Introduction to Algebric Geometry and Commutative Algebra
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Lecture 53
Topological Properties of Spec A

Welcome to this course on introduction to Algebraic Geometry and Commutative Algebra. In the last lecture we have seen the prime spectrum of a commutative ring. We have also defined Zariski topology on that and we have done little bit basics of these. For example, we have noted formal Hilbert rules and (())(0:52) and because of the general setting how easy it was to give a proof and proof he just commutative to algebra language converted into geometric language.

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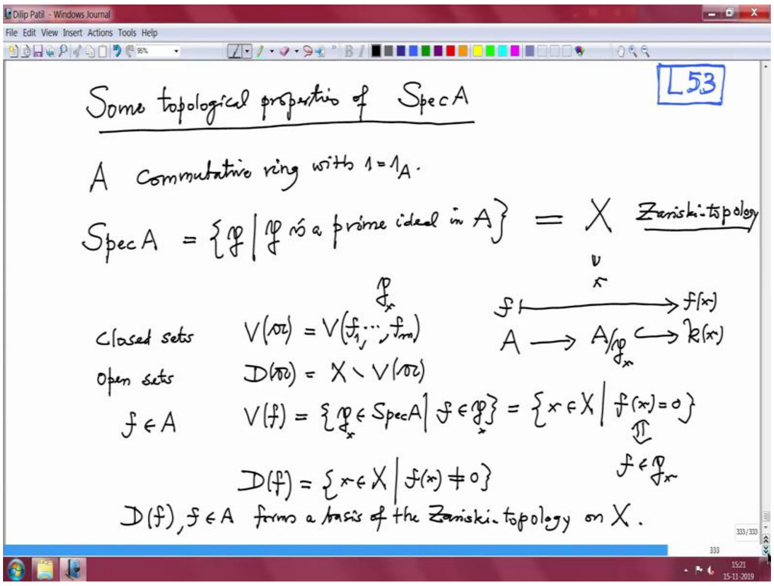

Now I am going to see today, some more properties of the spectrum. Some properties, some topological properties I would say, properties of the spectrum Spec of A, where A is a commutative ring with identity. So as it is, as usual, A is always our commutative ring with 1, 1 equal to 1A to be precise and we have defined these spectrum when I say the word spectrum, you one must remember the spectrum which appears in a optics.

So, I will I write that as an example, what is the connection between this spec and that. So this spec what we have defined is the set of all prime ideals P. P, where p is a prime ideal in A and remember notation, this we have denoted by X so, and on X we have this Zariski topology. So, whenever we see a word spectrum when that is always Zariski topology and our notation is if you have a point small x in X, when you want to switch to algebra, you should write P x.

Just to remember we have come back to algebra and we are going to deal with prime ideals maximal ideals intersection some etcetera and so on. So, first I will want to prove very basic property. So, what do we know, we know the closed sets, closed sets are precisely closed sets are precisely, V of an ideal A or this is also same thing as V of finitely many elements in the ring A f1 to fm and what are the open sets? Open sets are precisely D of A which are the complements this is X minus V A.

So, in particular when we have element f in the ring A, as I said last lecture so I will denote the elements of the ring by fg etc. because our model example of commutative ring with identity element is the polynomial ring over a field and those elements the elements of their that ring are precisely the polynomials and we denote polynomial by fg etc. Therefore, we I adopt this similar notation.

So, this is V f is by definition all those prime ideals, then P in spec A such that f belongs to P but this is same, this is Px our notation. So this you can think of same. These are all those, all those points x in X, such that f of x is 0, I keep saying f of x, but f of x remember f of x is the image of you have A here, you have A mod Px here, this is an integral domain. So it has a equation field that the kappa x, this is a quotient field of this integral domain and we have an element f here, this is a ring homomorphism.

So, the image of f under the ring homomorphism in this quotient field is denoted by f of x fx, I it is f of x but this is the image of f. So, that is 0 this is equivalent to saying that, because this map is injective if it is f of x is 0 here that means it is 0 here, but that will mean that it is in Px. So, this means f belongs to Px.

So, what is the complement df? df is precisely all those, directly write here x in X, such that f of x is nonzero. See, even when we were studying calculus, the usual standard thinking was being non zero is always an open property. So, therefore, these are the open sets and this is called basic open set.

So, df family d f as f in f f in A forms a basis of this Zariski topology on x. Now, these are the basic blocks like when you study for example calculus on Rn, then the basic open sets are precisely the open balls. So, these Df take the role of the open balls.

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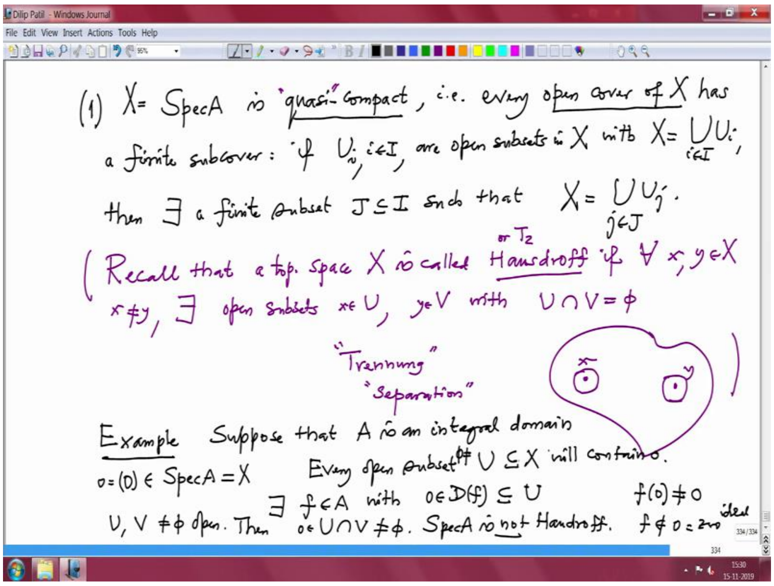

Now, the first one, I want to note that these spec A is quasi compact. I will recall what it means. So, that is every open cover let us call it x cover of x has a finite subcover, what is that mean let us write in symbols. So, this is the meaning in symbols, if Ui I in I are open subsets in X, with cover means, full X is covered by these Ui means x is a union of these Ui's, such a thing is called an open cover of x, then there exists a finite subset J of I, such that X is a union of U j.

So, that means, X is covered by finitely many of the Ui Uj's. So, such a thing is called quasi compact, before I go on, I will also explain the word why quasi compact, why this word quasi. Normally if you see standard books on topology the most of them will say compact, but, when usually one says a topological space is compact one usually assume that it is Hansdroff, Hansdroff means,

So, recall that these I am recalling for I will recall these quickly because we are not going to go too much into this. Recall that a topological space X is called Hansdroff if for every pair of points x and y in X, x not equal to y, there exists open subsets U and V which U contains X, y, y is contained in V with U intersection V is empty. See the, if you want to draw a picture, this is x something, there are two points here x and y, x not equal to y.

So, you can find a open set around this x And around this y they do not intersect such a thing is called Hansdroff or T2 it is also called T2, these there are many properties T0, T1, T2, T3 and half T4 etc. and they come from its T then abbreviation for Trannung, Trannung that means a separation, separation. So, these mean the two points you can separate them by neighborhoods, such a thing is called Hansdroff. But in our case, Zariski topology is almost never Hansdroff. So, let us write an example first, then we will come back to the proof of this.

So example suppose that A is an integral domain, then that means the 0 is a prime ideal in A and this I will write this as 0 only I will not write the bracket around that, 0 is a prime ideal, this is your X. And I say you need to open sets every open set. Note that every open set open subset U containing next will contain 0, why that? I what do I would show that it contains a 0 that means what? Okay, it is an open set U.

Therefore, it will contain a basic open set. So, there exist f in the ring A such that with Df is contained in U, but f cannot be 0, if f is 0, D 0 is everybody. So, if you have a proper open set U then this will contain this and 0 obviously belong to the basic open set f, D f, because what does that mean? This means f of 0 should not be 0 or in other words f should not belong to the ideal 0. This is the 0 ideal, this is the zero ideal, this is the primary ideal, because it is an integral domain.

Therefore, 0 will always belong there. Therefore, any proper open subset, no any set any open set proper even if it is equal to x, it will 0 will belong there. So, any open subset will continue basic open set and basic open set will always contain the prime ideal 0 that means, the point 0 in the topological space will always be there.

Therefore, when I take any two non empty open sets, I should say non empty, non empty open set if I take any two non empty open sets U and V both non empty open, then the intersection will always be non empty because 0 will always belong there because it belongs to both, therefore 0 always belong there.

So, therefore, there is no chance of separating two points at all because every two non empty open sets intersect. So therefore, this Spec A is not Hanstroff.

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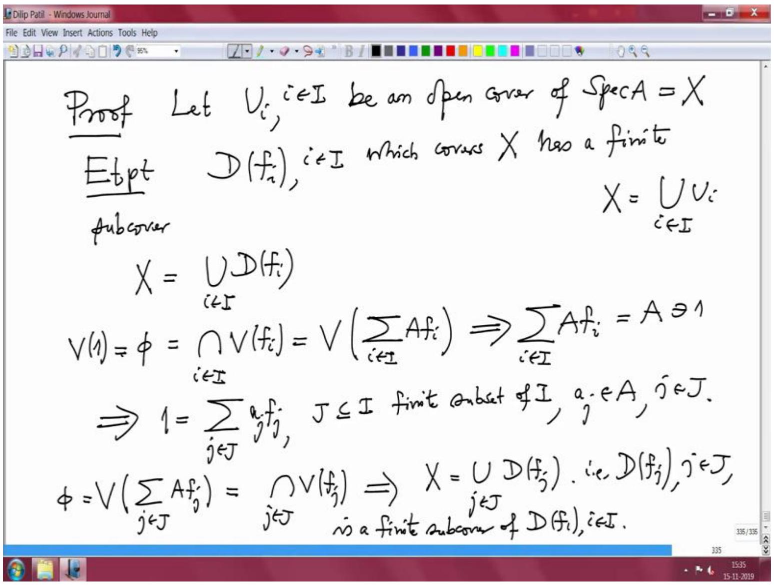

So now, we want to prove this assertion that they are quasi compact, means every open cover has a finite sub code so, we are proving that now, so proof. So, Let Ui i in I be an open cover of spec A which I will call it X and we want to check that finitely many Ui will cover X already. So, I can assume we may assume enough to prove that D fi I in I which covers X has a finite subcover.

Because X is a union of Ui's so every point in x in some Ui and I will concentrate on a basic neighborhood which contain that point. So that means the X will also be covered by basic open sets Df i and therefore, I will replace these Ui's by Dfi's then maybe the indexing sheet will change and then I want to extract now finite sub cover. What does that mean that x is the union of Dfi's? this means, this is this means, now I take the complements.

The complement will say that, the complement of X is empty set that will be this intersection will union will become intersection and the complements of these that is Vfi's, But we know the arbitrary intersection of closed sets is precisely the closed set, and closed set of whom? It is precisely V of the ideal generated by the fi's.

This is a property of V, we have checked in and we defend this, but this is V0, V1, sorry V1, V1 is empty and it is V of this, therefore, we know that these ideal must be unit ideal if, if V of this is empty that means, there is no prime ideal which contain these ideal in particular it has to be a unit ideal because if we do not the unit ideal, it will definitely be contained in a maximal ideal by Krull's theorem and that will be prime ideal and therefore, V cannot be empty.

So, this first of all implies the ideal generated by fi is a unit ideal. So, one will belong one belongs here, therefore, one will belong to this side. So, that will imply I can write one as a finite linear combination of fi. So, I will call it aj fj j is in j where J is a finite subset of I, and these coefficients aj they are in the ring A, j is in J but this will mean already that V of V of summation ideal generated by fj's is there finitely we know fj.

This is already empty because one belong there, again the same argument this is a unit ideal, therefore, it is not contained in any prime ideal therefore, V of that is empty set, but this is again intersection j in J V fj. Now, again take the complement of this. So, therefore, complement of empty set is X and the complement of the intersection will become union or j and then these V of j complement will become Dfj.

So, we have proved that X has this is a finite cover now, so that is D fj j in J is a finite sub cover of the original cover Dfi. So, that showed that it is quasi compact. So, we have seen it is quasi compact now when also, I will come back soon to the spectrum in optics, but before that I will write down few more properties.

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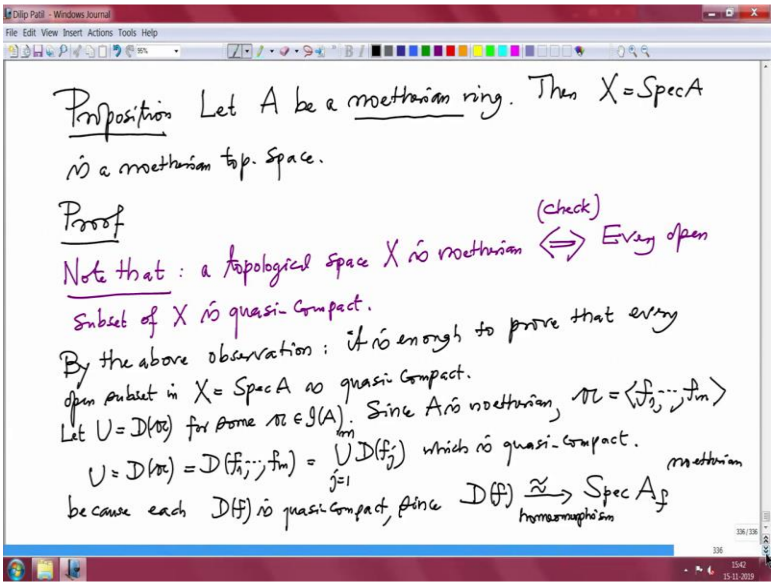

The next one if you like to call it a proposition. So, let A be a Noetherian ring then X equal to spec A is Noetherian topological space. So, recall that Noetherian topological space is the one which means that open subsets in X satisfies ACC or equivalently closed subsets in x satisfies DCC. So, that is the definition of Noetherian, we have seen last time or on can also say every non empty family of open sets in X has a maximal element or equivalently every non empty family of closed sets as a minimal element because this is because open sets and closed sets are the complements of each other.

Also one more property which I want to use in this proof is the following, X is a topological space. So, this is this I will write as a side mark note that this is very easy to check for topological space X is Noetherian if and only if, every open subset of X is quasi compact this will follow almost from a definition namely the non empty open subsets satisfies ACC, you can use that.

So, these I will just say check. So, we want to prove that this X is a Noetherian topology space. So, I did it precisely what I want to prove that every open subset is quasi compact. So, by the above observation it is enough to prove that every open subset in x equal to spec A as quasi compact. So that means what? How does the open subset look like so, start with an open subset, every open subset open subset will look like U which will be compliment of the closed set, that is look like D of A for some ideal A for some ideal A in A.

So, let this be an open subset and I want to prove that U is quasi compact, but you know, our ring is Noetherian. So, since A Noetherian, A is finally generated so that means A generated by finitely many elements f1 to fm, and what is Da then? Then U equal to Da which is equal to D of D of this f1 to fm but that is union of D of fj j in 1 to m. So, which is clearly quasi compact which is quasi compact.

This is because so, I will write because each Df is quasi compact, and why that? Since this Df we can identify with spec of A localized at f and A Noetherian, therefore this is Noetherian. And just now we have seen earlier that the spec A is quasi compact. Therefore, this this is a homomorphism. So therefore, it is quasi compact.

So, one more thing and then we will take a break these are the properties of the Zariski topology of the fi for the spectrum.

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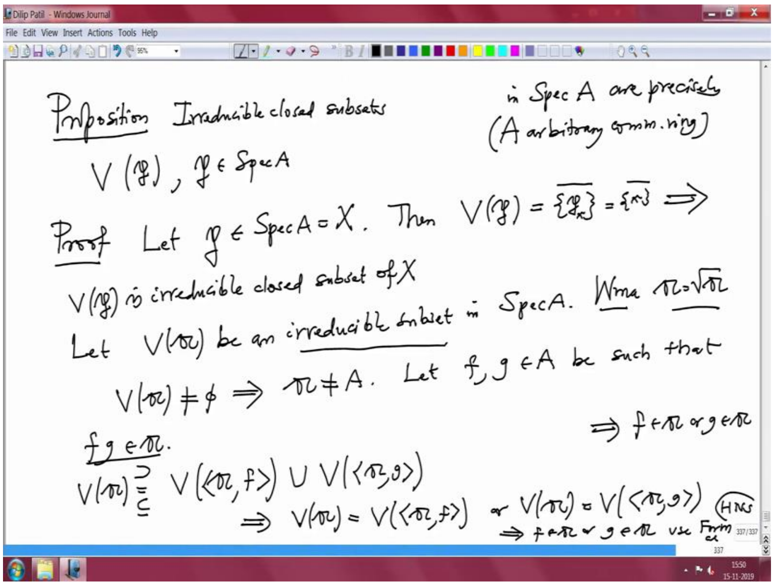

So next proposition, every closed irreducible subset in spec A is of the form V of p, where p is the prime ideal in spec A. So, we have closed irreducible means which you cannot break into two proper closed subsets, union of the two proper closed subsets. So proof so now, A is arbitrary commutative ring A is arbitrary commutative ring.

So, first of all start with any look at V of P V of P, first of all I am proving that if I take p in so, let p be a prime ideal in A which is we are calling it X, then what is VP? VP is all those prime ideals which contain P. So, I say this is precisely the closure of this P if you like Px because it contained P and it can this is close first of all and once it contain P it contained VP.

So this is a closure of x and we have seen in the last lecture closure of A irreducible subsidies again irreducible. Therefore, this is singleton therefore it is irreducible, therefore VP is irreducible, so, that proofs VP is irreducible closed subset of X. Conversely, actually what I am proving is irreducible, so let me be a little bit more precise irreducible, irreducible closed subsets in this are precisely this.

So, we had proved these are irreducible, conversely now I want to prove that every irreducible closed subset is a prime ideal P and it is of the form VP. So, start with a closed and irreversible, so, let V of A this is closed be an irreducible subset in spec A but do you remember we had proved that then the ideal has to be I will prove that this ideal has to be a prime ideal.

Now, we assume that is a radical ideal. Assume we may assume A equal to radical of A, because I replace A radical A this V will not change. Now, I will prove that A has to be prime ideal. So, it is irreducible subset. Therefore, V of A has to be non empty by definition of irreducible and now, I will prove that this proof first of all A is not the whole ring that is a requirement for need to be prime ideal.

So, to prove that it is a prime ideal. So, let f and g be two elements in the ring A, be such that f 10 g belongs to A then I am looking for proving either f is in A or g is in A that will prove that A is a prime ideal and that will finish the proof. So look at we have given this a look at V of A, but V of A then V of ideal generated by A and f union with ideal generated by A and g. This is we have checked this earlier also, but this is very easy to check f.

First of all the bigger the ideals, smaller the V therefore, this is this is very clear. Both are contained A therefore, the union is contained there. So, and now, now checks easily that from here, how do you check the inequality take any point here take any X here that takes will every element A in f in A not f, some f in A will vanish at that there therefore, this union is a product.

So, therefore, this is f times g is contained in A therefore, this is very easy to check the this inclusion that U use the fact that f 10 g is in this, but these are both are closed and this is union and therefore, one conclude from here that V of A because it is irreducible, V of A is either equal to V of ideal generated by A and f or V of A equal to, V of ideal generated by A and g.

But if V’s are equal then, but what does this mean this will mean that when I take the ideal on both sides, this is a by (())(35:09) that it will be the ideal A itself because A is radical ideal and this will this will contain a power of g and therefore, so from here you conclude that g will either f belongs to A or g belongs to A.

I will simply write use formal, formal HNS because it I do not say much that will follow from that. And therefore, we had proved that irreducible closed subsets in a spectrum are precisely V of p. And now I will we will have a break and after the break we will study some examples which are illuminating, so that will give us some more understanding about the spectrum. So thank you, we will meet after the break.