Introduction to Algebraic Geometry and Commutative Algebra
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Lecture 60
Cohen Seidenberg Theorem

Welcome back to this second half of today's lecture. Now, I will prove to the next theorem of Cohen Seidenberg that is called. So, let us write it as a theorem now theorem, this is called going up there.

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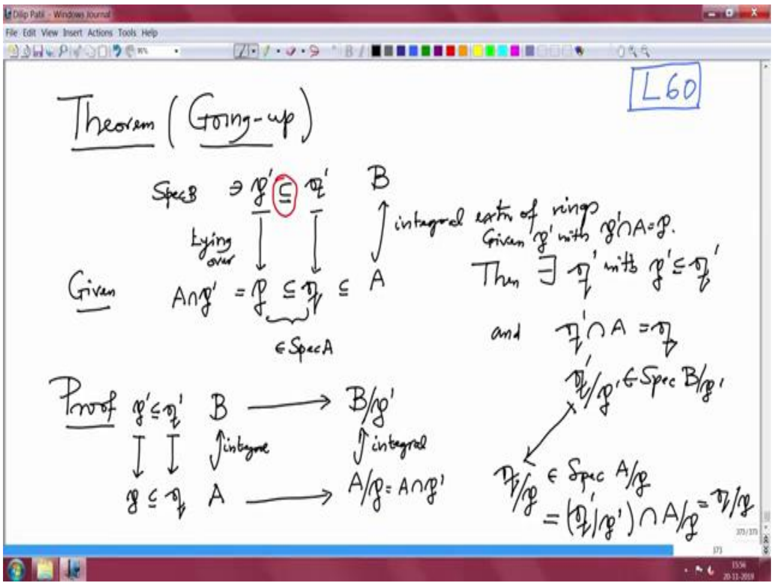

So, also it will explain the statement itself will explain why is it called Going up theorem and there should also be Going down theorem. So, we will come back first to the going up theorem. So, as usual we have B, this is integral extension of rings and given prime ideal where, p contained in q, these are the prime ideals in A both these are prime ideal in A and when is contained in other.

And we have seen that Lying over theorem, I can certainly find p prime in spec B such that this p prime is lying over this p that means this p is A intersection p prime, this is Lying over theorem. Now, we also know there is a p double prime or q prime. We also know there is a q prime in the spectrum again, which lies over this, but we do not know whether we can choose p prime is contained in q prime and the answer is yes.

We can choose q prime which lies over q and this contained here so, this is very important. So, that is precisely so then I will write the statement, then there exist q prime with q not q prime p prime, p prime containing q prime, and q prime intersection A is q. So, what is given? Given p prime with p prime lying over p we can find q prime which is lying over q and p prime is contained in q prime.

So, it is going up and we can keep gradually going as long as if you are given a chain of prime middle below in the ring and lying over theorem you choose one which is lying over p then by going up theorem you choose one lying over p that is why it is called Going up theorem.

Again the proof is very simple, proof. Now, earlier what we have done is we have localized now, we have to go module because we want somebody contained in this. So, B, A is here, this is integral and we are given p and given p prime. So, let us go mod p prime B mod p prime and this one will contain A by p now because p is A intersection p prime so, this is also inclusion and this is also integral, this we have noted earlier and now, this is a residue class map this is also residue class map.

Now, here this q by p is the prime ideal, this is in the spec of A by p because Spec of A by p precisely those prime ideal of A which will contain p, and q is one of them. And lying over theorem will tell there exist q prime ideal q prime in this ring by p prime in the spectrum of B by p prime.

So, there exists this which will lie over this so, that means this is lying over this, this mean this prime ideal is precisely this q prime by p prime intersected with A by p is equal to this one q by p. But what does this mean? There primary Lear mean now, when I pull it back to the Ring A Ring B, this p prime has to contain q prime.

So, p prime is contained in q prime and then this one is lying over p and then this one this way it is this will lie over q. So, we achieved this by going module so that that was more simpler than probably the Lying over theorem so, this is going up. So, this going up theorem will say that this will tell us I will just remark here.

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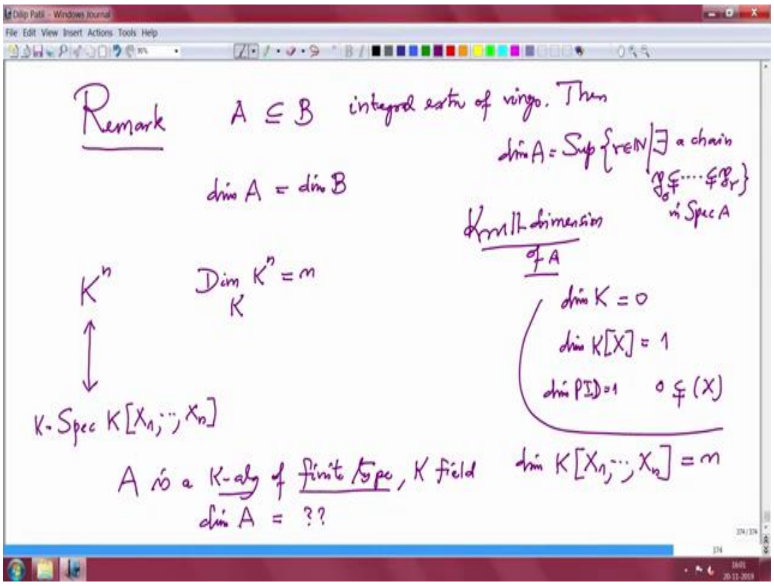

I do not have much time to elaborate on this. So, remarks says here that if A contained in B integral extension of rings then dimension of A will be equal to dimension of B where dimension of the ring A means supremum of r in N such that there exists a chain p 0 contained in not equal to P r.

Then the dimensions are equal because whenever you have a chain down, you can keep by going up theorem, we can extend chain to the ring b and whenever you chain up by contracting it, you will get a proper chain because no time there will be quality in between.

So, this is what the consequences of this, and fortunately, I cannot spend more time on this, this is called a Krull dimension of the ring A. Just to give you a flavour, what is the dimension of a field? Dimension of the field is 0 because there is only one prime ideal namely 0 and the moment you go on, then it will not be prime ideal, dimension of a polynomial X K polynomial X that is just 1 because 0 is one prime ideal.

Then next prime ideal is x is not equal and every nonzero prime ideal is maximal because this is a P I D therefore, the link cannot have more than two elements in A m. So, in particular dimension of a P I D is also 1. Now, dimension of the polynomial ring K X 1 to X n, one can prove this is n.

This will match with our usual earliest study that we have seen, if you look at the vector space K power n, this is n dimension of K power n and the k vector space is n. And this corresponds to the find space so this correspond in our situation this corresponds to the K spectrum of K X 1 to X n, so it matches with that.

And also to prove more generally we will be interested in proving if A is K algebra of finite type, then how do you write down what is the dimension? By the way here from this definition it is not even clear that this dimension is finite or not because it is supremum over the chains in spectrum.

Who knows if the chain may go on for longer and longer and we do not have any clue how does one prove the statement, but this is how do you compute for finite type algebra over a field, K is the field. This is interesting and this is very very important. In fact, it is very important that the algebraic geometry really starts from such things, we will match over this. Now, two more thing that I wanted to say, namely one what do you do with the finite type algebra over a field and how do you compute the dimension that is one.

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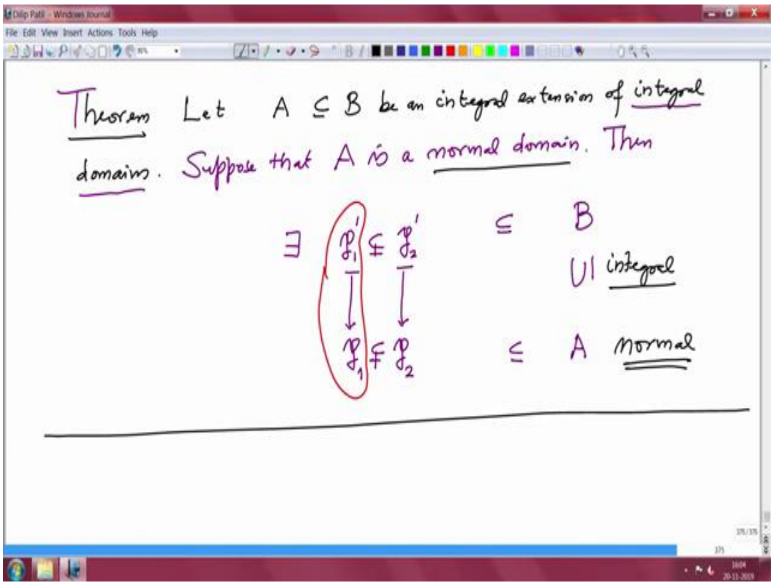

And secondly, as I mentioned going down going up and going down. So, let me state going down first, I will not prove it, but at least a statement we will do it. So, going down is let A containing B be an integral extension of integral domains. Now, we have to assume this is integral domain both are integral domain, this is added extra. So, I will just note what we have added extra assumptions. And also we need to assume suppose that A is normal domain that means the A is integrally close in its quotient field, then the going down holds then what is going down?

So, B is here, A is here. So, now given p 1 contained in not equal to p 2. Suppose I have given p 2 prime ideal in B, such that this lies over this then we can go down, then there exists p 1 prime, which is containing equal to this and then this lies over this, this is what the statement is. Then you can go down, but this has two extra assumptions than the going up, going up did not have any assumption. The only assumption we had for this is integral extension. This is integral that was given and then this is normal, both are domains and this one is normal, integrally closed in its quotients field, both the assumptions are very important.

And I will not be able to prove this because our time is coming to an end, but I will close this course with a very important statement, which will tell us how do you compute the dimension of a finite type algebra over a field and that one is very important theorem.

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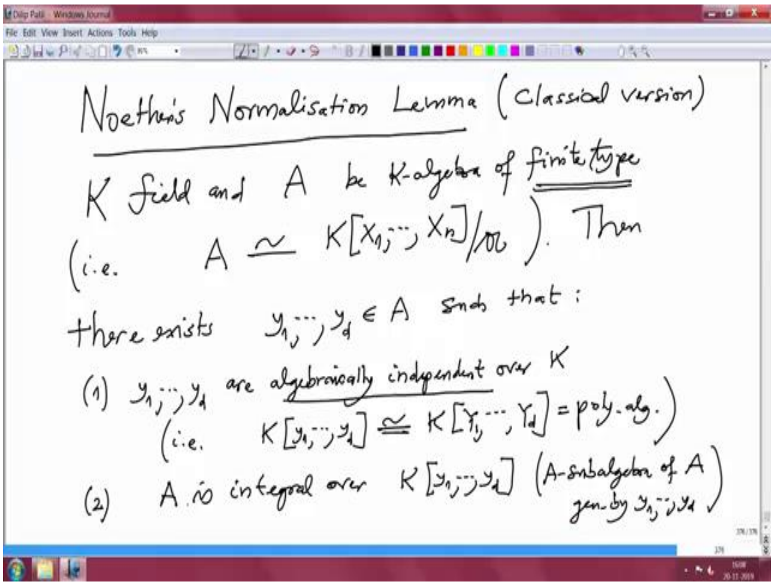

This one is called Noether's normalization Lemma, there are I am going to state only the classical version. So, K is a field, and A be a K algebra finite type that means what? So, that means that is A is a quotient of polynomial algebra in finitely many variables over the field K. Modular of ideal A that is typically an algebra finite type is algebra generated by finitely many elements.

It may not be finite K algebra, either module it may not be finitely generated, then there exists d elements y 1 to y d let me call this y 1 to y d in the ring A such that 1 y 1 to y d are algebraically independent over k. So, this means so that is they are, if I take the sub algebra of A generated by y 1 to y d, this one is isomorphic to the polynomial, this is a polynomial algebra.

That means they do not satisfy any nonzero polynomial with coefficients in K then such elements are called algebraically independent elements. And second, so here A is integral over the sub algebra generated by y 1 to y d. This is a sub of that because y 1 to y d this is A sub algebra of A generated by this y 1 to y d.

So, because of this one and two, we know that this is the integral so, whatever the dimension of A what we are looking for that will be dimension of the polynomial algebra. So, somehow if you would approve dimension of the polynomial algebra is d, the number of variables then our problem for the finite type algebra is also solved.

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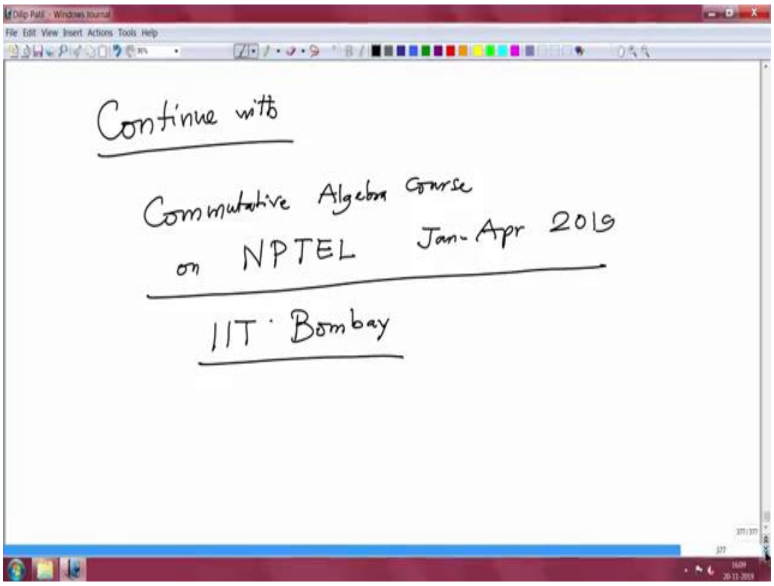

So, I will not have much time to prove this, but I would strongly recommend when you want to study further from this course, you should look at. So, continue with commutative algebra course on NPTEL, this was from January to April, this was recorded under IIT Bombay. This course in fact starts with Noether's normalization Lemma, I forgot to say that this was due to Noether, I said in the title, this is Noether's normalization Lemma, this is also abbreviated as NNL.

So, this course starts with that and further it goes to Commutative Algebra and more in to the homological methods in Commutative Algebra, which is also needed. So, with this I would like to start the course, and I must confess that there is not much enough algebraic geometry is done in this course, it only started because (())(20:19) was the cornerstone that we have proved it. And not only we have proved it, we have proved many equivalent versions.

So, this should give participants more courage to learn more abstract algebraic geometry from many books, which are written this way. And of course, Commutative Algebra will be very important and time to time it will need up gradation to study Algebraic geometry. With this I will stop and thank you very much.