Introduction Algebraic Geometry and Commutative Algebra
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Lecture 56
Elementwise Characterization of Integral Extensions

Welcome to this second half of today's lecture and in the last half I have left to implication to prove. We will finish those implications now.

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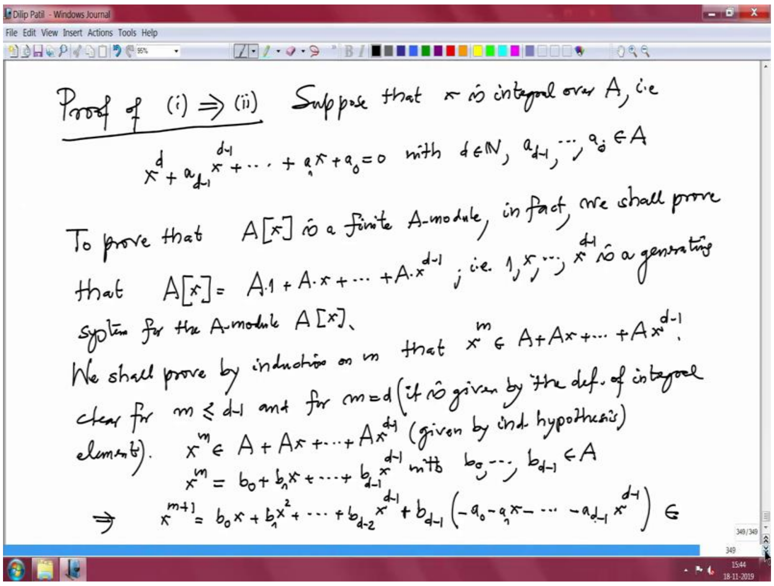

So, proof of 1 implies 2. This was left, so we will do that now. So, one was X is so given suppose that X is integral over A, that is it satisfies the monic polynomial with coefficient in A, that is X power d plus ad minus 1 X power d minus 1 etc etc plus a1X plus a0 is 0 with d is some integer, natural number and a d minus 1 etc etc a0 there elements in the ring A. This is we have given and what do we want to prove? So, I will write in it. To prove that, the sub algebra of b generated by X that is this is a finite A-module. That means we want to find the generating system for this module and that should be finite.

So, in fact we shall prove that this algebra A X generated by 1 2. So, it is generated by A times 1 plus A times 1 X plus plus plus A times X power d minus 1. That means these equality means every element here is a linear combination of 1X up to X power d minus 1. So, this means, so that is 1, X, X power d minus 1 is a generating system for the A module A X. Once you prove that then we are done. And how am I going to prove this? I am going to put these by induction.

So, we shall prove this, we shall prove by induction on m that X power m belongs to A plus AX etc etc plus AX power d minus 1. This is what induction on m. So, that this. So, for m equal to d this is m equal to were smaller than d minus, smaller equal to d minus 1 there is nothing to prove. So, clear for m less equal to d minus 1 and for m equal to d. It is given, this equation, I will just rearrange it. So, it is given by the definition of Integral element. So, now I have to prove it from inductive step only from I will assume for m and prove it for m plus 1.

So, what do you do is, you assume X power m belongs to A plus AX plus plus plus plus AX power d minus 1. So, what does that mean? So, this is given to us by induction hypothesis and what do you want to prove, the next one. But this is given means what? That means X power m equal do some b0 plus b1 X plus plus plus plus plus bd minus 1 X power d minus 1 with b0 etc etc bd minus 1 there elements in A.

Now, multiply this equation by X. So, multiplying by X we will get X power m plus 1 which is equal to b0X plus b1X square, you can go on till bd minus 2 X power d minus 1 and the next one is bd minus 1 and this d minus 1 multiplying X becomes Xd and from there, from this integral equation, I will substitute that, that would be minus a0 minus a1X etc minus ad minus 1 X power d minus 1 and now rewrite this equation from the combination between X, X1, X square and X power d minus 1. So, then these will belong to where we wanted.

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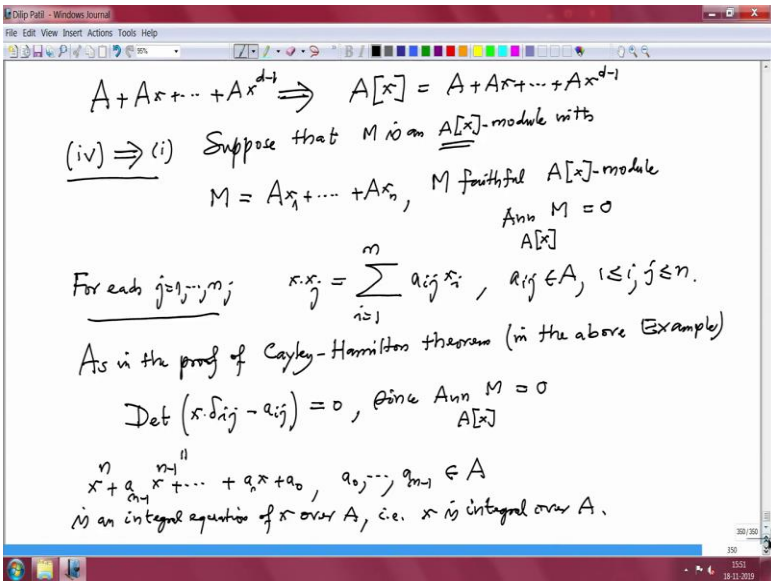

So, I will just write these will belong to A plus AX plus plus plus plus plus AX power d minus 1. So, that finishes the proof of induction inductive step and therefore we proved that every element, every generator of this A X is contained here. Therefore, this equality.

Because every element here is a polynomial index and we proved that all monomials belong to write hand side therefore finite elements also belongs to that side. So, that is all it.

So, that proves 1 implies 2 and the last one left was 4 implies 1 and what is given? So, 4 is given. Suppose that what was 4, 4 was there is a module m which is finitely generated over A. The module is finitely A module that is as a module over A it is finitely generated. So, we have elements X1 to Xn, so that it is module generated by X1 to Xn and M is faithful as A X module. So, suppose that M is an A X module with these two properties. It is finitely generated as the module over A and it is faithful as a A X module.

This means, Ann letter of M as a A X module is 0, at this two conditions you have given and from here we are looking for an integer equation for X. That means we are looking for the monic polynomial in X with coefficients in A which is 0 and now here is the point where we will imitate what example I did Cayley–Hamilton theorem we will emitted that.

So, it is a module over A X that means if I take X and there is a multiplication of X on M, because this is the module over X in particular X times any element of M makes sense it is a scalar multiplication by X. So, for each generator, these Xj j from 1 to n, when you multiply this X by Xj this is the scalar multiplication we have given and these are the elements in the module. This makes sense and this I can again write it as a A linear combination of X1 to Xn. So, we will write this as a sum, sum is ranging from 1 to n, aij Xi and this is true for every j from 1 to n and now imitate.

Therefore what did we get after this? So, you saw by as in the earlier proof, as in the proof of Cayley- Hamilton theorem in the above example we will get determinant of X times delta ij minus aij, this is a matrix, this multiplied by that xj, all of them, that will be 0. But now it is a faithful model so it cannot eliminate M. Therefore this has to be 0.

So, this if you like again, we have used Karima's rule and we have importantly we have used this assumptions. Since Ann letter of M as a AX module this is 0 and this is an element in the Ann letter. This analyte every j by Karima’s rule as we have seen in the earlier example and therefore this has to be 0 because this is a faithful module.

And but what is this? This is when you expand this determinant this is the degree n polynomial and monic polynomial and coefficient will come from some multiplication and additions from this aij and this aij are elements in the ring A. So, when we expand this, this will become X power n plus an minus 1 X power n minus 1, etc. plus a1X plus a0, this is 0 and we know that a0 to an minus 1 these are elements in the ring A. So, this one is an integral equation of X over A, over means the coefficients of A. So, that is X is integral over A. This is very economical and verifiable condition. Let me just show you again, look at this proposition.

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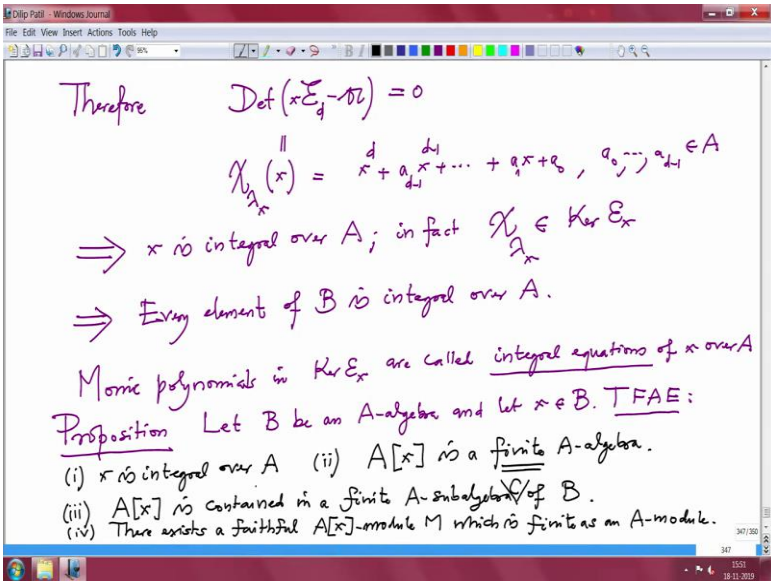

These are 4 parts. This is integral over A and our definition says that it satisfies the monic polynomial with coefficients in A and this one really verifiable whether A X is finite as an algebra or not, it is a finitely generated A module or not. And these 2 is very verifiable. So, I will show you in number of corollaries now how it is used this equivalence of integral extensions which is given by this propagation.

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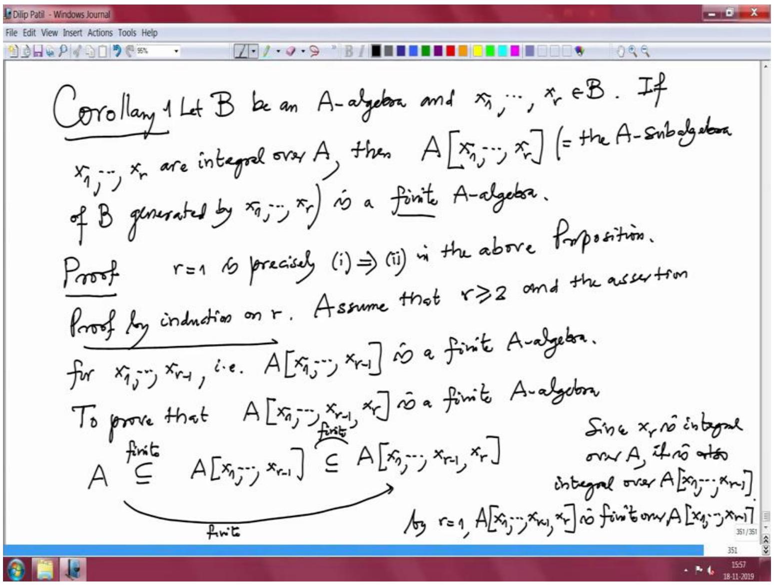

So, Corollary 1. So, as usual let B be an A- algebra, A is our fix base commutative ring and X1 to Xn or X1 to Xr are elements in the ring B. If X1 to Xr, all of them, are integral over A then the sub algebra of B generated by X1 to Xr, this is I will write into the bracket, this is the A sub algebra of B generated by X1 to Xr.

So, that means it is the smallest A sub algebra of B which contains all these elements X1 to Xr. Then this sub algebra is a finite A-algebra. That means there is algebra over A it is finitely generated as a module over A. That means finite A- algebra.

So proof, r equal to 1 is precisely the case proved in the proposition precisely 1 implies 2 in the proposition. So, therefore I will prove this assertion by induction on r, so proof by induction on r. Already r equal to 1 induction started so I only have to prove the state from r minus 1 to r. So, assume that r is bigger equal to 2 and assertion for elements X1 to Xr minus 1.

So, that means we are assuming that is we are assuming this sub algebra of B generated by X1 to Xr minus 1 is a finite A-algebra. That is what we are assuming by induction hypotheses and now to prove that AX1 to Xr minus 1 and Xr this is a finite A-algebra. But look here, we have A here and we have A X1 Xr minus 1 here. This is contained here.

This is given to be finite extension, finite A-algebra and then what we want to prove is the next one A X1, Xr minus 1 along with Xr we want to prove this is finite. This is what we want prove, this is finite. But we have given these elements Xr is integral over A therefore this will be integral over bigger ring since Xr is integral over A, it is also integral over A adjoint the sub algebra X1 to Xr minus 1.

But then by r equal to 1 case, this extension is finite, A X1 to Xr minus 1, Xr is finite over A X1 to Xr minus 1 and now it is if this is finite, this is finite, this will be finite. Because if this is you take the generating system for this and this module and take the generating system for this module as a module over A and you can multiply them and then you will get the generating system for the module, for the A module, this upper one.

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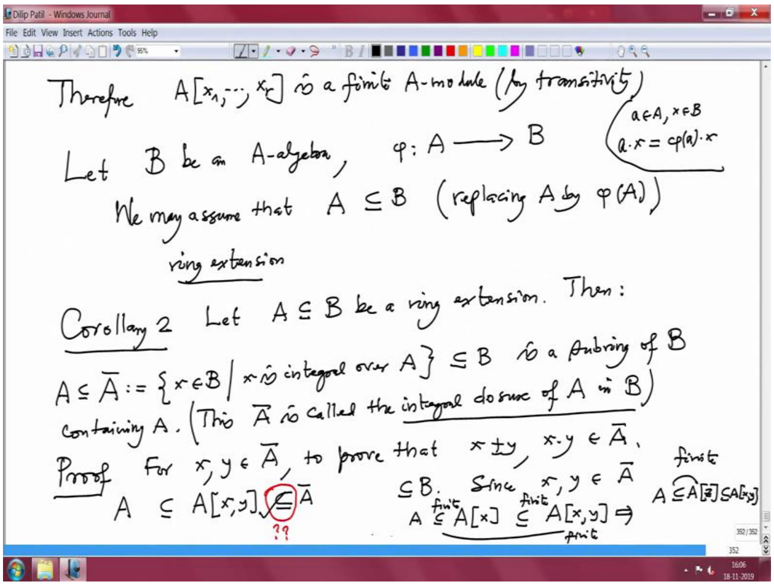

So, I will just say therefore A X1 to Xr is a finite A module by transitivity. This is very easy. So, now what we do is if I have, so if I have the next one I am preparing for the next one. So, let B be any algebra. That means we have a structure of phi from A to B and already you have been writing elements and multiplying by elements in B. That simply means we are when I write a times X, what are these mean? These means phi a times X for a in A or X in B this what do you have in writing.

So, this we may assume that this is actually a subring. So, we may assume in general, we may assume that A is a subring of B and the inclusion homomorphism is the structure homomorphism. This is if you like replacing A by phi of A and remember when we say subring that identity of the ring and identity of the subring they are same.

So, next corollary. Now, so this is also called the ring extension. That means a ring A, this ring B and this ring is got extended. So, the rings structure is same, the multiplication and addition in the ring A is a restriction of addition and multiplication in the ring B. Now, corollary 2.

So, let A containing B be a ring extension. Then A bar, A bar is precisely by definition all those elements of the upper ring B, so X in B such that X is integral over B integral over A contains A and this is containing B by definition and this is sub ring of B containing A. This A bar is called the integral closure of A in B. This I should write in the bracket.

What do we need to prove that it is a sub ring. That means we need to prove that, so first of all these inclusion is obvious because if you have element in A how do we write down integer equation for A. That is very simple. It just satisfy the equation A minus A, the coefficients are in the ring A. So, this is obvious and this is also I have given by definition. Now, to prove that it is a sub ring, I have to prove that. So, proof for X, Y in the A bar, to prove that X plus minus Y and X dot Y they also belong to A bar.

Once you prove that, it is a sub ring is clear because 1 is there, 1 of these are in 1 of B is same because A is a subring of B and so on. So, it is enough to prove this. Now, from here what do we know that X satisfy a monic equation, Y satisfy a monic equation over A and we want to check whether X plus Y, X minus Y, X times Y they also satisfy a monic equation with coefficients in A.

So, now if you try to find out by using the monic equation that X satisfies and that Y satisfies that will be very complicated. So, what will come in handy is the proof of equivalence of 1 and 2 in the above proposition. So, what do you have to prove, that so look at this, so A is here and X and Y are elements in A bar. So, I consider the sub algebra generated by X and Y. This is a sub algebra of B generated by X and Y.

And what do you want to prove? I want to prove that this is also containing in A bar. So, I will write in r red, this is what the proof we are looking for. That means what, we want to prove that every element of this is also integer over B, in particular the X plus Y, X minus Y and X Y. But what do I know? Since X and Y both are in A bar we know that these A X over A, this is finite.

This is finite and A X, Y is here and like in the above corollary if Y is integral over A Y is integral over this bigger ring. So, this is finite and this is also finite, both are integral. Therefore this will be finite. Once you know that then if you take any Z so that will imply A is containing A Z is containing A X, Y then this is also finite.

Because if you like you use 3 or equivalence of third one that some element is integral if the sub algebra generated by that is containing the bigger subring, which is also finite. So, that proves this inclusion. So, this inclusion is proved. So, therefore we proved the corollary 2.

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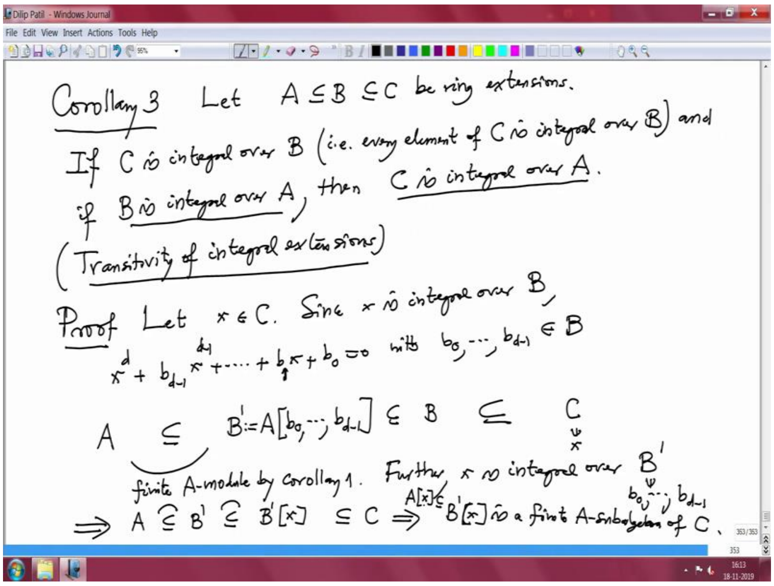

The next corollary, corollary 3. Again, what is a set up? Let we have a ring extension like this A containing B and B is contained in C, be a ring extensions. If C is integral over B so when do I say extension is integral that means every element of C is integral over B so I will write here one that is every element of C is integral over and if B is integral over A again the same that means every element B is integral over A then C is integral over A. This should be called as transitivity of integral extensions.

Proof, what do I need to prove? I need to prove that every element of C is integral over A. That is this. So, start with an element in C. Let X belongs to C and what are we looking for? We want to show that these X satisfies the monic polynomial coefficient in A. So, that is what we want to show. And what do we know? We have given the C is integral over B. So, that means since X is integral over B, it will satisfy the monic polynomial with coefficients in B.

So, we can write like that, X power d plus bd minus 1 X power d minus 1 plus plus plus plus plus b1X plus b0 equals 0 with b0 to bd minus 1 they are belonging to B. But we want the coefficients to be in A that is our aim. But now what do I do, A is here, C is here, and this is X was an element there and B was here and all these b1 to bdr in B. So, I can consider sub algebra of B generated by this.

So, that is B prime, let us call it B prime. This is the sub algebra of B generated over A by b0 to bd minus1. It is clear from our notation that this is a sub algebra of B over A generated by b0 to bd minus 1 and this is obviously A is a subring here. So, we are in this situation and what do we know B is integral over A. That is also given to us. That means every element of B is integral over A that means all these bi, b0 to bd minus 1, they are all integer over A and therefore this will be a finite A module by corollary 2. So, this one the finite A–module by corollary 1 or 2? 1.

Now, what do you want to prove? Now, if this X, now the integral equation of X has coefficients here b0 to bd minus 1, so they are coefficients in B prime because I put them in B prime. Further, X is integral over B prime because I already know this equation and B since b0 to bd minus 1 there elements in B and this equation is an integral equation of X over B prime. So, that means what? So, that means A is here containing B prime is here containing now take the sub algebra of B as of C generated over B prime by X.

This is containing C. This is finite because it is integral. This is finite already we noted and therefore that implies B prime X is a finite A sub algebra C. So, therefore I also found A X and A X is contained here. So, therefore by condition 3. So, I will write next.

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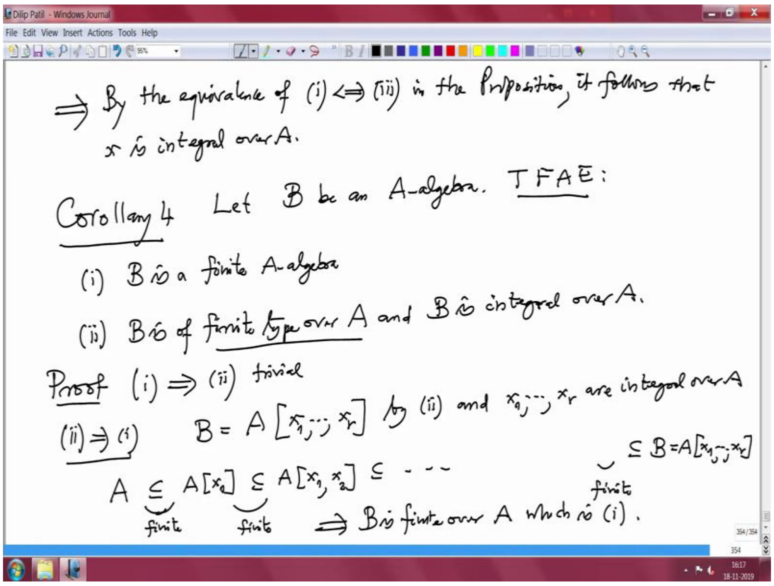

So, therefore by the implication, by the equivalence of 1 and 3 in the proposition it follows that X is integral over A and that is what we wanted to prove. So, that finishes the proof of the corollary 3. Now, one more corollary. Corollary 4, now the situation is let B be an A-algebra, A is our fixed base commutative ring, then the following are equivalent. One, this B is a finite A algebra that means B as a module over A finitely generated and two, B is of finite type over A and B is integral over A.

Proof, 1 implies 2 is trivial because if some algebra is finite as a module then it is finite type over A and finite then it is integral over A that we have seen in the proposition if you like equivalent of 1 and 3. Conversely, to prove that 2 implies 1, what is given, we have given that B finite over A. So, B is given to be and an algebra over A generated by finitely many elements X1 to Xr by 2.

And these all elements are integral over A, and X1 to Xr are integral over A. Now, from there it follows induction. So, A is here containing A X1 here, containing A X1, X2 here, contains so on contained B here, which is A X1 to Xr. At each stage it is finite. This is finite because X1 is integral over A. X2 is integral over A therefore it is integral over A, therefore this is finite. So, similarly so on.

So, all each stages they are finite extensions, finite algebras, that means they are the module over there is finite. So, if you have a tower of ring extensions where each stage is finite over the earlier one then the total B will be finite over A. So, that imply B is finite over here. So, which is 1. So, that proves the corollary.

And then next time I will give a number of examples. We will deal with examples and also I want to then eventually study the spectrums. What is the properties? When do you say extension is integral, what happens to the spectrum? So, that is we will do it in the next lecture. Thank you very much. We will continue integral extensions in the next lecture.