that is uniquely determined by the values of X1, X2, Xn so that I call it b1 b2 bn so this map is actually isomorphism.

This is an isomorphism bijective, there is no structural, I am only saying the map is bijective, so when time come we can identify these two sets. And there is nothing special about this finite n. So, let me write for arbitrary number of intuitive minutes so that is Hom A algebras from the polynomial ring now in many many variables index by i X i i in i this to sum algebra B A algebra B and where can I take it now here B power I and what is B power I? Think of B power I as maps from I to B these are maps from I to B.

So, each map will give you a tuple and each tuple will give you a map. So, either you can think of bi i in I or think it is a map from which map from I to B, i going to bi given any map from I to B you can get a tuple and given in tuple you can get a map. And better to think these maps. So, these are, these two sets are bijective. So, that is it, now I want to also. Next, observation I want to make.

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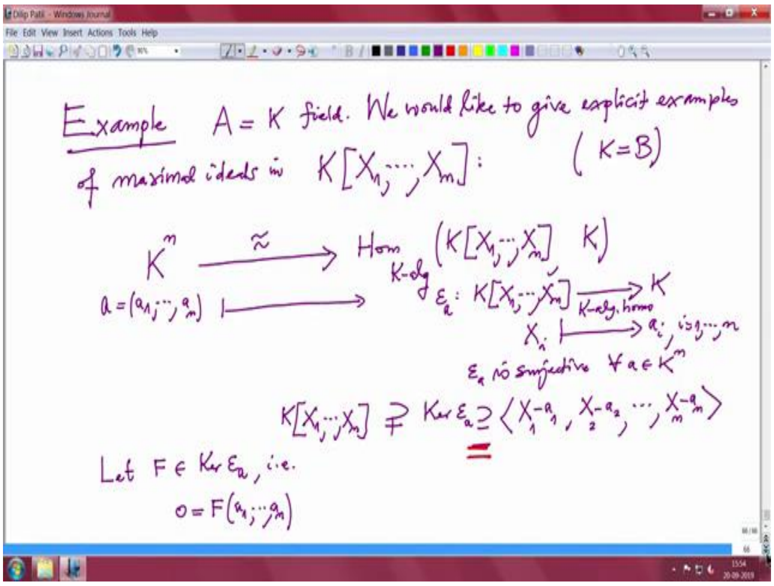

So, I want to use this observation to give you some examples maximal ideals for example in a polynomial ring. That is two lectures back we were discussing maximal ideals and we wanted to give more examples. So, how do we give more examples is this I will use this observation. So, this is example. So, now I will take my base ring to be a field K is a field and I want to give maximal ideals.

So, we would like to give explicit examples of maximal ideals where ideals in the polynomial ring. So, I fix I am going to, as you have notice this K power n, n tuples these are let me write them a equal to a1 to an and we have given a map from Hom A algebras, K algebra from the polynomial ring to where, in K. So, remember back of your mind that K equal to b in the above notation. And what is the bijection? Any element A, where does it go? Ist goes to epsilon a, epsilon a is a algebra homomorphism from K X1 to Xn to K.

Where X i are map to ai that was a map. So, given this A we have this epsilon a, epsilon a is a algebra homomorphism this is K algebra homomorphism and it is obviously surjective. Is it surjective, is it clear? Because 1 has to go to 1, 1 0 1 has to go to 1 and it is K linear because it is K algebras. So, this is this epsilon a is surjective map for every a in K power n, we know it is surjective.

So, what is a kernel let us compute the kernel. So, kernel of this I want to check what are the generators for the kernel this ideal, this is an ideal in the polynomial ring. So, this one I will list some elements which are obviously in the kernel. So, for example X1 minus A1 this polynomial this is free from all other variables, this is obviously in the kernel, because what we have to do to check in the kernel I have to check that when I substitute Xi is equal to Ai it becomes 0.

So, I have to substitute X 1 to a1, so it is 0 and remaining variables we do not have to, they do not appear, so nothing happens. Similarly, X2 minus A2 and so on. So, Xn minus an, all these linear polynomials they are in these kernel and therefore, the ideal generated by this polynomials it is the smallest ideal which contain this polynomial but these are also here therefore this inclusion is obvious.

So, kernel contains this ideal generated by this and this kernel because it is a algebra of homomorphism 1 has to go to 1. So, 1 cannot be in the kernel therefore, this cannot be unit ideal therefore this is a proper ideal of the polynomial algebra. And I want to prove now it is equal here I want to prove this is equal. This is equal I want to prove, so let us prove that so what do I have to proof? Every polynomial in the kernel is written as a combination of X1 minus a1, X2 minus a2 Xn minus an.

And coefficient should be in the polynomial ring, because we want it to be an ideal. So, take any polynomial in the kernel F. So, let F belong to the kernel, so that is F of when I substitute in F instead of variables these small ai a1, an equal to 0, because it is in the kernel. On the other hand this F is a function F is a continues function, it is a polynomial function therefore continues function.

I am offering one proof the pure algebra people mind find it little bit more difficult but it is obvious. So, think of it is a polynomial function and therefore it is a differentiable function and therefore, if you remember Taylor's theorem, what does it say? Taylor's expansion at, yes? So, this will be equal to what? So I want to expand, so what do I have to do, I want to expand not equal to this. So, this I want to expand, I want to write such a formula for arbitrary polynomial. So, let me give you the next page.

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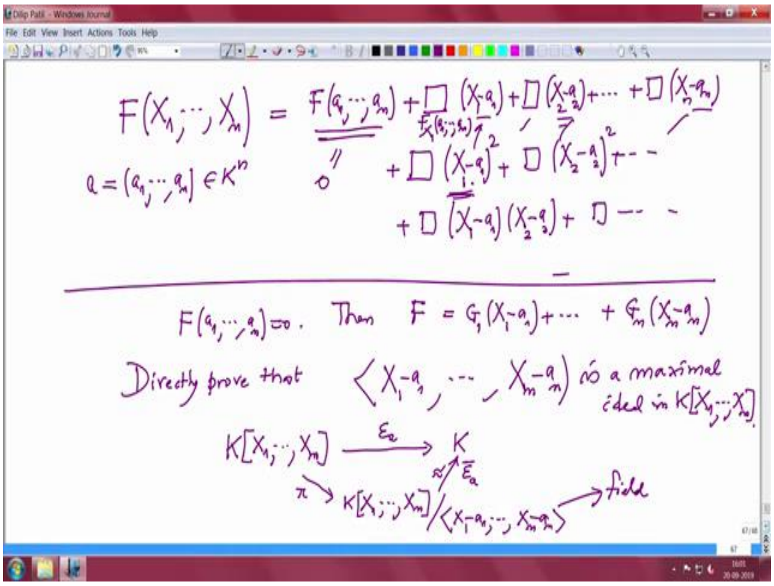

Given any polynomial F of X1 to Xn and a is any given point in K power n this equal to F evaluated a1 to an plus some coefficient X minus a1 some coefficient X minus a2 and so on plus some coefficient x minus an you can (())(33:29) what are the coefficient plus somebody now these are taken I have taken 1 at a time. Now, it will X minus a1 square, square terms and also the mixed terms plus somebody times X1 all this is X1.

This is X2, this is Xn, X2 minus a2 square and so on plus somebody X1 minus a1 X2 minus a2 two at a time and goes on like that. Now, this is a Taylor's expansion at a1 to an. So, now what are the coefficients here? This will be partial derivative of F with respective X1 evaluated at a1 this will be partial derivative. So, let me write this is F suffix X1 evaluated at a1 to an. Similarly, this similarly this and this and now in this if you (())(34:43) this term is given to be 0 for us.

So, where do F belongs to? Now, you can take out this, this is an X1 X2 Xn and this is this you can observe, this you can observe here because X1 minus a1 you can take out and remaining part you can observe in the coefficient similar this similarly so on. So, therefore, it is the equality is clear but his proof in all this formula but let me tell you also directly, so we want to check what? We want to check that, we want to check that if F of a1 to an is 0 then we have a expression like this F equal to some G1 times X1 minus a1 plus, plus, plus Gn Xn minus an.

This is what we want to prove, we can prove also directly this. So, or either this or directly prove this is equivalent to proving what? So, directly or directly proof that this ideal generated by X1 minus a1 Xn minus an is a maximal ideal in K X1 to Xn. Because if you prove this is a maximal ideal, this is a bigger ideal than that, and this is a proper ideal. So, they have no chance but this. So, that would prove the claim and how does one proof that this is a maximal ideal, how does one prove this is a maximal ideal?

You go (())(37:14) and check it is a field but you see we have this diagram K X1 to Xn to that K this is the substitution homomorphism epsilon a and then we want go (())(37:34) this maximal this ideal this is the residue map and this is the map induced by this in when we studied the residue class ring that is I denoted by bar this diagram is commutative but this map I have already noted this map is surjective therefore and we have (())(38:18) the kernel.

So, one of the theorem I proved one of the observation, one of the corollary I proved earlier that this has to be bijective, this has to be an isomorphism. So, therefore, not only get this quotient this residue class ring is a field. But this is isomorph is to K it is a specific field isomorph to K therefore this is maximal and therefore it proofs that. So, we get lots of examples of maximal ideals. So, what do we get let me summaries we get that all the points here they give you maximal ideals. So, let me note that we will use it for future.

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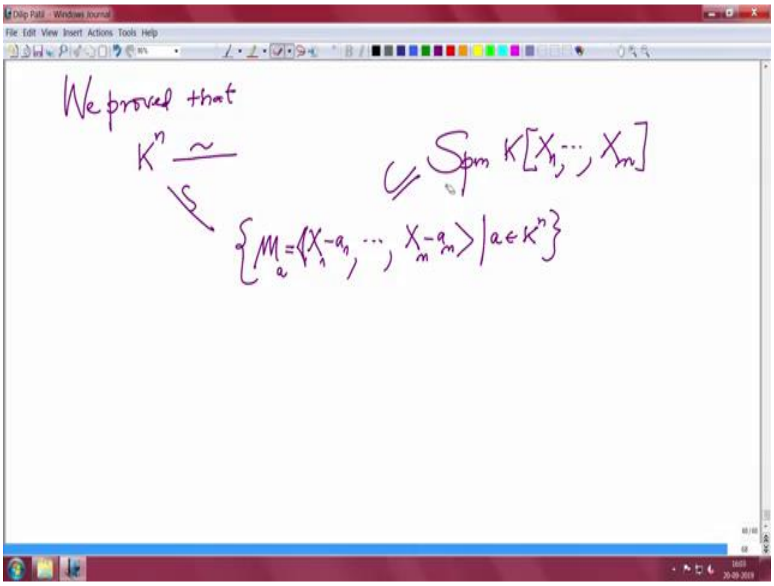

So, what we have proved we proved that Spm of K X1 to Xn this set of all maximal these are all maximal ideals and this K power n and K power n we have identified with ma this is generated by X1 minus a1 Xn minus an we just now check that this is maximal ideal and as a varies in K power n. So, let me clean up little bit maximal ideal ma this is X1 minus as a1, Xn minus an, this as a varies in K power n this set we have identified with this K power n and this is a subset here.

That is what we proved each one of them is maximal. So, we have ample number of maximal ideal but this may not be equal here I would like to be equal here but it may not be equal in general we will give example, so that this is not equal in general. So, I will continue this in a latter half, I will produce maximal ideal which is not of this type for some fields. So, after the break we shall meet to continue. Thank you.