Introduction to Algebraic Geometry and Commutative Algebra
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Lecture 58
Prime and Maximal ideals in integral extensions

Welcome to this second half of today's lecture, remember that we are studying Integral extensions. So, now I want to switch on to the study of prime ideals under Integral extensions. So, we are in the following situation.

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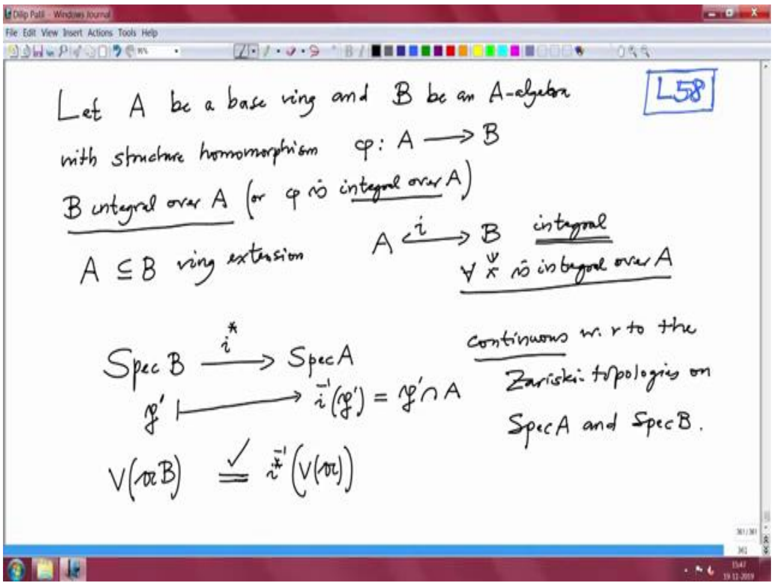

So let A be our base ring, and B be an algebra with structural homomorphism Phi from A to B. And then remember we have defined when is B integral over A that means every element of B satisfies a monic polynomial over A. But then this is also said sometimes also say or Phi integral homomorphism, Phi is integral over A, this he just Language and in this study we may also assume actually A is contained in B. So, this is by replacing A by its image in B, so this is a ring extension. That simply means that the inclusion ring homomorphism from A to B and then we say it is integral that means every element here x is integral over A very A that is our assumption.

Now, remember this any ring homomorphism will induce map on the spectrum. So, then we will have a map spec, if you call this as the inclusion map Spec of B to spec of A, this is a natural map, this remember we have called it i star, which is any primary ideal p prime Where did it go? Just pull it back contraction that is i inverse of p prime.

This is also denoted by p prime intersection A, this is again by use of notation, but this makes sense in this particular case. I would like to keep actually general Phi that does not matter. So, we want to study this map, now this map is a continuous map. This is a continuous map we have seen this is continuous with respect to the Zariski topologies on spec A and spec B.

So, that simply means that if I have so, one way to check continuous continuity is check that inverse image of a close set, it is a close set. And we know closer sets in a Zariski topology on the spectrum A if this notation V of an ideal A. And n what is the inverse image, so Iota star inverse of this set is precisely V of the extended ideal, V of A this is what we have checked earlier lectures so this contingencies we have checked.

Now, I am addressing to the question more. Now, we have this is so far for arbitrary ring extension or arbitrary homomorphism, now we have extra condition here integrals. So, what will these extra integralness will reflect some properties of these I star map? For example, is it subjective?

Can you describe the fibres and is it a close map or is it an open map and so on, such properties so, these are the topological properties. So, we are trying to convert the algebraic property of integral names into a topological property. So, this is what I want to study. So, for this study first I will do arbitrary ideals what happened to the arbitrary ideals? What can you say? Little more.

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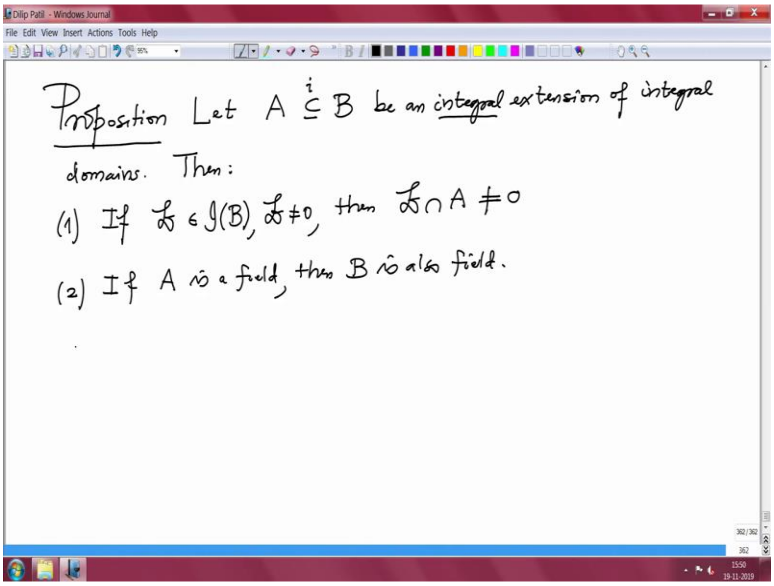

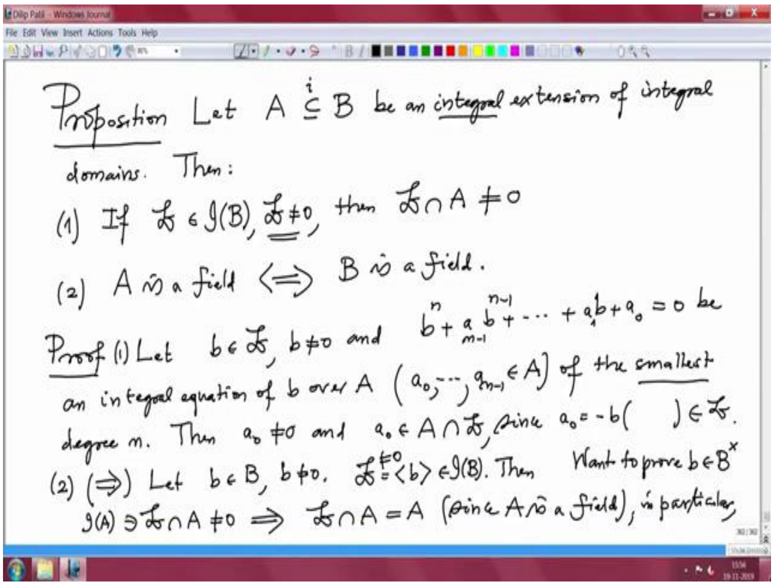

So, I want to prove a proposition first, so proposition, this proposition is very important in proving the properties of the spec map. So, proposition let A contained in B be an integral extension of integral domain that means, A and B are integral domains and this is an integration extension that means every element of B is integral over A means it satisfies a monic polynomial with coefficients in A.

Then one and so, and let me see if B is an ideal in B, use our notation I B, B is an ideal in B be nonzero, then the contraction of B to A, V intersection A. This is a contraction if you want disease Iota map, I map, but this is clear. So, this is also nonzero to if A is a field then B is also field. And 3 actually I should write in 2 only, if and only.

So, let us write that more General statement, A is a field if and only if B is a field alright. So, proof is very simple, I want to check that this ideal contraction of B is nonzero and what we have given ideal be is nonzero. So, start with a nonzero element there so, let b in the ideal be nonzero, there is such an element because the ideal B is given to be nonzero and now these extension is given to be integral that means, in particular this element b, which was in ideal B which is also an element in capital B.

It should satisfy an integral equation over A, and b power n plus a m minus 1, b power n minus 1 et cetra, b a 1 b plus a 0 equal to 0 be an integral equation of small b over A. So, that simply mean this A 0 to a n minus 1, there elements in the ring A. So, this is a monic polynomial with coefficients in A b satisfies that. Now what? Then it is very clear then what do you get?

I chose the smallest one, b is the integral equation of b over A of the smallest degree m. This makes sense, because among the equation then we choose the smallest degree equation, because n is varying in the natural numbers and so, there is one equations associated non empty (())(10:38) will be the smallest 10. Now, if it is a smallest degree, this equation I rewrite it.

So, I claim that then a naught has to be nonzero, because if a naught is 0, then this equation will have b in common and b is a nonzero element in integral domain so will cancel it and therefore I get a smaller degree integral equation for b so a 0 is nonzero and vary this a naught. And a naught belongs to A that is because it is a coefficient. On the other hand, it belongs to the ideal it is a multiple of b, therefore it is in the ideal b because b is in B, therefore, any multiple of b is also there.

So, this is in A intersection b, since a 0 equal to minus b times whatever remaining part, this is in b, so one is easy. So, these were the proof of one, two, I want to prove that a is the field if and only if b is the field. So first, I will prove this in a, if A is a field than upper bigger ring is also field provided it is integral.

Alright, so start with any element in the ring B, it is a domain so let b be in B I assume b is nonzero, and what we want to prove? We want to prove b belongs to B cross that means it is a unit in B. Then we would approve that every nonzero element is unique therefore, it is a field alright. So, how do you prove that?

Now, let us take the ideal B. B is an ideal generated by b. This is an ideal in B and it is a nonzero ideal. This is a nonzero ideal because nonzero element belongs there. And what does one say? One say that if I take the contraction of these ideal B to A then it becomes a nonzero ideal, then B intersection A is non 0 ideal, it is an ideal in a and it is a nonzero ideal, but A is a field given so, what are the ideals in the field? Either 0 or whole thing so that will imply B intersection A equal to A since A is a field, but in particular one will belong there. So, in particular one will belong to right side therefore, it belongs to the left side.

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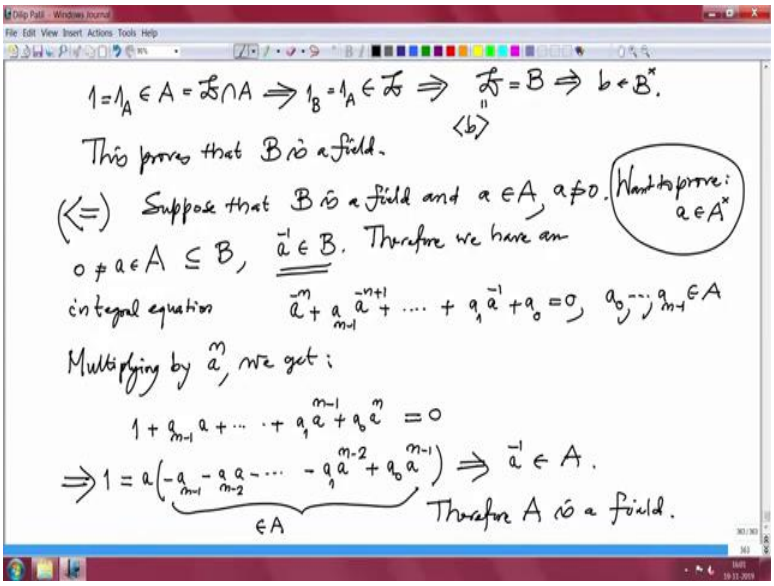

That means it belongs to 1, which is 1 A, which belongs to A which is B intersection A. So, 1 A belongs to the ideal B, but 1 A is the same thing as 1 B because A is the sub ring of B. So, that implies in particular that B is a unit ideal in capital B, but these B generated by b therefore, B has to be unit so B belongs to B cross.

So, we have proved that b is a field. So, this proves that B is a field. Now, we are doing this way, so suppose that B is a field and a arbitrary element in A, a is a nonzero element. And what is to prove? You want to prove this element A belongs to the units, this is what we want to prove.

Now, here is A is here, B bigger ring, A is an element here, A is a nonzero element here and this is a field. So, this is a nonzero element in the field B, so a inverse definitely belongs to B and I actually we want to prove that a inverse belongs to A, a inverse is an element in B but B is integral over A so, a inverse will satisfy a monic polynomial with coefficients in A.

So, therefore we have an integral equation so that is a power minus n plus a n minus 1 a power minus n plus 1 plus a 1 minus 1 plus a 0 equal to 0, where this a 0 to a n minus 1, they are elements in the ring A. These are the powers of minus 1 you see, minus 1 minus a power minus 1 power n minus 1 and so on, so we have such an integral equation.

Now, you can multiply this equation by a power n on both sides. So, what do we get, we get here multiplying by a power n we get. So, here it will be 1, here it will be plus a n minus 1 and a power n I multiplied so, it will be a plus 1 so on so powers will keep increasing now, a 1 power a power n minus 1 plus a 0 times a power n this is 0.

But now it is obvious what is the inverse of a. Then that implies minus, I keep 1 on this side and shift everybody the other side. So, 1 will be equal to and I take common a from there, a times minus a n minus 1 minus a n minus 2 a minus minus a 1 a power n minus 2 plus a 0 a power n minus 1. So, this is obviously, an element in the ring a and it is an inverse of A.

So, that implies a inverse belongs to the ring A. So, that proves that it is so therefore, we have proved that A is a field. So, this is very useful for studying prime ideals or maximal ideals because you want to check some ideal is prime then you have to check the residue class ring is a integral domain if you want to check some ideal is maximal that is equivalent to checking the residue class ring is a field. So, let us now go to the study of prime ideals and maximal ideals.

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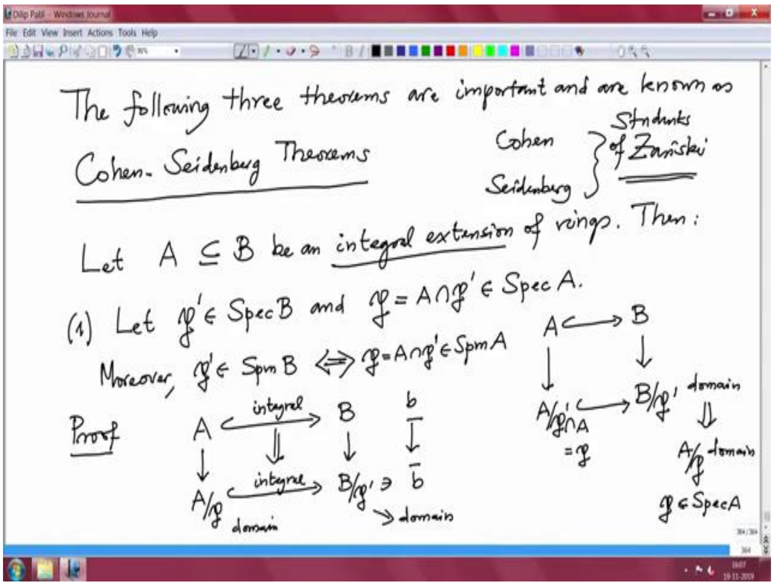

So, this is a very important theorem, so these theorems the following, we will prove following three assertion following three theorems. Maybe I will not be able to finish all the proofs today, but we will carry over to the next lecture. Are important and are known as Cohen Seidenberg theorems.

So, both these Cohen and Seidenberg they are students of Zariski, in fact, Cohen was the youngest student of that Zariski who unfortunately died very young. So, what are the theorems? So, our assumption is always let A contained in B be an integral extension of rings. Then, 1 let P prime be a prime ideal in B. And P be the contraction of P prime, A intersection P prime, this is obviously prime ideal that is no big deal because contraction of a primary ideal is always a primary ideal.

In fact, so this is because you see here we can draw a little diagram, A contained in B and then this P prime is given A, so B to B by P prime, this is an integral domain that is given to us and then when you contract what happens? When you contract A by P prime intersection A, this map induced by this will be also an inclusion map that is very simple if somebody goes to 0 here it will go to 0 here and you can check.

So, this inclusion map so this integral domain therefore, this will be integral domain so this is P so because this is a domain that will imply A by P is also domain and that will mean that the contraction ideal P is primarily linear. So, that is the proof of contraction of a primary ideal is a prime ideal, no big deal.

Moreover, P prime belongs to S p m B that is P prime is maximal, if and only if P which is contraction of p is a maximal ideal in A. This was one of the difficulty when we were studying arbitrary ring extensions, contraction of a maximal ideal may not be maximal, but it was true for finite type algebra over a field, but in integral extension also it is true that contraction of a maximal is maximal.

Proof is very simple so let me finish the proof itself. So, proof so, what happens again? So, again draw a diagram A to B this was a given integral extension then we have A by P here, this P is the contraction that to B by P prime, this is the residue maps and this is also inclusion map because we have gone mod kernel of this so this is also inclusion map.

And I claim this is also integral, this integral imply this residue map is also integral because if I take any element B bar here that will come in from some b here because these map is surjective and this B will satisfy integration over A, now take that integral equation and read mod, then you will get coefficients here read mod P, then you will get back so, this is also integral. Now, what is the advantage?

Now, this is a domain we know, therefore this is also domain we know and indeed, therefore if P were maximal, if P prime were maximal, that means this is a field. Then by the earlier lemma of preposition, then this will also be a field, conversely if this is a field, this is also a field, so I will write it down.

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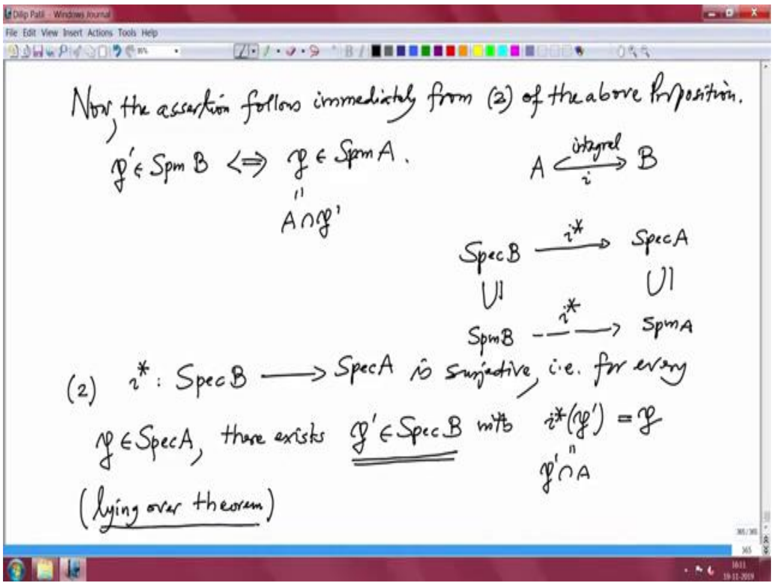

Now, the assertion follows immediately from two of the above preposition, namely P is maximal, P prime is maximal in B if and only if P is maximal in A, where P is the contraction alright. So, this means what? This means, the original map is you have A to B, this is our original ring homomorphism which we are assuming integral, then we have a map from spec B to spec A, this is i star, this is i and we have checked that, under this there is S pm B here, there is S pm A here, this is a subset here, this is subset here, under this this subset will be mapped onto this subset. So, that means i star will induce this map maximal spectrum on the maximum this is very useful.

Now, the next one, the second statement is this map i star this is from spec B to spe A is surjective. What does that mean? That is for every P in the spectrum of A there exists P prime in spec B with Iota star of P prime is P. So, this is by definition P prime intersection A. So, that means, there is at least one prime ideal P prime in B which will lie over P so, this is called Lying over theorem this theorem is called Lying over theorem. I will draw a diagram on the next page that will explain why is it called Lying over theorem. So, what is it? So, let us draw a diagram.

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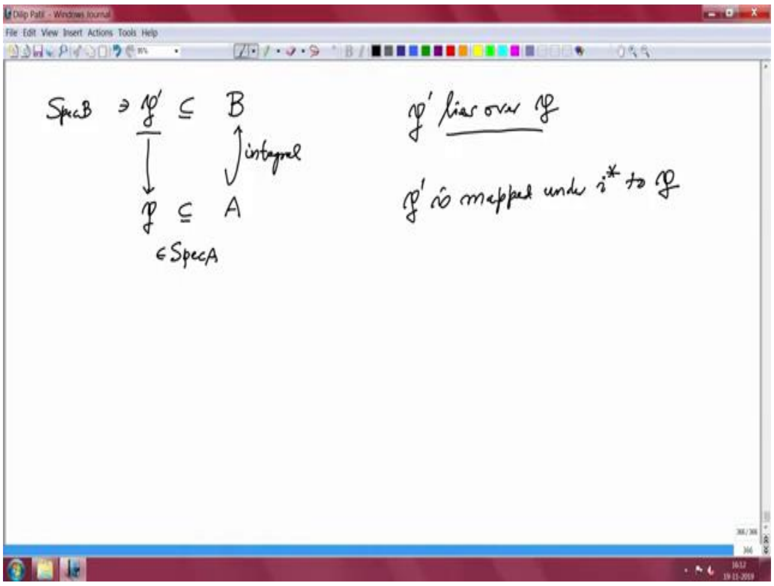

So, we have here B and here we have A and this is an integral extension and we have given P here, this is a prime ideal here, this is a prime ideal here spec A, then what do the theorem says, there exists P prime which is a prime ideal here, this is in spec B such that when I contracted, it goes to P. So, there exists a prime ideal above when I contracted to the ring A, it precisely P that is why he here we say the language is used is P prime lies over P. In this spec meme that mean P prime is mapped on to P under Iota star to P, this is the surjectivity.

So, I will stop here and we will prove the statement in the next lecture. Moreover also we will discuss that, how many can do, how many will lie over, how many prime ideals will lie over the same given prime ideal. That means, we will describe the fibre. So, what happened to the fibre over P, so that is the next goal.

And then this lying over theorems are these coincide lying over theorems are very important especially to prove what is dimension? How do you compare dimensions of the rings under integral extensions? When I say dimension means cool dimension, again this concept of dimension is very important, but I think we will not be able to touch in this course, but again I have proved starting with dimension I have proved many things in the next level committed to Algebra course which was in 2018 NPTEL course, which was from IIT Bombay. So, with this I will stop here and we will continue this study in the next lecture. Thank you.