Introduction to Algebraic Geometry and Commutative Algebra
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Lecture 55
Integral Extensions

Welcome to this lectures on Introduction to Algebraic Geometry and Commutative Algebra. So have been studying the spectrum of a commutative ring and some of its topological properties. Today I will start a new topic called Integral Extensions, this is the beginning it will be algebraic and eventually I will connect to the spectrum of the rings. Those who like algebra they will like this topic very much.

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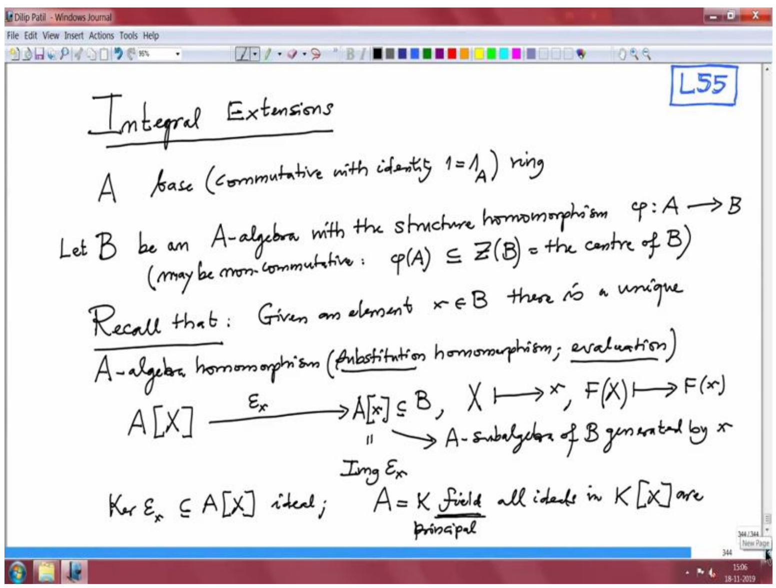

So let us start, so this is integral extensions, so to setup the notation as usual I will say that A is our base ring and just to remember it is commutative always with identity and that identity easily denoted by 1, if there is no confusion if there is a chance of confusion I will denote it by 1A. Ring and B be an A algebra, so that is our setup and to be very specific I will write here with the structure homomorphism phi from A to B and I will just say that we will allow A B to B may be non-commutative.

So I will just mention here may be non-commutative, and in that case we remember that the structure of homomorphism has the property that phi of A is contained in the center of B this is the center of B.

This will happen not so often because very often we will have the ring also be commutative. So, recall that this we have seen earlier also so I will just say recall that given an element X in B there exist we have there is a unique A algebra homomorphism from AX to B which maps X to small x. And this I am denoting by Ex, so this is called substitution homomorphism, some people also call it evaluation.

That is because where will be arbitrary polynomial will go? Arbitrary polynomial with coefficients in A this will go to F evaluated at X this make sense. We have seen earlier also the image of this map is precisely the sub algebra of B A sub algebra of B generated by X and that we are denoting by this. This is A sub algebra of B generated by that element X.

That we have seen earlier also, and now so we will look at kernel of this because image we know this is precisely the image of this Ex, what about kernel? Kernel of this no this is an ideal. This is contained in AX and this is an ideal that is because this epsilon x is ring morphism this is an ideal. Unfortunately, we do not know exactly what is the structure of the ideals in general if you have a general commutative ring and a polynomial ring over that in one variable, we do not know the structure of ideal there.

However, we knew this when A is actually a field K then we know all ideals in KX are principle and in fact generated by unique polynomial if you choose to be a monic polynomial.

And that is where this field play a very important role, but we unfortunately we are dealing with arbitrary commutative ring so do not have this privilege at our hand. So we have to deal with that.

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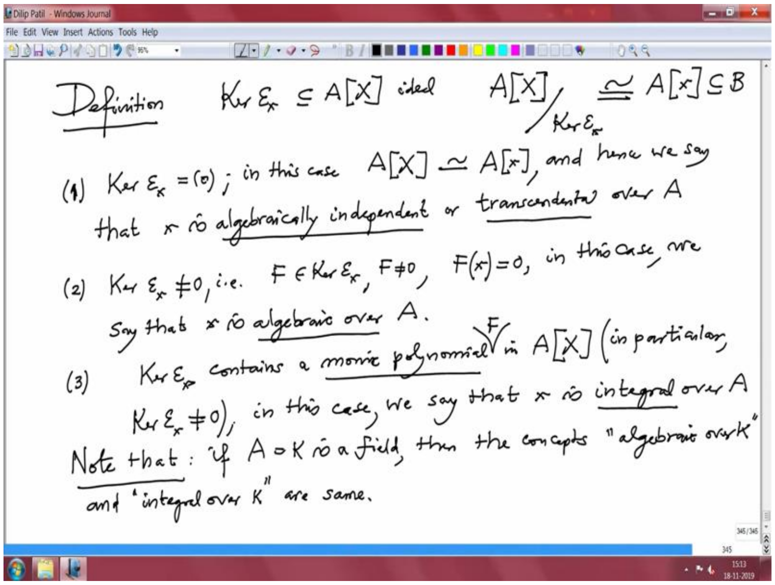

So now let me define, so definition, I use the above notation so what can happen we are denoting this kernel of Ex. This is an ideal in AX and we also know AX therefore, by isomorphism theorem this Ax modular this kernel of Ex this is isomorphic to the image which is an ace of algebra of B generated by X.

So what can happen? There are three cases can happen, one is kernel of this is 0, 0 ideal. In this case, AX the polynomial algebra is a isomorphic too the A sub algebra of B generated by that small x but, in this case and hence we say that small x is algebraically independent or also some people say or transcendental over the ring A.

So, in second case kernel of epsilon x is non zero that is f belongs to the kernel of Ex and f is non zero and because it is in the kernel X, kernel of Ex we know that f of f evaluated at x is 0. In this case we say that X is algebraic over K algebraic over A, that means it satisfy only a non zero polynomial.

But obviously how do we find a polynomial actually that is a big task. So third possibility is this kernel of Ex contains a monic polynomial in AX. So, in particular in particular a kernel is non zero kernel is Ex is non zero and this is a string assumption monic polynomial, here we did not say f is monic f is just a non-zero polynomial and it contains non-zero polynomial here, it contains a monic polynomial.

In this case, so let me give a name to that monic polynomial f, in this case we say that X is integral over A. So, this is very important so know we note that we have so note that if the base ring A is a field is a field, then the concept algebraic over K and integral over K these are same, because any non-zero polynomial over a field we can make monic by multiplying by the inverse of m of leading coefficients which is possible because we have a field therefore all non-zero elements of that are equivalent, therefore you can do that.

But if the base ring is not a field then both these concepts are not equivalent and one more thing I will mention here normally when one studies what is called field theory or Galva theory one assumes the field the base ring is a field and also the algebra is also field. But here algebra is arbitrary algebra here we are allowing even non commutative, even I should mention that it is worthwhile to study field theory where only the base ring is a field. The upper ring one should allow arbitrary commutative ring if not non-commutative. Non-commutative is also useful sometimes on especially when you are dealing with endomorphism rings of modules.

So now, I want to before I go on to study properties of integral elements and more about them or how to find out some elementies integral or not that is also not so easy or for example sum of two elements sum of two integral elements is integral or not product whether do you form a sub ring all this questions we will answer one by one.

But just to connect it with the earlier study what one had done, especially linear algebra I would like to write this example.

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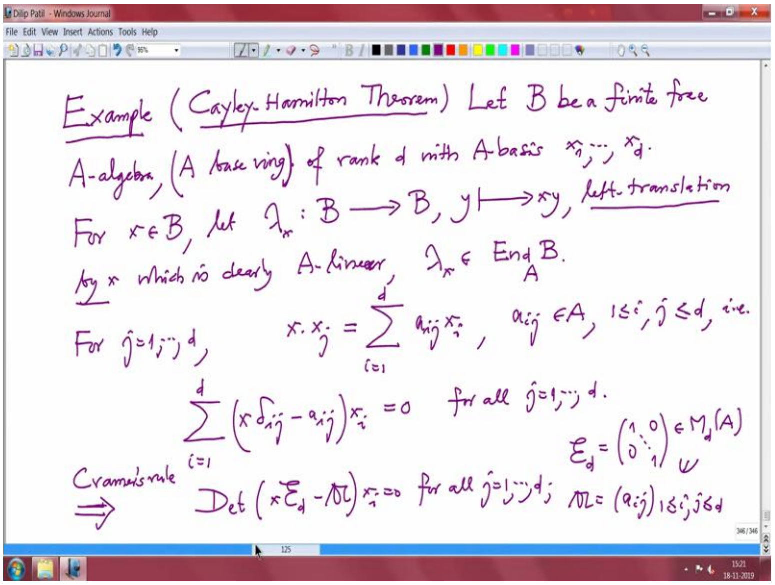

Example, this is basically what we have studied in linear algebra Cayley Hamilton theorem and this I will write down in a general setup, so again let B be a finite free A algebra and A is arbitrary A is base ring commutative always. I will not write that, so what are the admin finite means it is finite as a module over A and free means free other module over A normally all these adjective when we adjective we use it for algebra that means that as a module. If you say finite type, then it is finite as an algebra finitely generated as an algebra.

So, this I have been saying throughout the course sometimes reputation is also useful. So, I would put this in a base ring as a bracket a finite free algebra of rank D with basis is with a basis is X1 to Xt. As I said this B is a finitely generated free module of rank D and this is basis. Let me also say remind you that not all algebras are finite not all algebras may be free they may not be free.

So this is our given thing and for arbitrary element X in B, let lambda X this is from B to B map left multiplication by X on B. So that is any Y here in B is map to YX, XY. This is left translation by X.

Which is clearly A-linear, which is clearly A-linear, that means what this lambda X is an endomorphism of the free module B over A so this. Now, we have more than summation on B is more it is an algebra, that means it is a ring also, so you can multiply by elements of the ring B and get another element of the ring B. So for j equal to 1 to d x times xj, this is also an element in

B, so and B this basis so this element we write it in the form A-linear combination of xi's, i is from 1 to d again.

And where this aij's they are elements in the ring A 1 less equal to i, less equal to. And this aij's are defined for, j less equal to d. So therefore, we can when you rewrite this equation when you shift to this side and use chronical symbol. So therefore, what do we get? So I will just say that is summation I is from 1 to d and this term I want to take it here and as serve here. So, that is simple x delta ij minus aij and whole thing multiplied by xi this is 0 for all j is from 1 to d.

But immediately when you remember your linear algebra this means, so that will imply by Cramer's rule, I will write here Cramer's rule. What will it imply? Determinant of x small x times this big Ed minus a times xi equal to 0 for all j's from 1 to d, and where is Ed? It would bring Ed is an identity matrix 1 1 1 1 1 1 d cross d, so this is in MdA and this matrix A is this aij matrix 1 less equal to I small j less equal to d, this is also matrix in with entries in d.

Remember, we cannot use this m the matrices with coefficients in a, they are more difficult to handle when a is not a field. For example, to check the invertibility etc. is more difficult to handle. So, this determinant varies this element, so determinant is an element when we expand it, it is an element, it is a when you expand this is your monic polynomial in x of degree d. And remaining coefficient which will come from multiplications and additions from this matrix aij which is clearly an elements are in A.

Therefore, this determinant is a matrix. This determinant is a polynomial in small x with coefficients in the ring A and that polynomial kills all these xi's, but B is generated as a module by x1 to xd and identity element of B is also linear combination of this x1 to xd. So, if this determinant kills all the xi's it will contain it will kill one also in particular what we will know is. So this I will write on the next slide.

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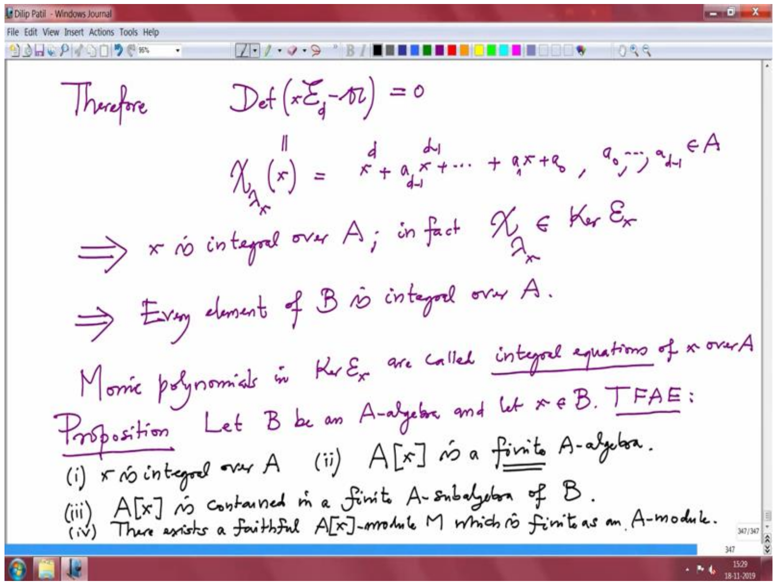

So, therefore determinant x times Ed minus a, this is 0, but what is this determinant? This determinant is precisely the characteristic polynomial of lambda x evaluated at x. So and what is this? This is monic polynomial in x of degree d, so x power d plus some coefficients x power d plus ad minus 1 x power d minus 1 etc. plus plus plus plus a1 x plus a0 where a0 ad minus 1 are elements in the ring A. So we got a monic polynomial which is 0.

So therefore, we know that so this implies that x is integral over A, because it satisfy monic polynomial. In fact, what we proved is, in fact this characteristic polynomial of lambda x we proved this is in the Kernel of evaluation map or substitution homomorphism. So, we proved that in every element we started with every element, so that means we proved that every element of B is integral over A, this what we proved.

So, our problem is this was a very natural which we inherited from linear algebra but, we want to find a good criterion which is verifiable economically to check that when the element are integral. How do you checks of elements are integral? So, that will be in a next proposition but also I want to introduce a language the polynomials the non-zero or so monic polynomials in a kernel of Ex are called integral equations of x over A.

And we want to know when can we find or when can we see that this kernel contains at least one integral equation and how do we find the integral equation. The above process shows under the very strong assumption that the algebra is finite and free over the ring A. And we do not even know whether this kernel is a prime ideal maximal ideal all this information we do not know in case of a field we can get a better information because we are dealing with a polynomial ring in one variable over a field which we know it is a pid and we also know more knowledge about prime ideals maximal ideals, they are generated by the prime polynomials and so on.

So, the next proposition is very important that we that will give us a criterion to check when some element is integral or not. So, this is a proposition, so let B be an A algebra, now we do not have any assumption finite, free etc. And let x be an element in b, then the following are equivalent. This is our standard short form we have been using it all the time. So, whatever the equivalent conditions, one of them has to be integral.

So, one x is integral over A, two the sub algebra of B generated by x over A, A sub algebra of B generated by x is a finite A algebra. That means, it is generated by finitely element finitely many element as a module over A. There we use the term finite. Third one, the sub algebra generated by x, A sub algebra of B generated by x is contained in a finite A sub algebra of B. This see note the difference between second and third condition, here second say that it is a finite A algebra.

This one says it is contained in a finite sub algebra of B. So, fourth one, there is there exists a faithful module over the sub algebra generated by x, A x of module A x module faithful A x module M which is finite as an A module. So, again let us look at the meaning properly. So there exist a module over this sub algebra and faithful means it is an i later of that is 0, I will write it on the next page, and which is when you restrict these module structure of M to A, that is finite as a module over A. So let me write it faithful means.

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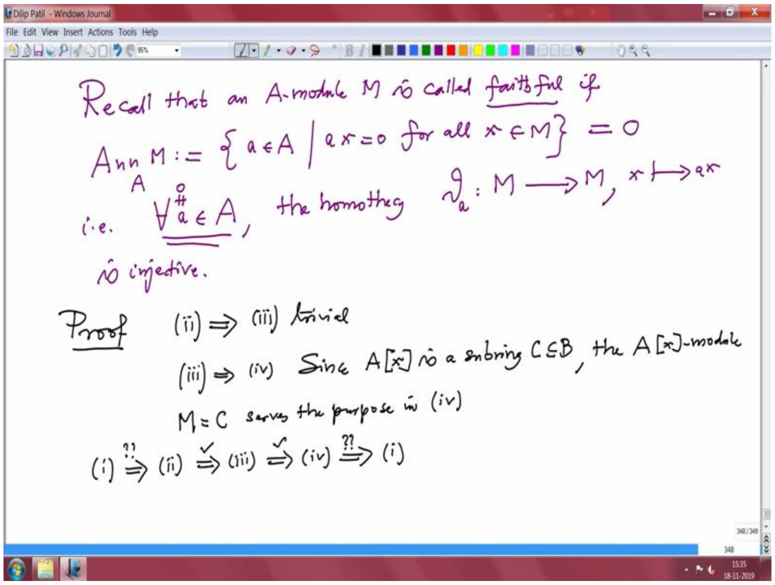

So, we will prove this preposition soon. So recall that an A module in general I am writing in general A module M is called faithful, if an I later of M over A which is by definition all those elements A in the ring, such that A times x is 0 for all x in M, then you call that is (())(31:44) the module M. Every element is unrelated so every all the elements in M are unrelated, so this is if this is 0 you call it faithful if this is 0.

So in other words, that is, for each A in the ring A the homothesy theta A which is a map from M to M defined by any x going to x this is obviously A-linear. This map is injective. This happens for every A, that means there is nobody here other than 0, for every A not equal to 0, I should have written, so I will write here not equal to 0. Then, such a module is called faithful.

Now we come back to the proof, I will write down the easy part and then the difficult part we will do it after the break. So, for example, two implies three, as I mentioned earlier this is trivial. Two implies three is trivial and three implies four. Three say that, so let me just go and check three say that this Ax is so first of all two implies three is trivial, because if this is a finite the two if this is a finite A algebra, I would take this we want it contained in a finite A sub algebra of B but I will take that itself.

So two implies three is trivial, and three implies four I want a faithful module M which is finite A module, but three says that this is contained in a finite sub algebra of B. So now, I will take that sub algebra which is finite, finite over A. So I will take that M equal to that sub algebra and it is a finite over A and it is faithful because one belong there so nobody else will kill everybody in M.

So, I would say here for this proof, take since Ax is a subring of B, Ax the Ax module. So, this is a subring of, not B, so or the, I should say here subring of C which is contained in B, the x module M equal to C serves the purpose in four. See, I just want to show you four, I want a faithful module over the Ax ring which is finite as a module over A. And we have given that it is containing a finite sub algebra, let me write here C.

So, this is a finite sub algebra and this is contained there, so this is a ring in particular subring of B, so it contains one. So therefore, that C will not be will be faithful module over Ax and that becomes very clear. So now it just matter of time your concept together to understand. And now to complete the proof, what we will prove is, one implies two, this we need to prove. Two implies three already we have proved above or it is trivial. Three implies four is also mentioned above.

And to complete the proof I have to prove four implies one. So, this is this two implications we will prove after the break. So, we will resume after the break. Thank you very much.